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带有非线性阻尼的非线性发展方程的时间依赖吸引子

刘亭亭，马巧珍
（西北师范大学数学与统计学院，兰州 730070）

摘 要：本文考虑带有非线性阻尼项的非线性发展方程解的长时间行为. 基于时间依赖空间中的吸引子理论,本文利用压缩函数方法和一些估计技巧证明了带有临界非线性项的非线性发展方程时间依赖吸引子的存在性.

关键词：非线性发展方程；非线性阻尼；时间依赖吸引子；压缩函数.
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Time-dependent global attractor of nonlinear evolution equation with nonlinear damping

LIU Ting-Ting, *MA Qiao-Zhen*
(College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China)

Abstract: In this paper, we consider long-time behavior of solutions for the nonlinear evolution equation with nonlinear damping. With the theory of process on time-dependent spaces and some detailed estimates, we prove the existence of the time-dependent attractor for the nonlinear evolution equation with critical nonlinearity by using the contractive functions.

Keywords: Nonlinear evolution equation; Nonlinear damping; Time-dependent attractor; Contractive functions
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1 Introduction

Let Ω be an open bounded set of $\mathbf{R}^3$ with smooth boundary $\partial\Omega$. We consider the following equations

$$\begin{cases} u_{tt}-\Delta u+a(x)g(u_t)-\varepsilon(t)\Delta u_{tt}+f(u)= \\ \quad h(x),\ x\in\Omega,\ t>\tau, \\ (u(x,\tau),u_t(x,\tau))=(u_0(x),u_1(x)),\ x\in\Omega, \\ u|_{\partial\Omega}=0,\ x\in\partial\Omega \end{cases} \tag{1}$$

where $u(x,t)$ is an unknown function, $h(x)\in L^2(\Omega)$, $\varepsilon=\varepsilon(t)$ is a decreasing bounded function along with

$$\lim_{t\to+\infty}\varepsilon(t)=0 \tag{2}$$

and there exists a constant $L>0$ such that

$$\sup_{t\in\mathbf{R}}[|\varepsilon(t)|+|\varepsilon'(t)|]\leqslant L \tag{3}$$

The function $a(x)$ satisfies

$$a(x)\in L^\infty,\ a(x)\geqslant\alpha_0>0 \tag{4}$$

in Ω, where α_0 is a constant. The nonlinear damping $g\in C^1(\mathbf{R})$, $g(0)=0$, here g is strictly increasing, and satisfies

$$\liminf_{|s|\to+\infty}g'(s)>0 \tag{5}$$

$$|g(s)|\leqslant C_0(1+|s|^p),1\leqslant p<5 \tag{6}$$

The nonlinear term $f\in C^2(\mathbf{R})$, $f(0)=0$, and satisfies

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作者简介：刘亭亭(1991－)，女，硕士研究生，主要研究方向为无穷维动力系统. E-mail：1171638929@qq.com
通讯作者：马巧珍. E-mail：maqzh@nwnu.edu.cn

$$|f'(s)| \leqslant C_1(1+|s|^4),\ \forall s \in \mathbf{R} \tag{7}$$

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1,\ \forall s \in \mathbf{R} \tag{8}$$

where λ_1 is the first eigenvalue of the strictly positive Dirichlet operator $A = -\Delta$, C_0 and C_1 are positive constants.

Equation (1), which models the vibration of a nonlinear elastic rod, is used to represent the propagation problems of lengthways-wave in nonlinear elastics rods and ion-sonic of space transformation[1-3].

When ε is a positive constant independent of time t and the damping term is linear, the long-time behavior of solutions to equation (1) has been treated in many papers[4-8]. When the damping term is nonlinear, there were also several works devoted to this topic under the different conditions and spaces such as Refs. [13-15,17].

In the case when ε is a positive decreasing function which vanishing at infinity, the problem (1) becomes more complex and interesting. One of the reason is that the dynamical system associated with (1) is still understood under the non-autonomous framework even though the forcing term in the equation is not dependent on the time t. In order to solve these problems, Conti, Pata and Temam[9] presented a notion of time-dependent attractor based only on the minimality with respect to the pullback attraction property, and generalized the recent theory of attractors in time-dependent spaces developed in Ref. [16]. Meanwhile, they exploited the new framework to study the long-term behavior of the solutions to the following weak damped wave equation with time-dependent speed of propagation

$$\varepsilon(t)u_{tt} - \Delta u + \alpha u_t + f(u) = g(x) \tag{9}$$

and obtained the existence of the time-dependent global attractor, which converges in a suitable sense to the attractor of the parabolic equation $\alpha u_t - \Delta u + f(u) = g(x)$ (see Ref. [10]). Recently, in Ref. [11], the authors continued to show the asymptotic structure of time-dependent global attractor to the following specific one-dimensional wave equation

$$\varepsilon(t)u_{tt} - u_{xx} + [1+\varepsilon f'(u)]u_t + f(u) = h \tag{10}$$

To the best of our knowledge, the asymptotic behavior of the solution for the wave equation with nonlinear damping $g(u_t)$ on the time-dependent space was first paid attention to by Meng and Zhong[12], and they structured a new technical method to verify compactness of the process via contractive functions and obtained the existence of time-dependent global attractor to the corresponding problem.

Motivated by Refs. [9～12], we study the existence of time-dependent global attractor for (1) by using the methods borrowed from Ref. [12]. It is worth mentioning that we make use of a weaker dissipative condition (8) in present paper, because the conditions (13) and (14) the authors exploited in Ref. [12] were derived by $\liminf_{|s|\to\infty} f'(s) > -\lambda_1$ (see Ref. [18]).

2 Preliminaries

In this section, we iterate some notations and abstract results.

Without loss of generality, set $H = L^2(\Omega)$, and equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For $0 \leqslant s \leqslant 2$, we define the hierarchy of (compactly) nested Hilbert spaces

$$H^s = D(A^{\frac{s}{2}}), \langle w, v \rangle_s = \langle A^{\frac{s}{2}} w, A^{\frac{s}{2}} v \rangle,$$
$$\| w \|_s = \| A^{\frac{s}{2}} w \|.$$

Then for $t \in \mathbf{R}$ and $0 \leqslant s \leqslant 2$, we introduce the time-dependent spaces

$$H_t = H^1 \times H^1 \text{ with } \{a, b\}^2_{H_t} = \| a \|^2_1 + \| b \|^2 + \varepsilon(t) \| b \|^2_1.$$

Note that the spaces H_t are all the same as linear spaces, and the norm $\| z \|^2_{H_t}$ and $\| \cdot \|^2_{H_\tau}$ are equivalent for any fixed $t, \tau \in \mathbf{R}$.

For nonlinear function g (see Refs. [14,15]), by condition (6) we have

$$|g(s)|^{\frac{p+1}{p}} = |g(s)|^{\frac{1}{p}} + |g(s)|^p \leqslant C'_0(1+|s|)|g(s)| \leqslant C'_0 + C'_0 g(s)s \tag{11}$$

Furthermore, there holds

$$|g(s)| \leqslant C + C(g(s)s)^{\frac{p}{p+1}} \tag{12}$$

Set $F(u) = \int_0^u f(s)\mathrm{d}s$. The following results are immediately obtained.

Lemma 2.1[4,5,7] From (8) it is easy to obtain that for $0<\lambda<\lambda_1$,

$$\int_\Omega F(u)\mathrm{d}x \geqslant -\frac{\lambda}{4}\|u\|^2 - C_0|\Omega| \tag{13}$$

$$\int_\Omega f(u)u\mathrm{d}x \geqslant -\frac{\lambda}{2}\|u\|^2 - C_0|\Omega| \tag{14}$$

Lemma 2.2[12,14,17] Let $g(\cdot)$ satisfies condition (6). Then for any $\delta>0$ there exists a positive constant C_δ, such that $|u-v|^2 \leqslant \delta + C_\delta(g(u)-g(v))(u-v)$ for all $u,v\in\mathbf{R}$.

Theorem 2.3[12] Let $U(\cdot,\cdot)$ be a process in a family of Banach space $\{X_t\}_{t\in\mathbf{R}}$. Then $U(\cdot,\cdot)$ has a time-dependent global attractor $U^*=\{A_t^*\}_{t\in\mathbf{R}}$ satisfying $\{A_t^*\}_{t\in\mathbf{R}}=\bigcap_{s\leqslant t}\overline{\bigcup_{\tau\leqslant s}U(t,\tau)B_\tau}$ if and only if

(i) $U(\cdot,\cdot)$ has a pullback absorbing family $B=\{B_t\}_{t\in\mathbf{R}}$;

(ii) $U(\cdot,\cdot)$ is pullback asymptotically compact.

Theorem 2.4[12] Let $U(\cdot,\cdot)$ be a process onand has a pullback absorbing family $B=\{B_t\}_{t\in\mathbf{R}}$. Moreover, assume that for any $\varepsilon>0$ there exist $T(\varepsilon)\leqslant t$, $\varphi_T^t\in C(B_T)$ such that

$$\|U(t,T)x-U(t,T)y\|\leqslant\varepsilon+\varphi_T^t(x,y),\quad \forall x,y\in B_T,$$

for ant fixed $t\in\mathbf{R}$. Then $U(\cdot,\cdot)$ is pullback asymptotically compact, where $C(B_T)$ denotes the set of all contractive function on $B_T\times B_T$.

3 Existence of the time-dependent global attractor

3.1 Well-posedness and Time-dependent absorbing set

Global existence of solution u to (1) is classical, by using the standard Galerkin approximation method[4,7,18], that is, if (2)~(8) hold, then the problem (1) has a weak solution u with

$$u\in C([\tau,t],H^1),\ u_t\in C([\tau,t],H^1).$$

Uniqueness of solution will then follow by the continuous dependence estimate (15) in Lemma 3.1. Therefore, we can define the solution process with $[\tau,t]$, $t\geqslant\tau\in\mathbf{R}$. $U(t,\tau):H_\tau\to H_t$ acting as $U(t,\tau)z(\tau)=\{u(t),u_t(t)\}$, where u is the unique solution of (1) with initial time τ and initial condition $z=\{u_0,u_1\}\in H_\tau$.

For brevity, we denote C and C_i, $i=1,2,\cdots$ be a family of positive constant, which will change in the different line, even in the same line.

Lemma 3.1 Under the assumptions (2)~(8), for every pair of initial data $z_i(\tau)=\{u_0^i,u_1^i\}\in H_\tau$, $i=1,2$, such that $\|z_i(\tau)\|_{H_\tau}\leqslant R$, $i=1,2$, the difference of the corresponding solutions of (1) satisfies

$$\|U(t,\tau)z_1(\tau)-U(t,\tau)z_2(\tau)\|_{H_t}\leqslant \mathrm{e}^{C(t-\tau)}\|z_1(\tau)-z_2(\tau)\|_{H_\tau},\ \forall t\geqslant\tau \tag{15}$$

for some $C=C(R)\geqslant 0$.

Proof Let $z_1(\tau),z_2(\tau)\in H_\tau$ such that $\|z_i(\tau)\|_{H_\tau}\leqslant R$, $i=1,2$, and denote by C a generic positive constant depending on R but independent of $z_i(t)$. We first observe that the energy estimate in Lemma 3.2 below ensures

$$\|U(t,\tau)z_i(\tau)\|_{H_t}\leqslant C \tag{16}$$

We call $\{u_i(t),\partial_t u_i(t)\}=U(t,\tau)z_i(\tau)$ and denote $\bar{z}(t)=\{\bar{u}(t),\bar{u}_t(t)\}=U(t,\tau)z_1(\tau)-U(t,\tau)z_2(\tau)$. Then the difference between the two solutions satisfies

$$\begin{cases}\bar{u}_{tt}-\Delta\bar{u}+a(x)(g(u_{1t})-g(u_{2t}))-\\ \quad\varepsilon(t)\Delta\bar{u}_{tt}+f(u_1)-f(u_2)=0,x\in\Omega,t>\tau,\\ (\bar{u}(x,\tau),\bar{u}_t(x,\tau))=z_1-z_2,\\ \bar{u}|_{\partial\Omega}=0.\end{cases}$$

Multiplying the above equation by $2\bar{u}_t$ and integrating over Ω we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{z}\|_{H_t}^2-\varepsilon'(t)\|\overline{u_t}\|_1^2+2\int_\Omega a(x)(g(u_{1t})-g(u_{2t}))\bar{u}_t\mathrm{d}x=-2[f(u_1)-f(u_2),\bar{u}_t].$$

Since g is strictly increasing, together with (4) we have

$$2\int_\Omega a(x)(g(u_{1t})-g(u_{2t}))\bar{u}_t\mathrm{d}x\geqslant 0.$$

By exploiting conditions (7), (16) and Hölder Young inequality and $H^1\to L^6(\Omega)$, it yields

$$-2[f(u_1)-f(u_2),\bar{u}_t]\leqslant C\int_\Omega(1+|u_1|^4+|u_2|^4)\cdot\bar{u}\cdot\bar{u}_t\mathrm{d}x\leqslant C\|\bar{u}_t\|_1^2+C\|\bar{u}\|_1^2\leqslant \frac{C}{\varepsilon(t)}[L\|\bar{u}_t\|^2+\varepsilon(t)\|\bar{u}_t\|_1^2+L\|\bar{u}\|_1^2]\leqslant\frac{C(L+1)}{\varepsilon(t)}\|\bar{z}\|_{H_t}^2.$$

Thus, thanks to $\varepsilon'(t)<0$, we end up with the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{z}(t)\|_{H_t}^2\leqslant\frac{C(L+1)}{\varepsilon(t)}\|\bar{z}\|_{H_t}^2,$$

then applying the Gronwall Lemma on $[\tau,t]$, we obtain

$$\|\bar{z}(t)\|_{H_t}^2\leqslant \mathrm{e}^{C(L+1)\int_\tau^t\frac{1}{\varepsilon(s)}\mathrm{d}s}\|\bar{z}(\tau)\|_{H_\tau}^2,$$

where $C>0$ is a constant depending on R.

Lemma 3.2 Under the assumptions (2)~(8), for any initial data $z(\tau)\in B_\tau(R)\subset H_\tau$, there exists $R_0>0$, such that the family $B=\{B_t(R_0)\}_{t\in\mathbf{R}}$ is a time-dependent absorbing for the process $U(t,\tau)$ corresponding to (1).

Proof Denote

$$E_0(t)=\frac{1}{2}\|U(t,\tau)z\|_{H_t}^2+\int_\Omega F(u)\mathrm{d}x-\int_\Omega hu\,\mathrm{d}x.$$

Multiplying (1) by u_t and integrating over Ω, we achieve

$$\frac{\mathrm{d}}{\mathrm{d}t}E_0(t)+\int_\Omega a(x)g(u_t)u_t\mathrm{d}x-\frac{\varepsilon'(t)}{2}\|u_t\|_1^2=0 \tag{17}$$

According to the conditions of $a(x)$, $g(u_t)$ and $\varepsilon(t)$, we have

$$\int_\Omega a(x)g(u_t)u_t\mathrm{d}x-\frac{\varepsilon'(t)}{2}\|u_t\|_1^2>0.$$

Integrating (17) over $[\tau,t]$, we get

$$E_0(t)\leqslant E_0(\tau),\ \forall t\geqslant\tau \tag{18}$$

From (13) and Sobolev's embeddings we deduce

$$E_0(t)\geqslant\frac{\varepsilon(t)}{2}\|u_t\|_1^2+\frac{1}{2}\|u_t\|^2+\left(\frac{1}{2}-\frac{\lambda}{4\lambda_1}\right)\|u\|_1^2-C_0|\Omega|-\frac{\lambda}{4}\|u\|^2-\frac{1}{\lambda}\|h\|^2\geqslant\frac{\varepsilon(t)}{2}\|u_t\|_1^2+\frac{1}{2}\|u_t\|^2+\frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right)\|u\|_1^2-(C_0|\Omega|+\frac{1}{\lambda}\|h\|^2)\geqslant-(C_0|\Omega|+\frac{1}{\lambda}\|h\|^2) \tag{19}$$

By virtue of (8) and Sobolev's embeddings there exist positive constants $C_i>0$, $i=1,2,3,4$, such that

$$C_1\|U(t,\tau)z\|_{H_t}^2-C_2\leqslant E_0(t)\leqslant C_3\|U(t,\tau)z\|_{H_t}^2+C_4.$$

Thus, together with (17) and (19), it follows that

$$\int_\tau^t\int_\Omega a(x)g(u_t)\cdot u_t\mathrm{d}x-\frac{1}{2}\int_\tau^t\varepsilon'(t)\|u_t(s)\|_1^2\mathrm{d}s=E_0(\tau)-E_0(t)\leqslant E_0(\tau)+(C_0|\Omega|+\frac{1}{\lambda}\|h\|^2) \tag{20}$$

On the other hand, combining with (12), (19) and the Hölder inequality, there holds

$$\left|\int_\Omega a(x)g(u_t)u\mathrm{d}x\right|\leqslant C\int_\Omega a(x)|u|\mathrm{d}x+C\int_\Omega a(x)(g(u_t)u_t)^{\frac{p}{p+1}}|u|\mathrm{d}x\leqslant C\int_\Omega a(x)|u|\mathrm{d}x+\eta\|u\|_1^2+C_\eta\|u\|_1^{\frac{p-1}{p}}\int_\Omega a(x)g(u_t)u_t\mathrm{d}x \tag{21}$$

where $\eta>0$ is a small enough constant, which will be determined later.

Multiplying (1) by $u_t+\delta u$ and integrating over Ω, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\delta(t)+I(t)=0 \tag{22}$$

where

$$E_\delta(t)=E_0(t)+\delta[u_t,u]+\delta\varepsilon(t)[\nabla u_t,\nabla u],$$

$$I(t)=\frac{|\varepsilon'(t)|}{2}\|u_t\|_1^2-\delta\|u_t\|^2-\delta\varepsilon(t)\|u_t\|_1^2+\delta\|u\|_1^2+\delta\langle f(u),u\rangle-\delta\langle h,u\rangle+\int_\Omega a(x)g(u_t)(u_t+\delta u)\mathrm{d}x-\delta\varepsilon'(t)\langle\nabla u_t,\nabla u\rangle.$$

In the light of (3), (19), Hölder and Young inequalities, taking δ small enough, we arrive at

$$E_\delta(t)\geqslant\frac{\varepsilon(t)}{2}\|u_t\|_1^2+\frac{1}{2}\|u_t\|^2+\frac{1}{2}(1-\frac{\lambda}{\lambda_1})\|u\|_1^2-(C_0|\Omega|+\frac{1}{\lambda}\|h\|^2)-\left[\frac{\delta}{2}\|u_t\|^2+\frac{\delta}{2\lambda_1}\|u\|_1^2\right]-\left[\frac{1}{4}\varepsilon(t)\|u_t\|^2+\delta^2L\|u\|_1^2\right]\geqslant\frac{\varepsilon(t)}{4}\|u_t\|_1^2+\frac{1}{2}(1-\delta)\|u_t\|^2+\frac{1}{2}(1-\frac{\lambda}{\lambda_1}-\frac{\delta}{\lambda_1}-2\delta^2L)\|u\|_1^2-(C_0|\Omega|+\frac{1}{\lambda}\|h\|^2)\geqslant\frac{1}{8}(\|u_t\|^2+\varepsilon(t)\|u_t\|_1^2+\|u\|_1^2)-$$

$$(C_0|\Omega| + \frac{1}{\lambda}\|h\|^2) \tag{23}$$

Thanks to (5) and (6), there exist $\delta > 0$ and $C_\delta > 0$ such that

$$\int_\Omega a(x)g(u_t)\cdot u_t \mathrm{d}x \geqslant 2\delta\|u_t\|^2 - C_\delta|\Omega|.$$

Moreover,

$$-\delta\varepsilon'(t)[\nabla u_t, \nabla u] \geqslant -\frac{1}{4}|\varepsilon'(t)|\ \|u_t\|_1^2 - \delta^2 L\|u\|_1^2.$$

Hence, combining with (14), (21) provided η and δ mall enough, we conclude

$$\begin{aligned}I(t) \geqslant &(\frac{|\varepsilon'(t)|}{2} - \delta\varepsilon(t))\|u_t\|_1^2 - \delta\|u_t\|^2 + \\ &\delta\|u\|_1^2 - \delta\frac{\lambda}{2}\|u\|^2 - \delta C_0|\Omega| - \\ &\delta\left[\frac{\lambda}{2\lambda_1}\|u\|_1^2 + \frac{1}{2\lambda}\|h\|^2\right] + 2\delta\|u_t\|^2 - \\ &C_\delta|\Omega| - \delta\left[C\int_\Omega a(x)|u|\mathrm{d}x + \eta\|u\|_1^2 + \right.\\ &\left.C_\eta\|u\|_1^{\frac{p-1}{p}}\int_\Omega a(x)g(u_t)u_t\mathrm{d}x\right] - \\ &\left[\frac{1}{4}|\varepsilon'(t)|\ \|u_t\|_1^2 + \delta^2 L\|u\|_1^2\right] \geqslant \\ &\frac{\delta}{4}(\|u_t\|^2 + \varepsilon(t)\|u_t\|_1^2 + \|u\|_1^2) - \\ &C_\delta E_0(\tau)\int_\Omega a(x)g(u_t)u_t\mathrm{d}x - \\ &C_\delta(|\Omega| + \|h\|^2) - \delta C_0|\Omega| - \\ &(C\|a\|_{L^\infty})^2 \end{aligned} \tag{24}$$

Now, integrating (22) from τ to t yields

$$E_\delta(t) = E_\delta(\tau) - \int_\tau^t I(s)\mathrm{d}s.$$

Then by (20), (23), and (24), we get

$$\begin{aligned}&\frac{1}{8}(\|u_t\|^2 + \varepsilon(t)\|u_t\|_1^2 + \|u\|_1^2) - M_2 \leqslant \\ &-\int_\tau^t\left[\frac{\delta}{4}(\|u_t\|^2 + \varepsilon(t)\|u_t\|_1^2 + \right.\\ &\left.\|u\|_1^2) - M_1\right]\mathrm{d}s\end{aligned} \tag{25}$$

where

$$\begin{aligned}M_1 = &C_\delta(|\Omega| + \|h\|^2) + \delta C_0|\Omega| + \\ &(C\|a\|_{L^\infty})^2, M_2 = (C_0|\Omega| + \frac{1}{\lambda}\|h\|^2) + \\ &C_\delta E_0(\tau)(E_0(\tau) + C_0|\Omega| + \frac{1}{\lambda}\|h\|^2) + CE_0(\tau).\end{aligned}$$

Therefore, for any $R_0 = \frac{4M_1}{\delta}$, there exists a $t_0 \geqslant \tau$ such that

$$\|u_t(t_0)\|^2 + \varepsilon(t_0)\|u_t(t_0)\|_1^2 + \|u(t_0)\|_1^2 \leqslant R_0.$$

Set $B_\tau = \{(u_0, u_1) \in H_\tau: \|u_1\|^2 + \varepsilon(\tau)\|u_1\|_1^2 + \|u_0\|_1^2 \leqslant R_0\}$. so we have B_τ is a bounded time dependent absorbing set. Define

$$B_t = \bigcup_{t\geqslant\tau} U(t,\tau)B_\tau.$$

Then B_t is also a bounded time-dependent absorbing set.

3.2 A priori estimate

The main purpose of this part is to establish (35)~(37), which will be used to obtain the asymptotic compactness of the process.

Let $(u_i(t), u_{i_t}(t))$ be the corresponding solution of (1) with initial datum $(u_0^i(\tau), v_0^i(\tau)) \in \{B_\tau\}_{\tau\in\mathbf{R}}$. For convenience, as in Ref. [12], we introduce notations

$$g_i(t) = g(u_{i_t}(t)), f_i(t) = f(u_i(t)), i = 1, 2$$

and

$$w = u_1(t) - u_2(t).$$

Then $w(t)$ satisfies

$$\begin{cases} w_{tt} - \Delta w + a(x)(g_1(t) - g_2(t)) - \\ \quad \varepsilon(t)\Delta w_{tt} + f_1(t) - f_2(t) = 0, t > T, \\ w(x,T) = u_0^1(T) - u_0^2(T), w_t(x,T) = \\ \quad v_0^1(T) - v_0^2(T), \\ w|_{\partial\Omega} = 0 \end{cases} \tag{26}$$

Denote

$$E_w(t) = \frac{1}{2}[\|w_t\|^2 + \|w\|_1^2 + \varepsilon(t)\|w_t\|_1^2].$$

Taking the inner product (26) with w_t in $L^2(\Omega)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}E_w(t) + [a(x)(g_1(t) - g_2(t)), w_t] - \frac{\varepsilon'(t)}{2}\|w_t\|_1^2 + [f_1 - f_2, w_t] = 0 \tag{27}$$

Integrating (27) over $[s,t]$, we have

$$\begin{aligned}&E_w(t) - E_w(s) + \\ &\int_s^t[a(x)(g_1(\xi) - g_2(\xi)), w_t(\xi)]\mathrm{d}\xi - \\ &\frac{\varepsilon'(t)}{2}\int_s^t\|w_t(\xi)\|_1^2\mathrm{d}\xi + \\ &\int_s^t[f_1 - f_2, w_t(\xi)]\mathrm{d}\xi = 0\end{aligned} \tag{28}$$

Thanks to $\varepsilon'(t) < 0$, from (28) we have

$$\int_s^t[a(x)(g_1(\xi) - g_2(\xi)), w_t(\xi)]\mathrm{d}\xi \leqslant$$

$$E_w(s)-\int_s^t[f_1-f_2,w_t(\xi)]\mathrm{d}\xi.$$

From (3) and Lemma 2.2, we get that, for any $\delta>0$ there exists $C_\delta>0$ such that

$$\varepsilon(\xi)|w_t|^2\leqslant L|w_t|^2\leqslant \delta L+LC_\delta(g_1(\xi)-g_1(\xi))w_t(\xi).$$

Then

$$\int_T^t\varepsilon(\xi)\|w_t\|^2\mathrm{d}\xi\leqslant\delta L|\Omega|(t-T)+\frac{C_\delta L}{\alpha_0}\varepsilon_w(T)-\frac{C_\delta L}{\alpha_0}\int_T^t[f_1-f_2,w_t(\xi)]\mathrm{d}s \quad (29)$$

Multiplying (26) by w and integrating over $\Omega\times[T,t]$ we obtain

$$\int_T^t\|w\|_1^2\mathrm{d}s+\langle w_t,w\rangle+\varepsilon(t)\langle-\Delta w_t,w\rangle=\langle w_t(t),w(T)\rangle+\langle\varepsilon(T)\nabla w_t(T),\nabla w(T)\rangle+\int_T^t\|w_t\|^2\mathrm{d}s+\int_T^t\varepsilon'(s)\langle-\Delta w_t(s),w(s)\rangle\mathrm{d}s+\int_T^t\varepsilon(s)\|w_t\|_1^2\mathrm{d}s-\int_T^t\langle f_1-f_2,w(s)\rangle\mathrm{d}s-\int_T^t\langle a(x)(g_1-g_2),w(s)\rangle\mathrm{d}s \quad (30)$$

Therefore, plus (29) and (30) yields

$$2\int_T^tE_w(s)\mathrm{d}s\leqslant2\delta(L+1)|\Omega|(t-T)+\frac{2C_\delta(L+1)}{\alpha_0}E_w(T)-\frac{2C_\delta(L+1)}{\alpha_0}\int_T^t\langle f_1-f_2,w_t\rangle\mathrm{d}s+\langle w_t(T),w(T)\rangle+\varepsilon(T)\langle\nabla w_t(T),\nabla w(T)\rangle-\langle w_t,w\rangle-\varepsilon(t)\langle\nabla w_t,\nabla w\rangle+\int_T^t\varepsilon'(\xi)\langle\nabla w_t,\nabla w(\xi)\rangle\mathrm{d}\xi-\int_T^t\langle a(x)(g_1-g_2),w(\xi)\rangle\mathrm{d}\xi-\int_T^t\langle(f_1-f_2),w(\xi)\rangle\mathrm{d}\xi \quad (31)$$

On the other hand, integrating (28) over $[T,t]$ we have

$$(t-T)E_w(t)+\int_T^t\int_s^t\langle a(x)(g_1(\xi)-g_2(\xi)),w_t\rangle\mathrm{d}\xi\mathrm{d}s-\frac{1}{2}\int_T^t\int_s^t\varepsilon'(\xi)\|w_t\|_1^2\mathrm{d}\xi\mathrm{d}s=-\int_T^t\int_s^t\langle(f_1(\xi)-f_2(\xi)),w_t(\xi)\rangle\mathrm{d}\xi\mathrm{d}s+\int_T^tE_w(s)\mathrm{d}s \quad (32)$$

Since

$$\int_\Omega a(x)(g_1(\xi)-g_2(\xi))w_t(\xi)\mathrm{d}x-\frac{1}{2}\varepsilon'(\xi)\|w_t\|_1^2>0,$$

then together with (31),(32), we conclude

$$(t-T)E_w(t)\leqslant\delta(L+1)|\Omega|(t-T)+\frac{C_\delta(L+1)}{\alpha_0}E_w(T)+\frac{1}{2}\langle w_t(T),w(T)\rangle+\frac{1}{2}\varepsilon(T)\langle\nabla w_t(T),\nabla w(T)\rangle-\frac{1}{2}\langle w_t,w\rangle-\frac{1}{2}\varepsilon(t)\langle\nabla w_t,\nabla w\rangle+\frac{1}{2}\int_T^t\varepsilon'(s)\langle\nabla w_t(s),\nabla w(s)\rangle\mathrm{d}s-\frac{C_\delta(L+1)}{\alpha_0}\int_T^t\langle f_1(s)-f_2(s),w_t(s)\rangle\mathrm{d}s-\frac{1}{2}\int_T^t\langle a(x)(g_1(s)-g_2(s)),w(s)\rangle\mathrm{d}s-\frac{1}{2}\int_T^t\langle f_1-f_2,w(s)\rangle\mathrm{d}s-\int_T^t\int_s^t\langle f_1-f_2,w_t(\xi)\rangle\mathrm{d}\xi\mathrm{d}s \quad (33)$$

Next, we will deal with $\int_T^t\int_\Omega(g_1(\xi)-g_2(\xi))w\mathrm{d}x\mathrm{d}\xi$. Multiplying (1) by u_{i_t} and integrating over Ω we achieve

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\|u_{i_t}\|^2+\|u_i\|_1^2+\varepsilon(t)\|u_{i_t}\|_1^2\rangle-\frac{\varepsilon'(t)}{2}\|u_{i_t}\|_1^2+\langle a(x)g(u_{i_t}),u_{i_t}\rangle+\langle f(u_{i_t}),u_{i_t}\rangle=\langle h,u_{i_t}\rangle,$$

which combining with (17)~(19) and the existence of time-dependent absorbing set, we get

$$\int_T^t\langle a(x)g(u_{i_t}),u_{i_t}\rangle\mathrm{d}s\leqslant E_0(T)-E_0(t)\leqslant C_T,$$

where the constant C_T depends on T. Then, it follows that

$$\left|\int_T^t\int_\Omega a(x)g(u_{i_t})w(x,s)\mathrm{d}x\mathrm{d}s\right|\leqslant\left(\int_T^t\int_\Omega a(x)(C_0'+C_0'g(u_{i_t})\cdot u_{i_t})\mathrm{d}x\mathrm{d}s\right)^{\frac{p}{p+1}}\left(\int_T^t\int_\Omega a(x)|w(x,s)|^{p+1}\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{p+1}}\leqslant\left[C_0'|\Omega|\|a\|_{L^\infty}(t-T)+C_0'\int_T^t\int_\Omega a(x)g(u_{i_t})\cdot u_{i_t}\mathrm{d}x\mathrm{d}s\right]^{\frac{p}{p+1}}\left(\int_T^t\int_\Omega a(x)|w(x,s)|^{p+1}\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{p+1}}\leqslant[C_0'|\Omega|\|a\|_{L^\infty}(t-T)^{\frac{p}{p+1}}+$$

$$(C_0'C_T)^{\frac{p}{p+1}}]\left(\int_T^t\int_\Omega a(x)\,|w(x,s)|^{p+1}\,\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{p+1}} \tag{34}$$

Now, collecting (33) and (34) we get

$$\begin{aligned}(t-T)E_w(t)\leqslant{}&\delta(L+1)|\Omega|(t-T)+\frac{C_\delta(L+1)}{\alpha_0}E_w(T)+\frac{1}{2}\varepsilon(T)\langle\nabla w_t(T),\\&\nabla w(T)\rangle-\frac{1}{2}\langle w_t(T),w(T)\rangle-\\&\frac{1}{2}\varepsilon(t)\langle\nabla w_t,\nabla w\rangle+\frac{L}{2}\int_T^t|\nabla w_t|\cdot\\&|\nabla w|\,\mathrm{d}s-\frac{C_\delta(L+1)}{\alpha_0}\int_\Omega\langle f_1(s)-\\&f_2(s),w_t\rangle\mathrm{d}s-\frac{1}{2}\int_T^t\langle f_1(s)-\\&f_2(s),w(s)\rangle\mathrm{d}s-\int_T^t\int_s^t\langle f_1-\\&f_2,w_t(\xi)\rangle\mathrm{d}\xi\mathrm{d}s+\\&A\left(\int_T^t\int_\Omega a(x)\,|w(x,s)|^{p+1}\,\mathrm{d}x\mathrm{d}s\right)^{\frac{p}{p+1}},\end{aligned}$$

where

$$A=\frac{[(C_0'|\Omega|)\,\|a\|_{L^\infty}(t-T))^{\frac{p}{p+1}}+(C_0'C_T)^{\frac{p}{p+1}}]}{2}.$$

Set

$$\begin{aligned}\varphi_T^t((u_0^1(T),v_0^1(T)),(u_0^2(T),v_0^2(T)))={}&-\frac{1}{t-T}\int_T^t\int_s^t\langle f_1-f_2,w_t(\xi)\rangle\mathrm{d}\xi\mathrm{d}s-\\&\frac{1}{2(t-T)}\langle w_t,w\rangle-\frac{1}{2(t-T)}\varepsilon(t)\langle\nabla w_t,\nabla w\rangle+\\&\frac{L}{2(t-T)}\int_T^t|\nabla w_t||\nabla w|\,\mathrm{d}s-\\&\frac{C_\delta(L+1)}{\alpha_0(t-T)}\int_T^t\langle f_1(s)-f_2(s),w_t\rangle\mathrm{d}s-\\&\frac{1}{2(t-T)}\int_T^t\langle f_1(s)-f_2(s),w\rangle+\\&\frac{A}{(t-T)}\left(\int_T^t\int_\Omega a(x)\,|w(x,s)|^{p+1}\,\mathrm{d}x\mathrm{d}s\right)^{\frac{p}{p+1}}\end{aligned} \tag{35}$$

and

$$\begin{aligned}C_M={}&\delta(L+1)|\Omega|(t-T)+\\&\frac{C_\delta(L+1)}{\alpha_0}E_w(T)+\frac{1}{2}\langle w_t(T),w(T)\rangle+\\&\frac{1}{2}\varepsilon(T)\langle\nabla w_t(T),\nabla w(T)\rangle\end{aligned} \tag{36}$$

Then we deduce

$$E_w(t)\leqslant\frac{C_M}{t-T}+\varphi_T^t((u_0^1(T),v_0^1(T)),(u_0^2(T),v_0^2(T))) \tag{37}$$

3.3 Asymptotically compact

Theorem 3.3 Under the assumption (2)~(8), for any fixed $t\in\mathbf{R}$ bounded sequence $\{x_n\}_{n=1}^\infty\subset X_{\tau_n}$ and any $\{\tau_n\}_{n=1}^\infty\subset\mathbf{R}^{-t}$, with $\tau_n\to-\infty$ as $n\to\infty$, sequence $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ has a convergent sequence.

Proof For any $\varepsilon>0$, and fixed t let $T<t$, and $t-T$ so large enough, such that

$$\frac{C_M}{t-T}<\frac{\varepsilon}{2}.$$

Hence, thanks to Theorem 2.4, we only need to verify that $\varphi_T^t\in C(B_t)$ for each fixed T.

Let (u_n,u_{n_t}) be the solution corresponding to the initial data $(u_0^n,v_0^n)\in B_T$ for the problem (1). From (18), $\|u_t\|^2+\|u\|_1^2+\varepsilon(\xi)\|u_t\|_1^2$ is bounded, where the bound depends on the T, furthermore, $\|u_n\|_1^2$ is bounded. Moreover, by (2), (3) for fixed T, $\xi\in[T,t]$, $\varepsilon(\xi)$ is bounded, hence $\|u_{n_t}\|_1^2$ is bounded.

According to the Alaoglu Theorem, without loss of generality (at most by passing to subsequence), we assume that

(i) $u_n\to u$ *−weakly in $L^\infty(T,t;H_0^1(\Omega))$;

(ii) $u_{n_t}\to u_t$ *−weakly in $L^\infty(T,t;H_0^1(\Omega))$;

(iii) $u_n\to u$ in $L^{p+1}(T,t;L^{p+1}(\Omega))$;

(iv) $u_n(T)\to u(T)$ and $u_n(t)\to u(t)$ in $L^4(\Omega)$.

Here we take advantage of the compact embeddings $H_0^1(\Omega)\to L^{p+1}(\Omega)\,(p<5)$. Now, we will deal with each term in (35) one by one.

Firstly, from Lemma 3.2, (i), (ii) and (iv) we get

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega(u_{n_t}(t)-u_{m_t}(t))(u_n(t)-u_m(t))\mathrm{d}x=0 \tag{38}$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega\varepsilon(t)(\nabla u_{n_t}(t)-\nabla u_{m_t}(t))(\nabla u_n-\nabla u_m)\mathrm{d}x=0 \tag{39}$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega L(\nabla u_{n_t}(t)-\nabla u_{m_t}(t))(\nabla u_n-\nabla u_m)\mathrm{d}x\mathrm{d}s=0 \tag{40}$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega(f(u_n)-f(u_m))(u_n(s)-u_m(s))\mathrm{d}x\mathrm{d}s=0 \tag{41}$$

Similar to the proof of the Theoren 5.4 in Ref. [12], we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega (f(u_n)-f(u_m))(u_{n_t}(s)-u_{m_t}(s))\mathrm{d}x\mathrm{d}s=0 \quad (42)$$

At the same time, for each fixed t, $\left|\int_s^t\int_\Omega (u_{n_t}(\xi)-u_{m_t}(\xi))(\varphi(u_n(\xi)-\varphi(u_m(\xi)))\mathrm{d}x\mathrm{d}\xi\right|$ is bounded , then by the Lebesgue dominated convergence theorem there holds

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_s^t\int_\Omega (u_{n_t}(\xi)-u_{m_t}(\xi))(\varphi(u_n(\xi)-\varphi(u_m(\xi)))\mathrm{d}x\mathrm{d}\xi\mathrm{d}s=\int_T^t\Big(\lim_{n\to\infty}\lim_{m\to\infty}\int_s^t\int_\Omega (u_{n_t}(\xi)-u_{m_t}(\xi))(\varphi(u_n(\xi)-\varphi(u_m(\xi)))\mathrm{d}x\mathrm{d}\xi\Big)\mathrm{d}s=0 \quad (43)$$

Finally, by (iii), we derive

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{A}{t-T}\int_T^t\int_\Omega (|u_n(s)-u_m(s)|^{p+1}\mathrm{d}x)^{\frac{1}{p+1}}\mathrm{d}s=0 \quad (44)$$

Thus, from (38)～(44) we get that $\varphi_T^t\in C(B_t)$, so the proof is completed.

3.4 Existence of the time-dependent global attractor

Theorem 3.4 Under the conditions (2)～(8) the process $U(t,\tau):H_\tau\to H_t$ generated by problem (1) has a invariant time-dependent global attractor $U=\{A_t\}_{t\in\mathbf{R}}$.

Proof By means of Lemma 3.2, Theorem 3.3 and 2.3, we know that there exists a unique time-dependent global attractor $U=\{A_t\}_{t\in\mathbf{R}}$, Furthermore, due to the strong continuity of the process stated in Lemma 3.1, we can obtain that U is invariant.

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