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随机网络中三种渗流相变的交叉

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摘 要：基于一个简单的分段线性权函数，研究了参数 α 统一两种不同三临界的渗流模型，当调节 α 的值从 1 到 0 渗流相变从连续变化到多重不连续和不连续. 通过计算不同系统尺寸下序参量的相对方差，表明 $\alpha=0.6$ 时在热力学极限下跌落为一个奇异峰，$\alpha=0.5$ 时在某个超临界区交织在一起，$\alpha=0.4$ 时在一个延伸的区间上随系统尺寸的增加变得更大. 因此对于相变从连续变化到多重不连续和从多重不连续变化到不连续，三相点分别位于 0.6 和 0.5 以及 0.4 和 0 之间. 该框架为理解随机网络中不同类型相变之间的交叉行为提供了见解.

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Crossover of three percolation transitions types in random networks

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Abstract: Based on a simple piecewise linear weighted function, we unified two different types of tricritical percolation with a parameter α in one model, in which by tuning α from 1 to 0 the phase transition can switch from continuous to multiple discontinuous to discontinuous. By calculating the relative variance of the order parameter with different system sizes, we find that it collapses to a singular peak at the critical point in the thermodynamical limit at $\alpha=0.6$, and interlaces together on a supercritical interval at $\alpha=0.5$, and becomes larger on an extended interval at $\alpha=0.4$ with increasing system size, respectively. It shows that the tricritical value of α is between 0.6 and 0.5 as well as between 0.4 and 0 from continuous to multiple discontinuous as well as from multiple discontinuous to discontinuous respectively. Our framework provides insights into understanding the crossover behaviors between different types of phase transition in random networks.

Keywords: Percolation; Phase transition; Relative variance; Random networks

1 Introduction

Percolation, describing the onset of large-scale connectivity of networks as edges are added, is one of most studied problems in statistical physics and has been widely applied in various systems[1-6]. As one of the most classical percolation models, the Erdös-Rényi (ER) random graph[7] chooses one edge uniformly and randomly at each time step and the resulting phase transi-

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tion is continuous. Instead of selecting one edge at each time step, a competitive percolation process with a given number of candidate edges can lead to so-called explosive percolation[8] where the order parameter is seemingly discontinuous in a relatively large system size. Stimulated by this work, various competition rules[9-18] are proposed, and some authors introduced weighted rules[19-24] where the edges are added with a certain probability. Later, it is shown that the phase transition induced by Achlioptas process is continuous in the thermodynamical limit[25]. Moreover, they have made clear that the percolation phase transition can be discontinuous if the number of the competition edges grows with the system size[25].

Although any rule based on picking a fixed number of random nodes would lead to a continuous phase transition[25], the authors also pointed out that there are Achlioptas processes whose order parameter has random fluctuations even in the thermodynamic limit[26]. Nagler *et al*. analyzed a devil's staircase model, in which the order parameter generates a multiple discontinuous phase transition accompanied by an infinite number of discontinuous jumps in the supercritical region[27]. After that, Schröder *et al*. studied a fractional growth percolation model[28] that generates a multiple discontinuous phase transition where the locations and sizes of the jumps are randomly distributed in the supercritical region. Moreover, in the supercritical region the relative variance of the order parameter is tending to a nonzero constant in the thermodynamical limit, implying non-self-averaging effect[28]. It is also reported that the relative variance of the order parameter oscillates with amplitudes even in the thermodynamical limit, both in the subcritical and supercritical region[29].

Recently, continuous and multiple discontinuous and discontinuous phase transitions are observed in one model with a parameter α. In the model[30] with appropriate value of α, due to a decreasing edge on the interval (N^α, N) the clusters with linear size are suppressed to grow in the supercritical region and the mergence between them leads to a multiple discontinuous phase transition. In this paper, we consider a percolation model based on a rather simple piecewise linear weighted function with a rising edge on the interval [1, N^α] and a lever edge on the interval (N^α, N), in which the percolation phase transition can change from continuous to multiple discontinuous to discontinuous as the value of α is tuned from 1 to 0. Distinctly different from the physical mechanism in the paper [30], in this work the selected probability of each cluster with size on the interval (N^α, N) is equal due to a lever edge, which provides insights into understanding the crossover behaviors of multiple phase transitions.

2 Model

The network starts with N isolated nodes, and at each time step add an edge between two different nodes, and in the adding-edges process the intracluster edges are excluded. In order to add an edge to the network, two nodes without having been connected in the present network are sequentially selected with a certain probability, which depends on the size and the corresponding number of the cluster containing the node. If a cluster contains i nodes, this cluster is named as i-node cluster, and the number of the i-node clusters is written as n_i. Thus the selected probability of the node belonging to the i-node clusters can be expressed as $n_i f(i) / \sum_i n_i f(i)$, in which $f(i)$ is called the weighted function and defined as

$$f(i) = \begin{cases} \dfrac{i}{N^\alpha}, 1 \leqslant i < N^\alpha \\ 1, \ N^\alpha \leqslant i \leqslant N \end{cases} \tag{1}$$

where α is a continuously tunable parameter on the interval [0, 1]. In the percolation process, the occupied edge density is defined as $t = L/N$ where L indicates the number of the occupied edges, and the order parameter is written as $c_1 = C_1/N$ where C_1 represents the size of the largest cluster at each time step.

3　Results and analysis

Typical evolution of the order parameter c_1 is plotted in Fig. 1 with the system size $N=2^{20}$. At $\alpha=1$, the percolation model becomes the classical ER random graph model that leads to a continuous phase transition[22]. When $\alpha=0.6$ (the red point in Fig. 1), the order parameter exhibits some jumps in the supercritical region, but the phase transition is fact continuous in the thermodynamic limit, as will be explained in the later paragraph. When $\alpha=0.5$ (the blue point in Fig. 1) and $\alpha=0.4$ (the purple point in Fig. 1), due to the distinctly multiple jumps of the order parameter in the supercritical region, they are considered as multiple discontinuous phase transitions, which will be explained in detail later.

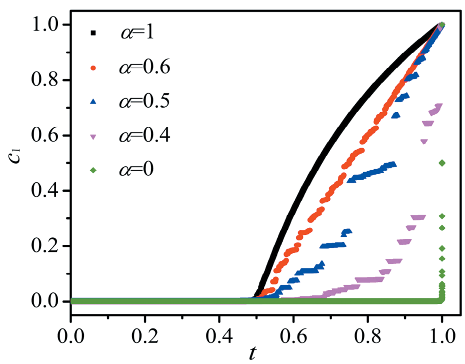

Fig. 1　Typical evolution of the order parameter c_1 with different α under $N=2^{20}$

We first show that the phase transition of $\alpha=0$ (the green point in Fig. 1) is discontinuous at the very end of the adding-edges process. As t increases from 0, the cluster size heterogeneity[31] (the number of distinct cluster sizes) increases till the lower pseudotransition point $t_l(N)$ when it becomes maximum and just after $t_l(N)$ many clusters with different sizes merge into one giant cluster with linear size. Fig. 2 shows the log-binned cluster size distribution with an average over 100 times at $t_l(N)$ and the corresponding fitting curves of different system sizes N. It is clear that the cluster size obeys the exponential distribution by the fitting curve equations, indicating a first order phase transition. On the other hand, according to the fitting curve equations with different system sizes N in Fig. 2, it is reasonable to conclude that $n_i \sim 4\exp(-\beta_i)$, in which β depends on the system size N. Due to the fact that $\sum_i in_i = N$, we immediately have $\sum_i 4\exp(-\beta_i)i \sim N$. By changing the summation symbol into integral in the above equation, one can get $\beta \sim 2N^{-0.5}$. Furthermore, let $n_{\mathrm{sum}}(t_l(N))$ denote the total number of the clusters at $t_l(N)$, and thus $n_{\mathrm{sum}}(t_l(N)) = \sum_i n_i \sim 2N^{0.5}$. Note that with one edge added to the network the total number of the clusters

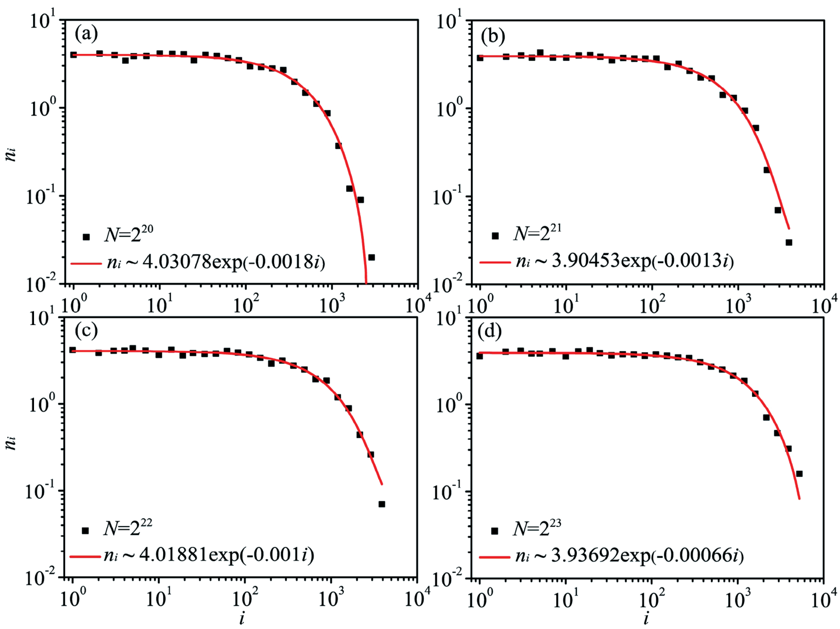

Fig. 2　For $\alpha=0$ log-binned cluster size distribution at the lower pseudotransition point and the corresponding fitting curves of different system sizes

accordingly reduces one, thus it is easy to obtain that $t_l(N)=1-n_{sum}(t_l(N))/N$. Combining with $n_{sum}(t_l(N))\sim 2N^{0.5}$, we find that the lower pseudotransition point $t_l(N)$ is converging to 1 in the thermodynamical limit. It is also worth noting that the percolation model with initially N isolated nodes becomes connected at $t=(N-1)/N$ converging to 1 in the thermodynamical limit. Therefore the percolation process at $\alpha=0$ leads to a first order phase transition at the very end of the adding-edges process in the thermodynamical limit.

According to the weighted function in equation (1) with appropriate value of α, two clusters with size i and j ($j>i>N^{\alpha}$) are generated at a certain occupied edge density t. From t it might be equal probability for i-node cluster and j-node cluster to be chosen in the subsequent adding-edges process, implying that the order parameter might be non-self-averaging even in the thermodynamical limit. Non-self-averaging is an important concept due to its applications in a broad range of real systems, ranging from spin grasses and neural networks to polymers and population biology[28]. In network percolation, non-self-averaging describes the phenomenon where the order parameter does not converge to a defined function of the occupied edge density t in the thermodynamical limit. Instead in the supercritical region, the order parameter has random fluctuations even in the thermodynamical limit[26-29]. Fig. 3 depicts several distinct realizations of the order parameter with $\alpha=0.5$ and $\alpha=0.4$. It is clear that the order parameter has tremendous variation from one realization to another in the supercritical region, indicating non-self-averaging phenomenon. Also note that the tail of the order parameter presents the shape of the continuous curve. The reason is, according to equation (1), $f(j)\gg f(1)$ if $j>N^{\alpha}$, and thus the largest cluster continuously absorbs small clusters with size approaching to one when the occupied edge density t is approaching to 1.

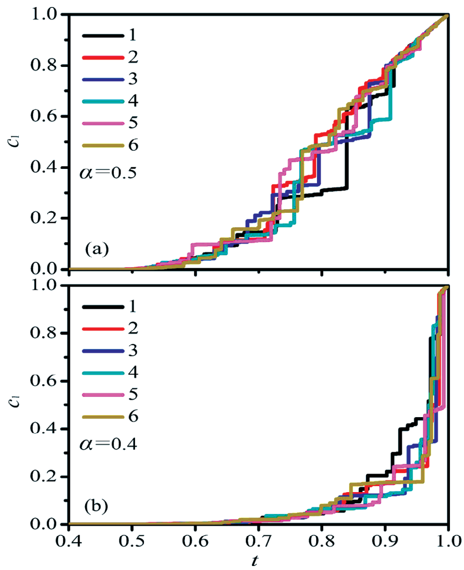

Fig. 3 For $N=2^{23}$ six simulations of the order parameter c_1 as a function of t for $\alpha=0.5$(a) and $\alpha=0.4$ (b)

Non-self-averaging is previously reported in some percolation models[26-29], where the relative variance of the order parameter is larger than zero in the supercritical region when the system size N is tending to infinite. To clearly illustrate the non-self-averaging effect in our model, we investigate the relative variance of the order parameter c_1, defined as

$$R_v=\frac{\langle c_1^2\rangle-\langle c_1\rangle^2}{\langle c_1\rangle^2} \tag{2}$$

where the brackets denote ensemble averaging. With an ensemble of 500 realizations Fig. 4 presents the relative variance R_v of the order parameter in dependence on the occupied edge density t at $\alpha=0.6$, 0.5 and 0.4. In Fig. 4(a) with $\alpha=0.6$, R_v is converging to zero in the supercritical region when the system size N becomes large. In Fig. 4(b) with $\alpha=0.5$, although R_v is rapidly converging to zero at the end of the occupied edge density when the system size N becomes large, but there exists a supercritical interval where R_v ($\neq 0$) of different system sizes interlaces together, indicating the non-self-averaging effect. In Fig. 4(c) with $\alpha=0.4$, with increasing system size R_v becomes larger on an extended interval indicating non-self-averaging effect. For continuous phase transition, it is universally known that large fluctuations in the relative variance R_v are observed only in the critical window, and at the critical point they collapse to a singular peak in the thermodynamical limit[5]. From the distinctly different supercritical behaviors of R_v in Figs 4(a) and 4(b) and 4(c), it is indicated that the tricritical value of α is between 0.6 and 0.5 for the phase transition from continuous to multiple discontinuous. Also note that the phase transition at $\alpha=0$ is discontinuous, thus the tricritical value of α is between 0.4 and 0 for the phase transition from multiple discontinuous to discontinuous.

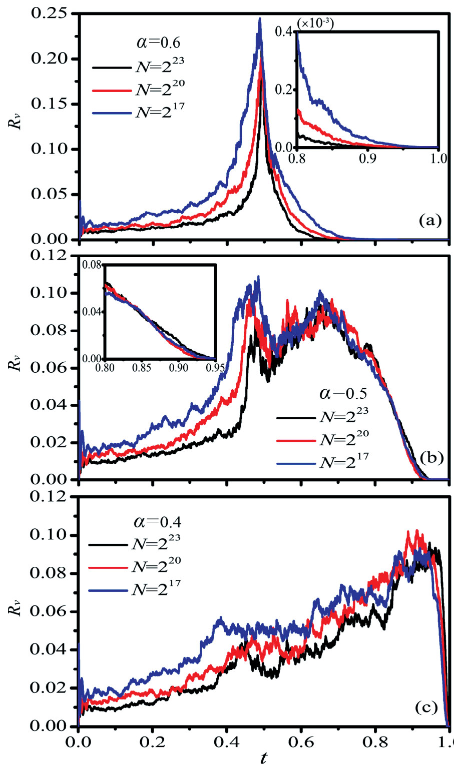

Fig. 4　The relative variance R_v of the order parameter in dependence on the occupied edge density t at $\alpha=0.6$ (a), $\alpha=0.5$ (b) and $\alpha=0.4$ (c)

4　Conclusions

In conclusion, based on a rather simple piecewise linear weighted functioncontinuous and multiple discontinuous and discontinuous phase transitions are unified into one percolation model with parameter α. At $\alpha=0$, we obtain an empirical formula of the cluster size distribution at the lower pseudotransition point, and based on the formula we show that the percolation process at $\alpha=0$ leads to a discontinuous phase transition at the very end of the adding-edges process in the thermodynamical limit. To understand how the phase transition changes from continuous to multiple discontinuous to discontinuous, we further calculate the relative variance of the order parameter with different system sizes. The results show that it collapses to a singular peak at the critical point in thermodynamical limit at $\alpha=0.6$, and interlaces together on a supercritical interval at $\alpha=0.5$, and becomes larger on an extended interval at $\alpha=0.4$ with increasing system size, respectively. Therefore it shows that the tricritical value of α is between 0.6 and 0.5 for the phase transition from continuous to multiple discontinuous, and that for the phase transition from multiple discontinuous to discontinuous the tricritical value of α is between 0.4 and 0.

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