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混合整数非线性规划问题的全局最优性条件

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摘　要：本文给出了带界约束的混合整数非线性规划问题全局极小点的必要条件，该问题包含连续优化问题和离散优化问题为特殊情形，得到了带界约束的混合整数非线性规划问题的充分全局最优性条件，其中规划问题的目标函数只需要二次连续可微.如果目标函数是二次的，则所得的全局最优性条件易于验证.数值例子说明了全局最优性条件的意义.

关键词：全局最优性条件；混合整数非线性规划问题；界约束
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Global optimality conditions for mixed integer nonlinear programming problems

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Abstract: In this paper, some necessary conditions for a given global minimizer applied to mixed integer nonlinear programming problem with bounded constraints which arises in continuous as well as discrete optimization are developed. Also, some sufficient global optimality conditions for mixed integer nonlinear programming problem with bounded constraints are established. The global optimality conditions readily apply to problems whose objective functions are generally twice continuously differentiable. If the objective functions are quadratic, then the global optimality conditions become verifiable. Some simple numerical examples can illustrate the significance of the optimality conditions.

Keywords: Global optimality conditions; Mixed integer nonlinear programming; Bounded constraints
(2010 MSC 90C46, 90C26, 90C30)

1　Introduction

Consider the following mixed integer nonlinear programming problem with bounded constraints:

$$(\text{MINP})\begin{cases}\min_{x\in\mathbf{R}^n} f(x),\\ \text{s.t. } x_i\in[u_i,v_i], i\in M,\\ x_j\in\{p_j,p_j+1,\cdots,q_j\}, j\in\mathbf{N},\end{cases}$$

where $f(x):\mathbf{R}^n\to\mathbf{R}$ is twice continuously differentiable on an open set containing the feasible set of the problem MINP, $M,N\subset\{1,2,\cdots,n\}$, $M\cap N=\varnothing$, $M\cup N=\{1,2,\cdots,n\}$, $u_i,v_i\in\mathbf{R}$ and $u_i<v_i$ for any $i\in M$, p_j,q_j are integers, and $p_j<q_j$ for all $j\in\mathbf{N}$.

Model MINP cover a broad range of optimization problems which arise in important applications of continuous as well as discrete optimization. Problem MINP has widespread practical applications, such as manufacturing, management science, operations research, engineering design,

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reliability networks, computational chemistry, facility planning and scheduling[1-5]. The approaches to solve problem MINP may be found in Ref. [6]. However, it is very difficult to solve problem MINP due to their nonlinear properties and mixed variables in the objective function. Furthermore, we even don't know whether the results we found are the global ones.

Recently, much attention has been focused on globally characterizing optimal solutions for various special cases of problem MINP. Beck *et al*.[7] have established a necessary global optimality condition for nonconvex quadratic optimization problems with binary constrains. Jeyakumar, Rubinov and Wu[8] have obtained necessary global optimality conditions, which are different from the Lagrange multiplier conditions for special classes of quadratic optimization problems. Jeyakumar *et al*. have obtained sufficient global optimality conditions for a quadratic minimization problem subject to box constraints or binary constraints in Ref. [8]. Li, Wu, and Quan[9] have established some necessary and sufficient global optimality conditions for quadratic integer programming problems. Jeyakumar *et al*.[10] have established some necessary and sufficient conditions for a given feasible point to be a global minimizer of some minimization problems with mixed variables. Wu *et al*.[11] have established some global optimality conditions and given global optimization methods for quartic polynomial optimization problems. Quan and Wu[12] have established some necessary global optimality conditions and some sufficient global optimality conditions for some classes of polynomial integer programming problems. Jeyakumar *et al*. presented necessary global optimality conditions for polynomial problems with box or bivalent constraints using separable polynomial relaxations in Ref. [13]. Zhao and Zhang[14] presented some global optimality sufficient conditions by combining the lagrangian function, L-subdifferential, L-regular cones.

In this paper, we establish some necessary global optimality conditions for a given global minimizer of mixed integer nonlinear programming problems with bounded constraints. We also derive some sufficient global optimality conditions for mixed integer nonlinear programming problems. Then necessary and sufficient global optimality conditions for the mixed integer quadratic programming problem are considered. Numerical examples are also given to illustrate the significance of our optimality conditions.

2 Necessary global optimality conditions for MINP

In this section, we derive the necessary global optimality conditions for problem MINP at a given global minimizer $\bar{x}$. Firstly, we present some notation that will be used throughout this paper. The real line is denoted by $\mathbf{R}$ and the n-dimensional Euclidean space is denoted by $\mathbf{R}^n$. For vectors $x, y \in \mathbf{R}^n$, $x \geqslant y$ means that $x_k \geqslant y_k$ for all $k = 1, 2, \cdots, n$. The notation $A \geqslant B$ means $A - B$ is a positive semidefinite matrix and $A \leqslant 0$ means $-A \geqslant 0$. For problem MINP, let

$$S := \{x = (x_1, x_2, \cdots, x_n)^T \mid x_i \in [u_i, v_i], i \in M; \; x_j \in \{p_j, p_{j+1}, \cdots, q_j\}, j \in \mathbf{N}\},$$

$$\bar{S} := \{x = (x_1, x_2, \cdots, x_n)^T \mid x_i \in [u_i, v_i], i \in M; \; x_j \in [p_j, q_j], j \in \mathbf{N}\}.$$

Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^T \in S$ for any $i \in M, j \in \mathbf{N}$, $k \in M \cup N$, we define

$$\hat{\bar{x}}_i = \begin{cases} -1, & \text{if } \bar{x}_i = u_i, \\ 1, & \text{if } \bar{x}_i = v_i, \\ \operatorname{sign}(\nabla f(\bar{x}))_i, & \text{if } u_i < x_i < v_i, \end{cases}$$

$$\hat{\bar{x}}_j = \begin{cases} -1, & \text{if } \bar{x}_j = p_j, \\ 1, & \text{if } \bar{x}_j = q_j, \\ \operatorname{sign}(\nabla f(\bar{x}))_j, & \text{if } p_j < x_j < q_j, \end{cases}$$

$$b_{\bar{x}_i} = \frac{\hat{\bar{x}}_i (\nabla f(\bar{x}))_i}{v_i - u_i},$$

$$b_{\bar{x}_j} = \max\left\{\frac{\hat{\bar{x}}_j (\nabla f(\bar{x}))_j}{1}, \frac{\hat{\bar{x}}_j (\nabla f(\bar{x}))_j}{q_j - p_j}\right\},$$

$$b_{\bar{x}} = (b_{x_1}, b_{x_2}, \cdots, b_{x_n})^T,$$

where

$$\operatorname{sign}(\nabla f(x))_k = \begin{cases} -1, & (\nabla f(x))_k < 0, \\ 0, & (\nabla f(x))_k = 0, \\ 1, & (\nabla f(x))_k > 0. \end{cases}$$

For a given vector $(\alpha_1, \alpha_2, \cdots, \alpha_n)^T$, we define

$Q=\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)$,

$\tilde{\alpha}_i=\min\{0,\alpha_i\}$,

$\forall i\in M, \tilde{\alpha}_j=\alpha_j, \forall j\in\mathbf{N}$,

$\tilde{Q}=\mathrm{diag}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)$.

Theorem 2.1 (Necessary global optimality condition) For problem MINP, we assume that there exists a diagonal matrix $Q=\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)$, such that $\nabla^2 f(x)-Q\leqslant 0$ for each $x\in\bar{S}$. If $\bar{x}\in S$ is a global minimizer of problem MINP, then the condition

$$[\mathrm{NC}]\ \mathrm{diag}(b_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}$$

holds.

Proof Let $h(x)=\frac{1}{2}x^TQx+(\nabla f(\bar{x})-Q\bar{x})^Tx$ and $\varphi(x)=f(x)-h(x)$. Then we have that

$$\nabla^2\varphi(x)=\nabla^2 f(x)-\nabla^2 h(x)=\nabla^2 f(x)-Q\leqslant 0, \forall x\in\bar{S}.$$

Thus $\varphi(x)$ is concave on $\bar{S}$ and

$$\nabla\varphi(\bar{x})=\nabla f(\bar{x})-\nabla h(\bar{x})=0.$$

So we get that $\varphi(x)\leqslant\varphi(\bar{x}), \forall x\in\bar{S}$ and

$$f(x)-f(\bar{x})\leqslant h(x)-h(\bar{x})$$

holds for all $x\in\bar{S}$. That is to say,

$$f(x)-f(\bar{x})\leqslant h(x)-h(\bar{x}), \forall x\in S.$$

So, if $\bar{x}$ is a global minimizer of problem MINP, then $h(x)-h(\bar{x})\geqslant 0, \forall x\in S$.

In the following, we prove

$$\sum_{k=1}^{n}\left[\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla f(\bar{x}))_k(x_k-\bar{x}_k)\right]\geqslant 0, \text{ for any } x\in S \quad (1)$$

if and only if, for any $k=1,2,\cdots,n$,

$$\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla f(\bar{x}))_k(x_k-\bar{x}_k)\geqslant 0, \text{ for any } x\in S \quad (2)$$

In fact, if there exists a $l_0\in M$ and a $y_{l_0}\in[u_l,v_l]$ such that

$$\frac{1}{2}\alpha_{l_0}(y_{l_0}-\bar{x}_{l_0})^2+(\nabla f(\bar{x}))_{l_0}(y_{l_0}-\bar{x}_{l_0})<0,$$

we let $x_{l_0}=y_{l_0}$ and $x_l=\bar{x}_l$ for all $l\in M, l\neq l_0$, $x_j=\bar{x}_j$ for all $j\in\mathbf{N}$. Then $x=(x_1,x_2,\cdots,x_n)^T\in S$ and we have

$$\sum_{k=1}^{n}\left[\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla f(\bar{x}))_k(x_k-\bar{x}_k)\right]=\frac{1}{2}\alpha_{l_0}(y_{l_0}-\bar{x}_{l_0})^2+(\nabla f(\bar{x}))_{l_0}(y_{l_0}-\bar{x}_{l_0})<0,$$

which contradicts to (1). If there exists a $j\in\mathbf{N}$ and a $y_{j_0}\in\{p_j,p_j+1,\cdots,q_j\}$ such that

$$\frac{1}{2}\alpha_{j_0}(y_{j_0}-\bar{x}_{j_0})^2+(\nabla f(\bar{x}))_{j_0}(y_{j_0}-\bar{x}_{j_0})<0,$$

we let $x_l=\bar{x}_l$ for all $l\in M$, $x_{j_0}=y_{j_0}$ and $x_j=\bar{x}_j$ for all $j\in\mathbf{N}, j\neq j_0$. Then $x=(x_1,x_2,\cdots,x_n)^T\in S$ and we have that

$$\sum_{k=1}^{n}\left[\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla f(\bar{x}))_k(x_k-\bar{x}_k)\right]=\frac{1}{2}\alpha_{j_0}(y_{j_0}-\bar{x}_{j_0})^2+(\nabla f(\bar{x}))_{j_0}(y_{j_0}-\bar{x}_{j_0})<0,$$

which contradicts to (1).

Now we verify that (2) is equivalent to the condition $\mathrm{diag}(b_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}$. We consider the following six cases:

Case 1. If $\bar{x}_l=u_l$, then (2) is equivalent to

$$\frac{1}{2}\alpha_l(x_l-\bar{x}_l)+(\nabla f(\bar{x}))_l\geqslant 0,$$

for any $x_l\in(u_l,v_l]$. If $\alpha_l\geqslant 0$, then (2) is equivalent to $(\nabla f(\bar{x}))_l\geqslant 0$; if $\alpha_l<0$, then (2) is equivalent to

$$(\nabla f(\bar{x}))_l\geqslant-\frac{(v_l-u_l)\alpha_l}{2}.$$

So if $\bar{x}_l=u_l$, (2) is equivalent to

$$\hat{\bar{x}}_l(\nabla f(\bar{x}))_l\leqslant\min\{0,\frac{(v_l-u_l)\alpha_l}{2}\}.$$

Case 2. If $\bar{x}_l=v_l$, then (2) is

$$\frac{1}{2}\alpha_l(x_l-\bar{x}_l)+(\nabla f(\bar{x}))_l\leqslant 0,$$

for any $x_l\in[u_l,v_l)$. If $\alpha_l\geqslant 0$, then (2) is equivalent to $(\nabla f(\bar{x}))_l\leqslant 0$; if $\alpha_l<0$, then (2) is equivalent to $(\nabla f(\bar{x}))_l\leqslant\frac{(v_l-u_l)\alpha_l}{2}$. So if $\bar{x}_l=v_l$, (2) is equivalent to

$$\hat{\bar{x}}_l(\nabla f(\bar{x}))_l\leqslant\min\{0,\frac{(v_l-u_l)\alpha_l}{2}\}.$$

Case 3. If $u_l<\bar{x}_l<v_l$, when $x_l\in(\bar{x}_l,v_l]$, then (2) is equivalent to $\frac{1}{2}\alpha_l(x_l-\bar{x}_l)+(\nabla f(\bar{x}))_l\geqslant 0$, when $x_l\in[u_l,\bar{x}_l)$, then (2) is equivalent to $\frac{1}{2}\alpha_l(x_l-\bar{x}_l)+(\nabla f(\bar{x}))_l\leqslant 0$, So if $u_l<\bar{x}_l<v_l$, (2) is equivalent to $(\nabla f(\bar{x}))_l=0$, $\alpha_l\geqslant 0$, so (2) is also equivalent to

$$\hat{\bar{x}}_l(\nabla f(\bar{x}))_l\leqslant\min\{0,\frac{(v_l-u_l)\alpha_l}{2}\}.$$

Case 4. If $\bar{x}_j=p_j$, then (2) is equivalent to

$$\frac{1}{2}\alpha_j(x_j-\bar{x}_j)+(\nabla f(\bar{x}))_j\geqslant 0,$$

for any $x_j\in\{p_j+1,p_j+2,\cdots,q_j\}$. If $\alpha_j\geqslant 0$, then (2) is equivalent to $(\nabla f(\bar{x}))_j\geqslant -\frac{\alpha_j}{2}$; if $\alpha_j<0$, then (2) is equivalent to $(\nabla f(\bar{x}))_j\geqslant -\frac{(q_j-p_j)\alpha_j}{2}$. So if $\bar{x}_j=p_j$, (2) is equivalent to

$$\hat{\bar{x}}_j(\nabla f(\bar{x}))_j\leqslant\min\{\frac{\alpha_j}{2},\frac{(q_j-p_j)\alpha_j}{2}\}.$$

Case 5. If $\bar{x}_j=q_j$, then (2) is equivalent to

$$\frac{1}{2}\alpha_j(x_j-\bar{x}_j)+(\nabla f(\bar{x}))_j\leqslant 0,$$

for any $x_j\in\{p_j,p_j+1,\cdots,q_j-1\}$. If $\alpha_j\geqslant 0$, then (2) is equivalent to $(\nabla f(\bar{x}))_j\leqslant\frac{\alpha_j}{2}$; if $\alpha_j<0$, then (2) is equivalent to $(\nabla f(\bar{x}))_j\leqslant\frac{(q_j-p_j)\alpha_j}{2}$. So if $\bar{x}_j=q_j$, (2) isequivalent to

$$\hat{\bar{x}}_j(\nabla f(\bar{x}))_j\leqslant\min\{\frac{\alpha_j}{2},\frac{(q_j-p_j)\alpha_j}{2}\}.$$

Case 6. If $\bar{x}_j\in\{p_j+1,\cdots,q_j-1\}$, when $x_j\in\{\bar{x}_j+1,\cdots,q_j\}$, (2) is equivalent to

$$\frac{1}{2}\alpha_j(x_j-\bar{x}_j)+(\nabla f(\bar{x}))_j\leqslant 0,$$

when $x_j\in\{p_j,\cdots,\bar{x}_j-1\}$, then (2) is equivalent to

$$\frac{1}{2}\alpha_j(x_j-\bar{x}_j)+(\nabla f(\bar{x}))_j\geqslant 0,$$

So if $\bar{x}_j\in\{p_j+1,\cdots,q_j-1\}$, (2) is equivalent to $\frac{\alpha_j}{2}\leqslant(\nabla f(\bar{x}))_j\leqslant\frac{\alpha_j}{2}$, $\alpha_j\geqslant 0$, so (2) is equivalent to

$$\hat{\bar{x}}_j(\nabla f(\bar{x}))_j\leqslant\min\{\frac{\alpha_j}{2},\frac{(q_j-p_j)\alpha_j}{2}\}.$$

By the above discussion, we know that if $\bar{x}$ is a global minimizer of problem MINP, then [NC] holds at $\bar{x}$.

***Remark* 1** Note that the objective function of problem MINP is generally twice continuously differentiable, while the objective functions of programming problem in Refs. [7, 9, 15] are quadratic, this result also extended the corresponding one in Ref. [12] to generally mixed integer cases.

We now provide a simple example where the necessary global optimality condition can be verified numerically.

Example 2. 2 Consider the problem

$$\min_{x\in\mathbf{R}^2}f(x):=$$

$$x_1^2+4x_1x_2-x_2^2-4x_1+2x_2-\frac{1}{16}x_1^4,$$

$$\text{s. t.}\begin{cases}x_1\in[-2,2],\\ x_2\in\{-2,-1,0,1,2\},\end{cases}$$

here

$$\nabla f(\bar{x})=$$

$$(-\frac{1}{4}x_1^3+2x_1+4x_2-4,4x_1-2x_2+2)^T,$$

$$\nabla^2 f(\bar{x})=\begin{pmatrix}2-\frac{3}{4}x_1^2 & 4\\ 4 & -2\end{pmatrix}.$$

By Example 2. 2, we know $\bar{x}=(2,-2)^T$ is a global minimizer, then $\nabla f(\bar{x})=(-10,14)^T$ and $b_{\bar{x}}=(-\frac{5}{2},-\frac{7}{2})^T$. Let $\boldsymbol{Q}=\begin{pmatrix}5 & 0\\ 0 & 4\end{pmatrix}$. We have

$$\nabla^2 f(\bar{x})-\boldsymbol{Q}=\begin{pmatrix}-3-\frac{3}{4}x_1^2 & 4\\ 4 & -6\end{pmatrix}\leqslant 0$$

for all $x_1\in[-2,2]$. So condition $\mathrm{diag}(b_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}$ holds. Therefore, condition [NC] holds at the global minimizer $\bar{x}$.

Now, we consider a special case of the problem MINP, mixed integer quadratic programming problem:

$$(\text{MIQP})\begin{cases}\min_{x\in\mathbf{R}^n}\frac{1}{2}x^TAx+a^Tx+c,\\ \text{s. t. }x_i\in[u_i,v_i],i\in M,\\ x_j\in\{p_j,p_j+1,\cdots,q_j\},j\in\mathbf{N},\end{cases}$$

where $A\in S^n$ and S^n is the set of all symmetric $n\times n$ matrices, $M,N\subset\{1,2,\cdots,n\}$, $M\cap N=\varnothing$, $M\cup N=\{1,2,\cdots,n\}$, $u_i,v_i\in\mathbf{R}$ and $u_i<v_i$ for any $i\in M$, p_j,q_j are integers and $p_j<q_j$ for all $j\in\mathbf{N}$. For given $\bar{x}=(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)^T\in S$, for any $i\in M$, $j\in\mathbf{N}$, we let

$$\hat{\bar{x}}_i=\begin{cases}-1,\text{ if }\bar{x}_i=u_i,\\ 1,\text{ if }\bar{x}_i=v_i,\\ \mathrm{sign}(a+A\bar{x})_i,\text{ if }u_i<x_i<v_i,\end{cases}$$

$$\hat{\bar{x}}_j=\begin{cases}-1,\text{ if }\bar{x}_j=p_j,\\ 1,\text{ if }\bar{x}_j=q_j\\ \mathrm{sign}(a+A\bar{x})_j,\text{ if }p_j<x_j<q_j,\end{cases}$$

$$b_{\bar{x}_i}=\frac{\hat{\bar{x}}_i(a+A\bar{x})_i}{v_i-u_i},$$

$$b_{\bar{x}_j}=\max\{\frac{\bar{x}\hat{}_j(a+A\bar{x})_j}{1},\frac{\hat{\bar{x}}_j(a+A\bar{x})_j}{q_j-p_j}\},$$

$$d_{\bar{x}}=(d_{x_1},d_{x_2},\cdots,d_{x_n})^T.$$

By Theorem 2.1, we have the following result which has been given in Ref. [15].

Corollary 2.3 For problem MIQP, we assume that there exists a diagonal matrix

$$Q=\text{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n),$$

such that $A-Q\leqslant 0$ for each $x\in\bar{S}$. If $\bar{x}\in S$ is a global minimizer of problem MINP, then the condition

$$[\text{QNC}]\ \text{diag}(d_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}$$

holds.

An example is now provided to illustrate that the global minimizer satisfies necessary optimality condition [QNC] of the problem MIQP.

Example 2.4 Consider the problem.

$$\min_{x\in\mathbf{R}^3} f(x):=\frac{3}{2}x_1^2+x_2^2-\frac{1}{2}x_3^2+2x_1x_2-x_1x_3+2x_2x_3+x_1-5x_2+x_3,$$

$$\text{s.t.}\begin{cases}x_1,x_3\in[0,2],\\ x_2\in\{0,1,2\},\end{cases}$$

here $A=\begin{pmatrix}3&2&-1\\2&2&2\\-1&2&-1\end{pmatrix}$, $a=(1,-5,1)^T$. By Example 2.4, we know $\bar{x}=(0,2,0)^T$ is a global minimizer, then

$$a+A\bar{x}=(5,-1,5)^T$$

and

$$d_{\bar{x}}=(-2.5,-0.5,-2.5)^T.$$

Let $Q=\begin{pmatrix}6&0&0\\0&6&0\\0&0&2\end{pmatrix}$, we have $A-Q=\begin{pmatrix}-3&2&-1\\2&-4&2\\-1&2&-3\end{pmatrix}\leqslant 0$. Condition [QNC] diag $(d_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}$ holds at the global minimizer $\bar{x}$.

3 Sufficient global optimality conditions for MINP

In this section we present sufficient global optimality conditions for problem MINP.

Theorem 3.1 (Sufficient global optimality condition) For problem MINP, we assume that there exists a diagonal matrix $Q=\text{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)\in S^n$, such that

$$(SC)\begin{cases}\nabla^2 f(x)-Q\geqslant 0,\ \forall x\in\bar{S},\\ \text{diag}(b_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}\end{cases}$$

holds, then $\bar{x}$ is a global minimizer of problem MINP.

Proof Let $h(x)=\frac{1}{2}x^TQx+(\nabla f(\bar{x})-Q\bar{x})^Tx$ and $\varphi(x)=f(x)-h(x)$. Then we have that

$$\nabla^2\varphi(x)=\nabla^2 f(x)-\nabla^2 h(x)=\nabla^2 f(x)-Q\geqslant 0,\ \forall x\in\bar{S}.$$

Thus $\varphi(x)$ is convex on $\bar{S}$ and

$$\nabla\varphi(\bar{x})=\nabla f(\bar{x})-\nabla h(\bar{x})=0.$$

So we get that $\varphi(x)\geqslant\varphi(\bar{x})$, $\forall x\in\bar{S}$ and

$$f(x)-f(\bar{x})\geqslant h(x)-h(\bar{x})$$

holds for all $x\in\bar{S}$. That is to say

$$f(x)-f(\bar{x})\geqslant h(x)-h(\bar{x}),\ \forall x\in S.$$

So if $h(x)-h(\bar{x})\geqslant 0$, $\forall x\in S$, then $\bar{x}$ is a global minimizer of problem MINP.

Similar to the proof of Theorem 2.1, we can prove

$$h(x)-h(\bar{x})=\sum_{k=1}^{n}\left[\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla f(\bar{x}))_k(x_k-\bar{x}_k)\right]\geqslant 0,$$

for any $x\in S$ is equivalent to the condition diag $(b_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}$. By the above discussion, we know that if condition [SC] holds at $\bar{x}$, then $\bar{x}$ is a global minimizer of problem MINP.

***Remark* 2** The objective functions of programming problem in Refs. [7～9, 14, 15] are quadratic, what discussed in Ref. [12] is some classes of polynomial integer programming problem, while the objective function of problem MINP is generally twice continuously differentiable, so this result extended the corresponding ones in Refs. [7～9,12,14,15].

The following example shows that the point which satisfies our sufficient global optimality condition is a global one.

Example 3.2 Consider the problem

$$\min_{x\in\mathbf{R}^2} f(x):=x_1^2+4x_1x_2-x_2^2-4x_1+$$

$$2x_2-\frac{1}{16}x_1^4,$$

$$\text{s. t.}\begin{cases}x_1\in[-2,2],\\x_2\in\{-2,-1,0,1,2\},\end{cases}$$

here

$$\nabla f(\bar{x})=$$

$$(-\frac{1}{4}x_1^3+2x_1+4x_2-4,4x_1-2x_2+2)^T,$$

$$\nabla^2 f(\bar{x})=\begin{pmatrix}2-\frac{3}{4}x_1^2 & 4\\ 4 & -2\end{pmatrix}.$$

Let $\bar{x}=(2,-2)^T$. Then $\nabla f(\bar{x})=(-10,14)^T$ and $b_{\bar{x}}=(-\frac{5}{2},-\frac{7}{2})^T$. Let $Q=\begin{pmatrix}-5 & 0\\ 0 & -7\end{pmatrix}$. We have

$$\nabla^2 f(\bar{x})-Q=\begin{pmatrix}7-\frac{3}{4}x_1^2 & 4\\ 4 & 5\end{pmatrix}\geqslant 0$$

for all $x_1\in[-2,2]$. And the condition $\mathrm{diag}(b_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}$ holds. So condition [SC] holds at $\bar{x}$, hence $\bar{x}=(2,-2)^T$ is a global minimizer of Example 3.2.

By Theorem 3.1, we have the following result which has been given in Ref.[15].

Corollary 3.3　For problem MIQP, we assume that there exists a diagonal matrix

$$Q=\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_n)\in S^n$$

such that

$$(\mathrm{QSC})\begin{cases}A-Q\geqslant 0,\ \forall x\in\bar{S},\\ \mathrm{diag}(d_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}\end{cases}$$

holds, then $\bar{x}$ is a global minimizer of problem MIQP.

An example is now provided to illustrate that conditions [QSC] can be used to identify a global minimizer of the problem MIQP.

Eexample 3.4　Consider the problem

$$\min_{x\in\mathbf{R}^3} f(x):=\frac{3}{2}x_1^2+x_2^2-\frac{1}{2}x_3^2+2x_1x_2-x_1x_3+2x_2x_3+x_1-5x_2+x_3,$$

$$\text{s. t.}\begin{cases}x_1,x_3\in[0,2],\\x_2\in\{0,1,2\},\end{cases}$$

here $A=\begin{pmatrix}3 & 2 & -1\\ 2 & 2 & 2\\ -1 & 2 & -1\end{pmatrix}$, $a=(1,-5,1)^T$. Let $\bar{x}=(0,2,0)^T$ and $a+A\bar{x}=(5,-1,5)^T$ and

$$d_{\bar{x}}=(-2.5,-0.5,-2.5)^T.$$

Let

$$Q=\begin{pmatrix}-5 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -5\end{pmatrix}.$$

We have

$$A-Q=\begin{pmatrix}8 & 2 & -1\\ 2 & 3 & 2\\ -1 & 2 & 4\end{pmatrix}\geqslant 0.$$

Then condition [QNC] $\mathrm{diag}(d_{\bar{x}})\leqslant\frac{1}{2}\tilde{Q}$ holds at $\bar{x}$. So $\bar{x}$ is a global minimizer of Example 3.4.

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