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弱耦合 Schrödinger 方程组的稳定性

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摘 要：本文旨在研究弱耦合 Schrödinger 方程组的稳定性问题. 为此，我们首先建立了关于弱耦合椭圆方程组的内插不等式，然后得到了耦合 Schrödinger 方程组的预解式估计，进而得到了相应的稳定性结果.

关键词：对数稳定性；内插不等式；预解式；弱耦合 Schrödinger 方程组.

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Stabilization of the weakly coupled Schrödinger system

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Abstract: This paper is devoted to analyzing the stabilization problems of the weakly coupled Schrödinger system. For this purpose, we establish the interpolation inequality for coupled elliptic system. Based on this, we obtain the resolvent estimate for the coupled Schrödinger system, thus obtaining the corresponding stabilization results.

Keywords: Logarithmic stability; Interpolation inequality; Resolvent; The weakly coupled Schrödinger system

1 Introduction

There exist some interesting results on the stabilization problems of Schrödinger system[1-9]. In Refs. [7, 9], the authors proved that a linear Schrödinger equation with time independent coefficients is exponentially stabilizable. In Refs. [5, 6], the authors obtained the polynomial decay of coupled Schrödinger system with variable coefficients.

However, to the best of the author's knowledge, there is no reference addressing the asymptotic behavior of the coupled Schrödinger equations. In this paper, we will show the logarithmic decay property for solutions of the weakly coupled Schrödinger system with only one dissipation mechanism.

Let Ω be a bounded domain of $\mathbf{R}^n$ with C^2 boundary. Set

$$Y=(y_1, y_2, \cdots, y_N)^T, Z=(y_1, 0, \cdots, 0)^T \quad (1)$$

Let us consider the following weakly coupled Schrödinger system:

$$\begin{cases} \mathrm{i}Y_t+\Delta Y+AY+\mathrm{i}d(x)Z=0 & \text{in} \quad \mathbf{R}\times\Omega, \\ Y=0 & \text{on} \quad \mathbf{R}\times\partial\Omega, \\ Y(0)=Y_0 & \text{in} \quad \Omega \end{cases} \quad (2)$$

here $d(\cdot)$ denotes the damping function and $A(\cdot)=(h^{jk}(\cdot))_{n\times n}$ denotes the coupling matrix satisfying

$$h^{jk}\in L^\infty(\Omega), h^{jk}=h^{kj}, j,k=1, 2, \cdots, n \quad (3)$$

and when $k\geqslant j$,

$$h^{jk}(\cdot)=\begin{cases} c_j(\cdot) & k=j+1 \text{ and } j=1,2,\cdots,n-1, \\ 0 & \text{else} \end{cases} \quad (4)$$

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We always assume that $c_j(\cdot), d(\cdot)$ are bounded Lebesgue measurable real valued functions satisfying

$$\begin{cases} 0\leqslant c_j(x)\leqslant c_j^1 \text{ in } \Omega \text{ and } 0\leqslant d(x)\leqslant d_1 \text{ in } \Omega, \\ 0< c_j^0\leqslant c_j(x) \text{ in } \omega_{c_j} \text{ and } 0<d_0\leqslant d(x) \text{ in } \omega_d \end{cases} \tag{5}$$

where ω_{c_j}, ω_d are any fixed non-empty open subsets of Ω, and c_j^0, c_j^1, d_0, d_1 are given constants. In what follows, we will use

$$C=C(\Omega,\omega_{c_j},\omega_d)(j=1,2,\cdots,n-1)$$

to denote a generic positive constant which may vary from line to line.

Put $H=(L^2(\Omega))^N$. Define an unbounded operator $B:D(B)\subset H\to H$, by

$$\begin{cases} D(B)=\{Y\in H \mid BY\in H, Y\mid \partial\Omega=0\}, \\ BY=\mathrm{i}\Delta Y+\mathrm{i}AY-d(x)Z \end{cases} \tag{6}$$

It is easy to show that B generate a C_0- semigroup $\{e^{tB}\}_{t\in\mathbf{R}^+}$ on H. Therefore, system (2) is well-posed in H. The energy of system (2) is defined as follows:

$$E(t)=\frac{1}{2}\int_\Omega |Y(t,x)|^2\mathrm{d}x= \frac{1}{2}\sum_{j=1}^N\int_\Omega |y_j(t,x)|^2\mathrm{d}x,\ \forall t>0 \tag{7}$$

It is easy to check that

$$E(t_2)-E(t_1)= -\int_{t_1}^{t_2}\int_\Omega d(x)|y_1(t,x)|^2\mathrm{d}x\mathrm{d}t,\ \forall t_2\geqslant t_1\geqslant 0 \tag{8}$$

Our main results is stated as follows:

Theorem 1.1 Let $c_j(\cdot)$ and $d(\cdot)$ satisfy (5). Suppose that $(\bigcap_{j=1}^{n-1}\omega_{c_j})\cap\omega_d\neq\varnothing$. Then solution $Y=e^{tB}Y^0\in C(\mathbf{R}^+;D(B))\cap C^1(\mathbf{R}^+;H)$ of system (2) satisfies

$$\|e^{tB}Y^0\|_H\leqslant\frac{C}{\ln(2+t)}\|Y^0\|_{D(B)},\ \forall Y^0\in D(B), \forall t>0 \tag{9}$$

It is now clear that, once a suitable resolvent estimate for the operator B is established, the existing result for C_0- semigroup can be adopted to yield the desired energy decay rate[1,2,8]. Hence, to prove Theorem 1.1, we only need to establish the following resolvent estimate for the operator B.

Theorem 1.2 Under the assumptions of Theorem 1.1, there exists a constant $C>0$ such that for any $\lambda\in C$ satisfying $\mathrm{Re}\lambda\in\left[-\frac{e^{-C\sqrt{\mathrm{Im}\lambda}}}{C},0\right]$, it holds

$$\|(\lambda I-B)^{-1}\|_{L(H)}\leqslant Ce^{C\sqrt{|\mathrm{Im}\lambda|}}$$

for $|\lambda|>1$.

The rest of this paper is organized as follows. In Section 2, we establish the interpolation inequality for coupled elliptic system. In Section 3, we give the resolvent estimate for the coupled Schrödinger system, thus obtaining the corresponding stabilization results.

2 Interpolation inequality for coupled elliptic system

We assume that ω_0 is a subdomain of Ω such that $\bar{\omega}_0\subset\omega_d\cap(\bigcap_{j=1}^{n-1}\omega_{c_j})$. Recalling that there exists a function $\hat{\psi}\in C^2(\overline{\Omega};\mathbf{R})$ such that[4]

$$\hat{\psi}>0, \text{in } \Omega,\ \hat{\psi}=0 \text{ on } \partial\Omega,\ |\nabla\hat{\psi}|>0 \text{ in } \overline{\Omega\backslash\omega_0} \tag{10}$$

With the aid of the function $\hat{\psi}$ defined above, we introduce a weight function as follows:

$$\theta=e^l, l=\lambda\varphi, \varphi=e^{\mu\psi}, \psi=\psi(s,x)= \frac{\hat{\psi}(x)}{\|\hat{\psi}\|_{L^\infty(\Omega)}}+b^2-s^2 \tag{11}$$

here $1<b\leqslant 2$ will be given later, λ,μ and s are parameters, $x\in\overline{\Omega}$. Let $(a_{jk})_{n\times n}=I_n$ in Ref. [3, Theorem 3.2], a short calculation can yield the following knowing result.

Lemma 2.1 Let $w\in C^2((-b,b)\times\Omega;C)$, and $l\in C^2((-b,b)\times\Omega;\mathbf{R})$ be given by (11). Then there is a constant $\mu_0>0$ such that for all $\mu\geqslant\mu_0$, one can find two constants $C=C(\mu)>0$ and $\lambda_2=\lambda_2(\mu)>0$ such that for all $w\in H_0^1((-b,b)\times\Omega)$ and $w_{ss}+\Delta w=f$ (in $(-b,b)\times\Omega$, in the sense of distribution) with $f\in L^2((-b,b)\times\Omega)$, and for all $\lambda\geqslant\lambda_2$, it holds that

$$\lambda\mu^2\int_{-b}^b\int_\Omega\theta^2\varphi(|\nabla w|^2+|w_s|^2+ \lambda^2\mu^2\varphi^2|w|^2)\mathrm{d}x\mathrm{d}s\leqslant C\{\int_{-b}^b\int_\Omega\theta^2|f|^2\mathrm{d}x\mathrm{d}s+$$

$$\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\varphi(|\nabla w|^2+ |w_s|^2+\lambda^2\mu^2\varphi^2|w|)\} \quad (12)$$

Based on Lemma 2.1, in this section we shall prove interpolation inequality for the following coupled elliptic system:

$$\begin{cases} P_{ss}+\Delta P+AP+\mathrm{i}d(x)R^0=G \\ \quad \text{in } Q=(-2,2)\times\Omega, \\ P=0 \text{ on } \Sigma=(-2,2)\times\partial\Omega \end{cases} \quad (13)$$

here $G=(g^1,g^2,\cdots,g_N)^T\in(L^2(Q))^N$, $P=(p_1,p_2,\cdots,p_N)^T$ and $R^0=(p_1,0,\cdots,0)^T$. In what follows, we will use the notations $Q_{\omega^*}=(-2,2)\times\omega^*$, $\omega^*=(\bigcap_{j=1}^{n-1}\omega_{c_j})\cap\omega_d$.

First, we have the following interpolation inequality for system (13).

Lemma 2.2 Under the assumptions in Theorem 1.1, there exists constants $C>0,\varepsilon_0>0$ such that for any $0<\varepsilon<\varepsilon_0$, every solution P of system (13) satisfies

$$\|P\|_{L^2(-1,1;H_0^1(\Omega))}\leqslant Ce^{\frac{C}{\varepsilon}}[\|G\|_{(L^2(Q))^N}+\|p_1\|_{L^2(Q_{\omega^*})}]+ Ce^{-\frac{2}{\varepsilon}}\|P\|_{H^1(-2,2;H_0^1(\Omega))} \quad (14)$$

The rest of this section is devoted to proving Lemma 2.2.

Proof of Lemma 2.2 The proof is based on the global Carleman estimate presented in Lemma 2.1. The main difficulty is to estimate the energy of the coupled system $P=(p_1,p_2,\cdots,p_N)$ localized in ω^* by $\int_{\omega^*}|p_1|^2\mathrm{d}x$. We divide the proof into four steps.

Step 1. Note that there no boundary conditions for P at $s=\pm2$ in system (13). Therefore, we introduce a cut-off function $\varphi=\varphi(s)\in C_c^\infty(-b,b)$ such that

$$\begin{cases} 0\leqslant\varphi(s)\leqslant1\,|s|<b, \\ \varphi(s)=1\quad |s|\leqslant b_0 \end{cases} \quad (15)$$

where $1<b_0<b\leqslant2$ are given follows:

$$b=\sqrt{1+\frac{1}{\mu}\ln(2+e^{\mu})},$$

$$b_0=\sqrt{b^2-\frac{1}{\mu}\ln(1+e^{-\mu})},\ \forall\mu>\ln2 \quad (16)$$

Put

$$\hat{P}=\varphi P=(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_N) \quad (17)$$

Then, noting that φ does not depend on x, by (13), it follows

$$\begin{cases} \hat{P}_{ss}+\Delta\hat{P}+A\hat{P}+\mathrm{i}d(x)\varphi R^0= \\ \quad \varphi_{ss}P+2\varphi_sP_s+\varphi G \text{ in } Q, \\ \hat{P}=0 \text{ on } \Sigma \end{cases} \quad (18)$$

For system (18), by using Lemma 2.1 (with w replaced by $\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_N$ respectively), we conclude that there is a $\mu_0>0$ such that for all $\mu\geqslant\mu_0$, one can find two co-nstants $C=C(\mu)>0$ and $\lambda_0=\lambda_0(\mu)>0$ so that for all $\lambda\geqslant\lambda_0$, it holds that

$$\lambda\mu^2\int_{-b}^{b}\int_{\Omega}\theta^2\varphi(|\nabla\hat{P}|^2+|\hat{P}_s|^2+ \lambda^2\mu^2\varphi^2|\hat{P}|^2)\mathrm{d}x\mathrm{d}s\leqslant C\{\int_{-b}^{b}\int_{\Omega}\theta^2\,|\varphi_{ss}P+2\varphi_sP_s+\varphi G- id(x)\varphi R^0-A\hat{P}|^2\mathrm{d}x\mathrm{d}s\}+ C\lambda\mu^2\int_{-b}^{b}\int_{\omega_0}\theta^2\varphi(|\nabla\hat{P}|^2+|\hat{P}_s|^2+ \lambda^2\mu^2\varphi^2|\hat{P}|^2)\mathrm{d}x\mathrm{d}s \quad (19)$$

Step 2. Let us estimate

$$\int_{-b}^{b}\int_{\omega_0}\theta^2(|\partial_s\hat{p}_j|^2+|\nabla\hat{p}_j|^2)\mathrm{d}x\mathrm{d}s$$

and

$$\int_{-b}^{b}\int_{\omega_0}\theta^2|\hat{p}_j|^2\mathrm{d}x\mathrm{d}s \text{ for } j=2,3,\cdots,N.$$

Recalling that $\hat{P}=(\hat{p}_1,\hat{p}_2,\cdots,\hat{p}_N)^T$, $R^0=(p_1,0,\cdots,0)$ and $G=(g^1,g^2,\cdots,g^N)$, by (3), (4), system (18) can be rewritten as

$$\begin{cases} \partial_{ss}\hat{p}_1+\Delta\hat{p}_1+c_1(x)\hat{p}_2+\mathrm{i}d(x)\hat{p}_1= \\ \quad \varphi_{ss}p_1+2\varphi_s\partial_sp_1+\varphi g^1 \text{ in } Q, \\ \partial_{ss}\hat{p}_j+\Delta\hat{p}_j+c_{j-1}(x)\hat{p}_{j-1}+c_j(x)\hat{p}_{j+1}= \\ \quad \varphi_{ss}p_j+2\varphi_s\partial_sp_j+\varphi g^j \text{ in } Q, \\ \quad j=2,\cdots,N-1, \\ \partial_{ss}\hat{p}_N+\Delta\hat{p}_N+c_{N-1}(x)\hat{p}_{N-1}=\varphi_{ss}p_N+ \\ \quad 2\varphi_s\partial_sp_N+\varphi g^N \text{ in } Q \end{cases} \quad (20)$$

We choose cut-off function $\eta\in C^\infty(\overline{\Omega};[0,1])$ satisfying

$$\begin{cases} \eta(x)=1,\ \forall x\in\omega_0, \\ 0<\eta(x)\leqslant1,\ \forall x\in\omega^*, \\ \eta(x)=0,\ \forall x\in\Omega\backslash\omega^* \end{cases} \quad (21)$$

Multiplying the second equation of (20) by $\theta^2\eta^{m_2}\overline{\hat{p}}_2$, it is to see that

$$\begin{aligned}&\theta^2\eta^{m_2}\overline{\hat{p}_2}[\partial_{ss}\hat{p}_2+\Delta\hat{p}_2]=\\&\quad(\theta^2\eta^{m_2}\overline{\hat{p}_2}\partial_s\hat{p}_2)_s-\theta^2\eta^{m_2}|\partial_s\hat{p}_2|^2-\\&\quad(\theta^2\eta^{m_2})_s\overline{\hat{p}_2}\partial_s\hat{p}_2+\\&\quad\sum_{j=1}^n[\theta^2\eta^{m_2}\overline{\hat{p}_2}\partial_{x_j}\hat{p}_2]_{x_j}-\theta^2\eta^{m_2}|\nabla\hat{p}_2|^2-\\&\quad\sum_{j=1}^n(\theta^2\eta^{m_2})_{x_j}\overline{\hat{p}_2}\partial_{x_j}\hat{p}_2\end{aligned}\qquad(22)$$

Integrating (22) on $(-b,b)\times\Omega$ and noting that $\hat{p}_2(b)=\hat{p}_2(-b)=0$ in Ω, by (11) and (20), we see that for big enough λ,

$$\begin{aligned}&\int_{-b}^b\int_\Omega\theta^2\eta^{m_3}(|\nabla\hat{p}_2|^2+|\partial_s\hat{p}_2|^2)\mathrm{d}x\mathrm{d}s\leqslant\\&\quad\frac{C_1}{\lambda^{l_2}}\int_{-b}^b\int_\Omega\theta^2|\varphi_{ss}p_2+2\varphi_s\partial_sp_2+\varphi g^2-\\&\quad c_1(x)\hat{p}_1-c_2(x)\hat{p}_3|^2\mathrm{d}x\mathrm{d}s+\\&\quad\frac{C_1}{\lambda^{l_2}}\int_{-b}^b\int_\Omega\theta^2|\nabla\hat{p}_2|^2\mathrm{d}x\mathrm{d}s+\\&\quad C_1\lambda^{2l_2}\int_{-b}^b\int_\Omega\theta^2\eta^{m_{21}-1}|\hat{p}_2|^2\mathrm{d}x\mathrm{d}s\end{aligned}\qquad(23)$$

Step 3. Let us estimate

$$\int_{-b}^b\int_\Omega\theta^2\eta^{m_2-1}|\hat{p}_2|^2\mathrm{d}x\mathrm{d}s.$$

It is easy to see that

$$\begin{aligned}&\theta^2\eta^{m_2-1}\overline{\hat{p}_2}[\partial_{ss}\hat{p}_1+\Delta\hat{p}_1]=\\&\quad\theta^2\eta^{m_2-1}\hat{p}_1[\partial_{ss}\overline{\hat{p}_2}+\Delta\overline{\hat{p}_2}]+\\&\quad[\theta^2\eta^{m_2-1}(\overline{\hat{p}_2}\partial_s\hat{p}_1-\hat{p}_1\partial_s\overline{\hat{p}_2})]_s-\\&\quad(\theta^2\eta^{m_2-1})_s\overline{\hat{p}_2}\partial_s\hat{p}_1+[(\theta^2\eta^{m_2-1})_s\overline{\hat{p}_2}\hat{p}_1]_s-\\&\quad[(\theta^2\eta^{m_2-1})_s\hat{p}_1]_s\overline{\hat{p}_2}+\\&\quad\sum_{j=1}^n[\theta^2\eta^{m_2-1}(\overline{\hat{p}_2}\partial_{x_j}\hat{p}_1-\hat{p}_1\partial_{x_j}\overline{\hat{p}_2})]_{x_j}-\\&\quad\sum_{j=1}^n(\theta^2\eta^{m_2-1})_{x_j}\overline{\hat{p}_2}\partial_{x_j}\hat{p}_1+\\&\quad\sum_{j=1}^n[(\theta^2\eta^{m_2-1})_{x_j}\overline{\hat{p}_2}\hat{p}_1]_{x_j}-\\&\quad\sum_{j=1}^n[(\theta^2\eta^{m_2-1})_{x_j}\hat{p}_1]_{x_j}\overline{\hat{p}_2}\end{aligned}\qquad(24)$$

On the other hand, multiplying the first equation of (20) by $\theta^2\eta^{m_2-1}\overline{\hat{p}_2}$, we have

$$\begin{aligned}&c_1(x)\theta^2\eta^{m_2-1}|\hat{p}_2|^2=\\&\quad-\theta^2\eta^{m_2-1}\overline{\hat{p}_2}[\partial_{ss}\hat{p}_1+\Delta\hat{p}_1]+\\&\quad\theta^2\eta^{m_2-1}\overline{\hat{p}_2}[\varphi_{ss}p_1+2\varphi_s\partial_sp_1+\varphi g^1-\mathrm{i}d(x)\hat{p}_1]\end{aligned}\qquad(25)$$

Now, integrating (25) on $(-b,b)\times\Omega$, noting that $\hat{p}_j(b)=\hat{p}_j(-b)=0\,(j=1,2)$ in Ω, by (5), (11), (20), (24), we find that for big enough λ,

$$\begin{aligned}&\int_{-b}^b\int_\Omega\theta^2\eta^{m_2-1}|\hat{p}_2|^2\mathrm{d}x\mathrm{d}s\leqslant\\&\quad C_1\mathrm{e}^{C_1\lambda}\Big[\int_{-b}^b\int_\Omega(|g^1|^2+|g^2|^2)\mathrm{d}x\mathrm{d}s+\\&\quad\int_{-b}^b\int_{\omega^*}|p_1|^2\mathrm{d}x\mathrm{d}s\Big]+\\&\quad C_2(\lambda)\int_{(-b,-b_0)\cup(b_0,b)}\int_\Omega\theta^2(|p_1|^2+\\&\quad|\partial_sp_1|^2+|p_2|^2+|\partial_sp_2|^2)\mathrm{d}x\mathrm{d}s+\\&\quad C_1\lambda^{2k_2}\int_{-b}^b\int_\Omega\theta^2\eta^{m_2-3}(|\partial_s\hat{p}_1|^2+\\&\quad|\nabla\hat{p}_1|^2)\mathrm{d}x\mathrm{d}s+\\&\quad\frac{C_1}{\lambda^{k_2}}\int_{-b}^b\int_\Omega\theta^2\eta^{m_2-3}(|\hat{p}_2|^2+|\hat{p}_3|^2)\mathrm{d}x\mathrm{d}s\end{aligned}\qquad(26)$$

Also, similar to (23), multiplying the first equation of (20) by $\theta^2\eta^{m_1}\overline{\hat{p}_1}$, integrating it on $(-b,b)\times\Omega$, using integration by parts and noting that $\hat{p}_1(b)=\hat{p}_1(-b)=0$ in Ω, by a simple calculation we conclude that for big enough λ,

$$\begin{aligned}&\int_{-b}^b\int_\Omega\theta^2\eta^{m_1}(|\partial_s\hat{p}_1|^2+|\nabla\hat{p}_1|^2)\mathrm{d}x\mathrm{d}s\leqslant\\&\quad\frac{C_1}{\lambda^{l_1}}\int_{-b}^b\int_\Omega\theta^2|\varphi_{ss}p_1+2\varphi_s\partial_sp_1+\varphi g^1-\\&\quad\mathrm{i}d(x)\hat{p}_1-c_1(x)\hat{p}_2|^2\mathrm{d}x\mathrm{d}s+\\&\quad\frac{C_1}{\lambda^{l_1}}\int_{-b}^b\int_\Omega\theta^2(|\partial_s\hat{p}_1|^2+|\nabla\hat{p}_1|^2)\mathrm{d}x\mathrm{d}s+\\&\quad C_1\mathrm{e}^{C_1\lambda}\int_{-b}^b\int_{\omega^*}|p_1|^2\mathrm{d}x\mathrm{d}s\end{aligned}\qquad(27)$$

where $m_i,l_1,k_2\in\mathbf{N}$ and $m_i\geqslant5,i=1,2$. They can be selected according to necessity.

Similarly, we can give the estimations of

$$\int_{-b}^b\int_{\omega_0}\theta^2(|\partial_s\hat{p}_j|^2+|\nabla\hat{p}_j|^2)\mathrm{d}x\mathrm{d}s$$

and

$$\int_{-b}^b\int_{\omega_0}\theta^2|\hat{p}_j|^2\mathrm{d}x\mathrm{d}s \text{ for } j=3,\cdots,N.$$

Step 4. Combining (15), (19), (23), (26), (27) and noting that $\hat{P}=\varphi P$, we have

$$\lambda\mu^2\int_{-1}^1\int_\Omega\theta^2\varphi(|\nabla P|^2+|P_s|^2+$$

$$\lambda^2\mu^2\varphi^2|P|^2)\mathrm{d}x\mathrm{d}s \leqslant C_3\mathrm{e}^{C_3\lambda}\left\{\int_{-b}^{b}\int_\Omega |G|^2\mathrm{d}x\mathrm{d}s + \int_{-b}^{b}\int_{\omega^*}|p_1|^2\mathrm{d}x\mathrm{d}s\right\} + C_4(\lambda)\int_{(-b,-b_0)\cup(b_0,b)}\int_\Omega \theta^2(|P|^2 + |P_s|^2)\mathrm{d}x\mathrm{d}s \tag{28}$$

for big enough λ. Next, recalling (11) and (16) for the definitions of φ and b, b_0, it is easy to check that

$$\begin{cases}\varphi(s,\cdot)\geqslant 2+\mathrm{e}^\mu, & |s|\leqslant 1,\\ \varphi(s,\cdot)\leqslant 1+\mathrm{e}^\mu, & b_0\leqslant |s|\leqslant b\end{cases} \tag{29}$$

Finally, denote $c_0 = 2+\mathrm{e}^\mu$. Fixing the parameter μ in (19), and using (29), one finds that

$$\lambda\mathrm{e}^{2\lambda c_0}\int_{-1}^{1}\int_\Omega(|\nabla P|^2 + |P_s|^2 + |P|^2)\mathrm{d}x\mathrm{d}s \leqslant C_5\mathrm{e}^{C_5\lambda}\left\{\int_{-2}^{2}\int_\Omega|G|^2\mathrm{d}x\mathrm{d}s + \int_{-2}^{2}\int_{\omega^*}|p_1|^2\mathrm{d}x\mathrm{d}s\right\} + C_6(\lambda)\mathrm{e}^{2\lambda(c_0-1)}\int_{(-b,-b_0)\cup(b_0,b)}\int_\Omega(|P|^2 + |P_s|^2)\mathrm{d}x\mathrm{d}s \tag{30}$$

From (30), one concludes that there exists an $\varepsilon_1 > 0$ such that the desired inequality (14) holds for $\varepsilon \in (0, \varepsilon_1]$. This completes the proof of Lemma 2.2. C_1, C_3, C_5 in this section are positive constants which are only related to μ, $C_2(\lambda)$, $C_4(\lambda)$, $C_6(\lambda)$ are polynomials about λ.

3　Proof of the main result

In this part, we shall give the proof of logarithmic decay results. As we mentioned before, we only need to establish the following resolvent estimate for the elliptic system. Thus, we only to prove the existence and the estimate of the norm of $(B-\lambda I)^{-1}$, $\mathrm{Re}\lambda \in \left[\frac{-\mathrm{e}^{-C\sqrt{|\mathrm{Im}\lambda|}}}{C}, 0\right]$ stated in Theorem 1.2.

We divide the proof into two steps.

Step 1. Let $F\in H$, $Y^0 = (y_1^0, y_2^0, \cdots, y_N^0)^T \in D(B)$. It is easy to see that the following equation

$$(B-\lambda I)Y^0 = F \tag{31}$$

is equivalent to

$$\begin{cases}\mathrm{i}\Delta Y^0 - d(x)Z^0 + \mathrm{i}AY^0 - \lambda Y^0 = F \text{ in } \Omega,\\ Y^0 = 0 \text{ on } \partial\Omega\end{cases} \tag{32}$$

where $Z^0 = (y_1^0, 0, \cdots, 0)^T$. Put

$$P = \mathrm{e}^{\sqrt{i\lambda}s}Y^0 = (p_1, p_2, \cdots, p_N)^T,$$

$$R^0 = \mathrm{e}^{\sqrt{i\lambda}s}Z^0 = (r_1, 0, \cdots, 0)^T \tag{33}$$

Therefore

$$\begin{cases}P_{ss} + \Delta P + \mathrm{i}d(x)R^0 + AP = -\mathrm{i}\mathrm{e}^{\sqrt{i\lambda}s}F \text{ in } Q,\\ P = 0 \text{ on } \Sigma\end{cases} \tag{34}$$

Step 2. By (33), we have the following estimates:

$$\{\|Y^0\|_{H_0^1(\Omega)} \leqslant Ce^{C\sqrt{|\mathrm{Im}\lambda|}}\|P\|_{L^2(-1,1;H_0^1(\Omega))},\ \|P\|_{H^1(-2,2;H_0^1(\Omega))} \leqslant Ce^{C\sqrt{|\mathrm{Im}\lambda|}}\|Y^0\|_{H_0^1(\Omega)},\ \|p_1\|_{L^2(Q_{\omega^*})} \leqslant Ce^{C\sqrt{|\mathrm{Im}\lambda|}}\|y_1^0\|_{L^2(\omega^*)} \tag{35}$$

Now, applying Lemma 2.2 into system (34), and combining (35), (19), we have

$$\|Y^0\|_{H_0^1(\Omega)} \leqslant Ce^{C\sqrt{|\mathrm{Im}\lambda|}}[\|F\|_{L^2(\Omega)} + \|y_1^0\|_{L^2(\omega^*)}] \tag{36}$$

By multiplying the first equation of (34) by $2\overline{y_1^0}$ and integrating it on Ω, it follows that

$$\int_\Omega\begin{pmatrix}-\Delta y_1^0 - \mathrm{i}\lambda y_1^0 - \mathrm{i}d(x)y_1^0 - \\ c_1(x)y_2^0\end{pmatrix}2\overline{y_1^0}\mathrm{d}x = -2\mathrm{i}\lambda\int_\Omega|y_1^0|^2\mathrm{d}x + 2\int_\Omega|\nabla y_1^0|^2\mathrm{d}x - 2\mathrm{i}\int_\Omega d(x)|y_1^0|^2\mathrm{d}x - 2\int_\Omega c_1(x)y_2^0\overline{y_1^0}\mathrm{d}x \tag{37}$$

Likewise, we obtain that

$$\int_\Omega(-\Delta y_j^0 - \mathrm{i}\lambda y_j^0 - c_{j-1}(x)y_{j-1}^0 - c_j(x)y_{j+1}^0)\cdot 2\overline{y_j^0}\mathrm{d}x = -2\mathrm{i}\lambda\int_\Omega|y_j^0|^2\mathrm{d}x + 2\int_\Omega|\nabla y_j^0|^2\mathrm{d}x - 2\int_\Omega c_j(x)y_{j+1}^0\overline{y_j^0}\mathrm{d}x - 2\int_\Omega c_{j-1}(x)y_{j-1}^0\overline{y_j^0}\mathrm{d}x,\quad j = 2,\cdots,N-1 \tag{38}$$

$$\int_\Omega(-\Delta y_N^0 - \mathrm{i}\lambda y_N^0 - c_{N-1}(x)y_{N-1}^0)\cdot 2\overline{y_N^0}\mathrm{d}x = -2\mathrm{i}\lambda\int_\Omega|y_N^0|^2\mathrm{d}x + 2\int_\Omega|\nabla y_N^0|^2\mathrm{d}x - 2\int_\Omega c_{N-1}(x)y_{N-1}^0\overline{y_N^0}\mathrm{d}x \tag{39}$$

Taking the imaginary part in the both sides of (37)~(39), we obtain

$$\int_\Omega d(x)|y_1^0|^2\mathrm{d}x = -\mathrm{Re}\lambda\|Y^0\|_{L^2(\Omega)} - \mathrm{Re}\int_\Omega F\overline{Y^0}\mathrm{d}x \tag{40}$$

Since $\omega^* \subset \omega_d$, $d(x) \geqslant d_0 > 0$ on ω_d, therefore

$$d_0 \int_{\omega^*} |y_1^0|^2 \mathrm{d}x \leqslant |\mathrm{Re}\lambda| \, \| Y^0 \|_{L^2(\Omega)}^2 + \int_\Omega |F| \, |Y^0| \, \mathrm{d}x \tag{41}$$

Combining (36) and (41), we arrive at

$$\| Y^0 \|_{H_0^1(\Omega)} \leqslant C e^{C\sqrt{|\mathrm{Im}\lambda|}} [\| F \|_{L^2(\Omega)} + |\mathrm{Re}\lambda| \, \| Y^0 \|_{H_0^1(\Omega)}] \tag{42}$$

When

$$C e^{C\sqrt{|\mathrm{Im}\lambda|}} |\mathrm{Re}\lambda| \leqslant \frac{1}{2},$$

we find that

$$\| Y^0 \|_{H_0^1(\Omega)} \leqslant C e^{C\sqrt{|\mathrm{Im}\lambda|}} \| F \|_{L^2(\Omega)} \tag{43}$$

By (43), we know that $B-\lambda I$ is injective. Therefore $B-\lambda I$ is bi-injective from $D(B)$ to H. Thus we can find a sufficiently large constant $C>0$ satisfying

$$\| B-\lambda I \|_{L(H)}^{-1} \leqslant C e^{C\sqrt{|\mathrm{Im}\lambda|}},$$

$$\mathrm{Re}\lambda \in \left[-\frac{e^{-C\sqrt{|\mathrm{Im}\lambda|}}}{C}, 0 \right], |\lambda| > 1.$$

This completes the proof of Theorem 1.2.

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