doi：10.3969/j.issn.0490-6756.2018.03.005

带非线性发生率的离散 SIR 模型的动力学行为

朱春梅[1]，李　燕[2]
(1. 四川大学数学学院，成都 610064；2. 西华大学理学院，成都 610039)

摘　要：本文探究了带有非线性发生率 $\lambda S^p I$ 的离散 SIR 传染病模型的动力学行为. 本文首先确定了无平衡点的拓扑类型，包括平衡点的存在性和稳定性，然后进一步地分析了无病平衡点的分岔情况. 通过中心流行定理和正规型理论，本文发现了限制在系统中心流行上的 flip 分岔以及 Neimark-Sacker 分岔，给出了各自的分岔方向. 最后，对所得的数学结果给出了相应的生物学解释.

关键词：SIR 传染病模型；中心流行；flip 分岔；Neimark-Sacker 分岔
中图分类号：O175.7　　**文献标识码**：A　　**文章编号**：0490-6756(2018)03-0445-07

Dynamics behavior of discrete SIR model with a nonlinear incidence rate

ZHU Chun-Mei[1], *LI Yan*[2]
(1. College of Mathematics, Sichuan University, Chengdu 610064, China;
2. School of Science, Xihua University, Chengdu 610039, China)

Abstract: In this paper we investigate the dynamics behavior of the discrete-time SIR epidemic model with a nonlinear incidence rate $\lambda S^p I$. Firstly, we determine the topological type of the endemic fixed point, including the existence and stability of the fixed point. Furthermore, we analyze the bifurcation situations, and discuss the flip bifurcation on the center manifold and the Neimark-Sacker bifurcation of this SIR system by center manifold theorem and normal form theory. Their bifurcation directions are given respectively. Finally, some biological explanations of our mathematical results are presented.

Keywords: SIR epidemic model; Center manifold; Flip bifurcation; Neimark-Sacker bifurcation
(2010 MSC 34C23, 37G10)

1 Introduction

Many kinds of epidemiological disease models have been developed by a large number of researchers to give a theoretical basis for disease prevention and government policies[1-9]. Usually, these models are of the continuous-time case because they are described by differential equations such as the following classical deterministic SIR bin model

$$\begin{cases} \dot{S} = A - dS(t) - \lambda S(t) I(t), \\ \dot{I} = \lambda S(t) I(t) - (d + \sigma + r) I(t), \\ \dot{R} = rI(t) - dR(t), \\ \dot{N} = S(t) + I(t) + R(t) \end{cases} \tag{1}$$

constructed by Kermack and McKendrick[8]. Here $S(t)$, $I(t)$, $R(t)$, $N(t)$ represent the numbers of susceptible, infected individuals, patients who gains

收稿日期：2017-09-26
基金项目：国家自然科学基金(11471228)
作者简介：朱春梅(1993－)，女，硕士，主要研究方向为差分方程的动力学行为. E-mail：18244299837@163.com
通讯作者：李燕. E-mail：liyan_xhu@163.com

immunity from illness and the total numbers of population at t respectively. A is the birth rate of the population, λ is contact coefficient between susceptible healthy individuals and patients, d is the natural mortality of the individuals, σ is the mortality from the disease, r is the rate of recovery of the affected individuals.

In recent years, more and more attention has paid to the case of discrete-time models[4] described by difference equations because of more wealthy dynamical behaviors and more convenient data collection than the continuous-time case. In general, by discretizing continuous-time SIR system (1) and using the forward Euler scheme[6] we obtain the discrete-time system

$$\begin{cases} S_{n+1}=S_n+h\{A-dS_n-g(S_n,I_n)\}, \\ I_{n+1}=I_n+h\{g(S_n,I_n)-(d+\sigma+r)I_n\}, \\ R_{n+1}=R_n+h\{rI_n-dR_n\}, \\ N_{n+1}=(1-hd)N_n+hA \end{cases} \quad (2)$$

where $g(S,I)$ denotes the incidence rate and $g(S,I)=\lambda SI$ in (1). As indicated [1], the incidence rate is the rate of new infection and play a key role in ensuring that the system does indeed give a reasonable qualitative description of the disease dynamics. In order to analyze system (2), it suffices to consider the dynamical behavior of (S_n, I_n) by the form of (2). That is, we only need to consider

$$\begin{cases} S_{n+1}=S_n+h\{A-dS_n-g(S_n-I_n)\}, \\ I_{n+1}=I_n+h\{g(S_n,I_n)-(d+\sigma+r)I_n\} \end{cases} \quad (3)$$

In Ref. [7], for system (3) with the bilinear incidence rate $g(S,I)=\lambda SI$, Hu *et al*. prove the existence and stability of fixed points, and show the occurrence of the flip bifurcation and the Neimark-Sacker bifurcation. Later, Du *et al*. [5] investigated the SIR model with the incidence rate $g(S,I)$ being the saturated contact rate $\lambda SI/(1+\alpha S)$.

In this paper we analyze the dynamical behavior of system (3) with nonlinear incidence rates $g(S,I)=\lambda S^p I$, i. e.,

$$\begin{cases} S_{n+1}=S_n+h\{A-dS_n-\lambda S_n^p I_n\}, \\ I_{n+1}=I_n+h\{\lambda S_n^p I_n-(d+\sigma+r)I_n\} \end{cases} \quad (4)$$

where $(h,A,d,\sigma,r,\lambda,p)\in B:=\{(h,A,d,\sigma,r,\lambda,p)\in \mathbf{R}^7: h,A>0, 0<d,\sigma,r,\lambda<1, p\geq 1\}$. Here we only consider the first quadrant in the phase space, i. e., $S_n, I_n \geq 0$ for all $n\in \mathbf{N}^+$. In section 2 we compute fixed points and determine their topological types. All possible one-codimensional bifurcations are analyzed in section 3 with h be the perturbed parameter. Finally, some biological explanations of our mathematical results are presented in the last of the paper.

2 Fixed points and their topological properties

By the method of Ref. [3], we compute the basic reproductive number R_0 [3] and get $R_0=\lambda A^p/d^p(d+\sigma+r)$ for system (4). Straight computation shows that system (4) has only a disease-free fixed point $E_1(A/d,0)$ when $0<R_0\leqslant 1$, a disease-free fixed point $E_2(S^*, I^*)$ when $R_0>1$, where

$$S^*=\left(\frac{d+\sigma+r}{\lambda}\right)^{1/p},$$

$$I^*=\frac{A\lambda^{1/p}-d(d+\sigma+r)^{1/p}}{(d+\sigma+r)\lambda^{1/p}}.$$

Because of the importance of the endemic fixed point in biology, by the classic dynamical analysis we give the topological type of E_2 shown in Table 1, where

$$\Delta:=d^2(1+p(R_0^{1/p}-1))^2- 4pd(d+\sigma+r)(R_0^{1/p}-1),$$

$$h_a:=\frac{d+pd(R_0^{1/p}-1)}{pd(d+\sigma+r)(R_0^{1/p}-1)},$$

$$h_b:=\frac{d+pd(R_0^{1/p}-1)-\sqrt{\Delta}}{pd(d+\sigma+r)(R_0^{1/p}-1)} \text{ for } \Delta\geq 0.$$

Tab. 1 Topological types of fixed point E_2 when $R_0>1$

Conditions		properties
$\Delta>0$	$0<h<h_b$	stable node
	$h=h_b$	non-hyperbolic
	$h_b<h<2h_a-h_b$	saddle
	$h=2h_a-h_b$	non-hyperbolic
	$h>2h_a-h_b$	unstable node
$\Delta=0$	$0<h<h_a$	stable node
	$h=h_a$	non-hyperbolic
	$h>h_a$	unstable node
$\Delta<0$	$0<h<h_a$	stable focus
	$h=h_a$	non-hyperbolic
	$h>h_a$	unstable focus

3 Bifurcation results

As given in section 2, fixed point E_2 is non-hyperbolic if and only if $(h,A,d,\sigma,r,\lambda,p)\in F_1\cup F_2\cup NS\cup R$, where

$$F_1:=\{(h,A,d,\sigma,r,\lambda,p)\in B:R_0>1,\Delta>0,h=h_b\},$$
$$F_2:=\{(h,A,d,\sigma,r,\lambda,p)\in B:R_0>1,\Delta>0,h=2h_a-h_b\},$$
$$NS:=\{(h,A,d,\sigma,r,\lambda,p)\in B:R_0>1,\Delta<0,h=h_a\},$$
$$R:=\{(h,A,d,\sigma,r,\lambda,p)\in B:R_0>1,\Delta=0,h=h_b\}.$$

As shown in Table 1, the hyperbolicity of E_2 changes if $(h,A,d,\sigma,r,\lambda,p)$ cross those hypersurfaces F_1,F_2,NS,R. In this section we are going to analyze some significant bifurcation phenomena with h being the unique perturbation parameter, i. e., $(h,A,d,\sigma,r,\lambda,p)$ do not change.

Theorem 3.1 Assume that $(h,A,d,\sigma,r,\lambda,p)\in F_1$. If $l_1 l_2\neq 0$, system (4) goes through a flip bifurcation at E_2, i. e., a stable 2-period orbit appears when $h=h_b-\mathrm{sgn}(l_1)$ if $l_2>0$ or a unstable 2-period orbit appears when $h=h_b+\mathrm{sgn}(l_1)$ if $l_2<0$, where $0<\varepsilon\ll 1$,

$$l_1=\{pd(d+\sigma+r)(R_0^{1/p}-1)h_b^2-2(d+pd(R_0^{1/p}-1))h_b-2(\omega_+-1)\}h_b^{-1}(\omega_++1)^{-1},$$
$$l_2=c_1^2-c_2(d+\sigma+r)(\omega_++1)h_b.$$

Here $\omega_+=1-2h_b(2h_a-h_b)^{-1}$, h_a,h_b are defined in section 2 and

$$c_1=\left\{\left[\frac{1}{2}dp(1-p)\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{2p-1}{p}}+\frac{1}{2}Ap(1+p)\lambda^{\frac{2}{p}}(d+\sigma+r)^{\frac{2p-2}{p}}\right]h_b^3-2p\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{2p-1}{p}}h_b^2\right\}\cdot\{\omega_++pd(R_0^{\frac{1}{p}}-1)h_b-(\sigma+r)h_b\},$$
$$c_2=b_1\{p(\omega_++2d-3)\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{2p-1}{p}}h_b^2+p(1+p)\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{2p-2}{p}}[A\lambda^{\frac{1}{p}}-d(d+\sigma+r)^{\frac{1}{p}}]h_b^3\}\cdot\{\omega_++pd(R_0^{\frac{1}{p}}-1)h_b-(\sigma+r)h_b-1\},$$
$$b_1=\left\{\left[\frac{1}{2}dp(1-p)\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{p-1}{p}}+\frac{1}{2}Ap(1+p)\lambda^{\frac{2}{p}}(d+\sigma+r)^{\frac{p-2}{p}}\right]h_b^2-2p\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{p-1}{p}}h_b\right\}\cdot\{2+[\sigma+r-pd(R_0^{\frac{1}{p}}-1)]h_b\}(\omega_+^2-1)^{-1}.$$

Proof Since $(h,A,d,\sigma,r,\lambda,p)\in F_1$, straight computation shows that $\omega_-=-1$ and

$$\omega_+=1-2h_b(2h_a-h_b)^{-1}\in(-1,1)$$

are two eigenvalues of the Jacobian matrix at E_2. Let $\xi_k=S_k-S^*$, $\eta_k=I_k-I^*$, $\mu_k=h-h_b$ for each $k=1,2,\cdots$. System (4) can be rewrote as

$$\begin{cases}\xi_{n+1}=a_{11}\xi_n+a_{12}\eta_n+a_{13}\xi_n^2+a_{14}\xi_n\eta_n+\\ \quad a_{15}\eta_n^2+b_{11}\xi_n\mu_n+b_{12}\eta_n\mu_n+b_{13}\xi_n^2\mu_n+\\ \quad b_{14}\xi_n\eta_n\mu_n+b_{15}\eta_n^2\mu_n+o((|\xi_n|+|\eta_n|)^2)\mu_n,\\ \mu_{n+1}=\mu_n,\\ \eta_{n+1}=a_{21}\xi_n+a_{22}\eta_n+a_{23}\xi_n^2+a_{24}\xi_n\eta_n+a_{25}\eta_n^2+\\ \quad b_{21}\xi_n\mu_n+b_{22}\eta_n\mu_n+\\ \quad b_{23}\xi_n^2\mu_n+b_{24}\xi_n\eta_n\mu_n+b_{25}\eta_n^2\mu_n+\\ \quad o((|\xi_n|)+|\eta_n|^2)\mu_n\end{cases}\tag{5}$$

where

$$a_{11}=1-(d+\lambda pS^{*\,p-1}I^*)h_b,$$
$$a_{12}=-(d+\sigma+r)h_b,$$
$$a_{13}=-\frac{1}{2}\lambda p(p-1)S^{*\,p-2}I^*h_b,$$
$$a_{14}=-\lambda pS^{*\,p-1}h_b,$$
$$a_{15}=0,a_{21}=\lambda pS^{*\,p-1}I^*h_b,$$
$$a_{22}=1,$$
$$b_{11}=-(d+\lambda pS^{*\,p-1}I^*),$$
$$b_{12}=-(d+\sigma+r),$$
$$b_{13}=-\frac{1}{2}\lambda p(p-1)S^{*\,p-2}I^*,$$
$$b_{14}=-\lambda pS^{*\,p-1},$$
$$b_{15}=0,$$
$$b_{21}=\lambda pS^{*\,p-1}I^*,$$
$$b_{22}=0,$$

and $a_{23}=-a_{13}$, $a_{24}=-a_{14}$, $a_{25}=-a_{15}$, $b_{23}=-b_{13}$, $b_{24}=-b_{14}$, $b_{25}=-b_{15}$. Obviously, $a_{12}\neq 0$. Through transformation

$$\begin{pmatrix}\xi_n\\ \mu_n\\ \eta_n\end{pmatrix}=\begin{pmatrix}a_{12} & 0 & a_{12}\\ 0 & 1 & 0\\ -1-a_{11} & 0 & \omega_+-a_{11}\end{pmatrix}\begin{pmatrix}u_n\\ \delta_n\\ v_n\end{pmatrix},$$

system (5) can be normalized as

$$\begin{pmatrix}u_{n+1}\\ \delta_{n+1}\\ v_{n+1}\end{pmatrix}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \omega_+\end{pmatrix}\begin{pmatrix}u_n\\ \delta_n\\ v_n\end{pmatrix}+\begin{pmatrix}f_1(u_n,\delta_n,v_n)\\ 0\\ f_2(u_n,\delta_n,v_n)\end{pmatrix} \quad (6)$$

where

$$f_1(u_n,\delta_n,v_n):=\bar{f}_1(a_{12}(u_n+v_n),\ \delta_n,(\omega_+-a_{11})v_n-(1+a_n)u_n)$$

and

$$f_2(u_n,\delta_n,v_n):=\bar{f}_2(a_{12}(u_n+v_n),\ \delta_n,(\omega_+-a_{11})v_n-(1+a_n)u_n).$$

Here

$$\begin{aligned}\bar{f}_1(\xi_n,\mu_n,\eta_n)=&\frac{(\omega_+-a_{11}+a_{12})a_{13}}{a_{12}(\omega_++1)}\xi_n^2+\frac{(\omega_+-a_{11}+a_{12})a_{14}}{a_{12}(\omega_++1)}\xi_n\eta_n+\frac{(\omega_+-a_{11}+a_{12})a_{15}}{a_{12}(\omega_++1)}\eta_n^2+\\&o((|\xi_n|+|\eta_n|)^2)+\frac{(\omega_+-a_{11})b_{11}-a_{12}b_{21}}{a_{12}(\omega_++1)}\xi_n\mu_n+\frac{(\omega_+-a_{11})b_{12}-a_{12}b_{22}}{a_{12}(\omega_++1)}\eta_n\mu_n+\\&\frac{(\omega_+-a_{11}+a_{12})b_{13}}{a_{12}(\omega_++1)}\xi_n^2\mu_n+\frac{(\omega_+-a_{11}+a_{12})b_{14}}{a_{12}(\omega_++1)}\xi_n\eta_n\mu_n+\frac{(\omega_+-a_{11}+a_{12})b_{15}}{a_{12}(\omega_++1)}\eta_n^2\mu_n+\\&o((|\xi_n|+|\eta_n|)^2)\mu_n,\end{aligned}$$

$$\begin{aligned}\bar{f}_2(\xi_n,\mu_n,\eta_n)=&\frac{(1+a_{11}-a_{12})a_{13}}{a_{12}(\omega_++1)}\xi_n^2+\frac{(1+a_{11}-a_{12})a_{14}}{a_{12}(\omega_++1)}\xi_n\eta_n+\frac{(1+a_{11}-a_{12})a_{15}}{a_{12}(\omega_++1)}\eta_n^2+\\&o((|\xi_n|+|\eta_n|)^2)+\frac{(1+a_{11})b_{11}+a_{12}b_{21}}{a_{12}(\omega_++1)}\xi_n\mu_n+\frac{(1+a_{11})b_{12}+a_{12}b_{22}}{a_{12}(\omega_++1)}\eta_n\mu_n+\\&\frac{(1+a_{11}-a_{12})b_{13}}{a_{12}(\omega_++1)}\xi_n^2\mu_n+\frac{(1+a_{11}-a_{12})b_{14}}{a_{12}(\omega_++1)}\xi_n\eta_n\mu_n+\frac{(1+a_{11}-a_{12})b_{15}}{a_{12}(\omega_++1)}\eta_n^2\mu_n+\\&o((|\xi_n|+|\eta_n|)^2)\mu_n.\end{aligned}$$

By Ref. [2] or Ref. [9], there is a center manifold $W^c_{loc}(O)$ for system (6). We can express locally it as

$$W^c_{loc}(O)=\{(u_n,\delta_n,v_n)\in\mathbf{R}^3: v_n=W(u_n,\delta_n),\ |u_n|<\varepsilon_1\ll 1,|\delta_n|<\varepsilon_2\ll 1,\ W(0,0)=W'_{u_n}(0,0)=W'_{\delta_n}(0,0)=0\}.$$

Clearly, we can assume that

$$W(u_n,\delta_n)=b_1u_n^2+b_2u_n\delta_n+b_3\delta_n^2+o((|\xi_n|+|\eta_n|)^2)$$

on the center manifold $W^c_{loc}(O)$. By (6), we obtain

$$v_{n+1}=\omega_+v_n+f_2(u_n,\delta_n,v_n)=\omega_+W(u_n,\delta_n)+f_2(u_n,\delta_n,W(u_n,\delta_n)) \quad (7)$$

Comparing the coefficients of (u_n,δ_n) on both sides of the equation in (7), we obtain the expression of b_1 as given in the statement of this theorem, $b_3=0$ and

$$b_2=\{2[d+pd(R_0^{1/p}-1)]h_b-pd\,(d+\sigma+r)^2(R_0^{1/p}-1)h_b^2-4\}h_b^{-1}(\omega_++1)^{-2}.$$

Confined on $W^c_{loc}(O)$ we get

$$u_{n+1}=-u_n+f_1(u_n,\delta_n,W(u_n,\delta_n))=:F(u_n,\delta_n),$$

where

$$F(u_n,\delta_n)=-u_n+\frac{c_1u_n^2+c_3u_n\delta_n+c_4u_n^2\delta_n+c_5u_n\delta_n^2+c_2u_n^3}{a_{12}(\omega_++1)}+o((|u_n|+|\delta_n|)^3).$$

Here

$$\begin{aligned}c_3=&(a_{12}b_{11}-b_{12}-a_{11}b_{12})(\omega_+-a_{11})-a_{12}^2b_{21}=(d+\sigma+r)\{2(\omega_+-1)+\\&2\{d+pd(R_0^{1/p}-1)\}h_b-pd(d+\sigma+r)(R_0^{1/p}-1)h_b^2\},\end{aligned}$$

$$\begin{aligned}c_4=&\{(a_{12}a_{14}b_2+b_{12}b_1)(\omega_+-a_{11})+a_{12}^2(2a_{13}b_2+b_{13})-\\&(a_{12}a_{14}b_2+a_{12}b_{14})(1+a_{11})\}(\omega_+-a_{11}+a_{12}),\end{aligned}$$

$$c_5 = b_2(b_{12}\omega_+ - b_{12}a_{11})(\omega_+ - a_{11} + a_{12}),$$

and c_1, c_2 as given in the statement of this theorem. Thus,

$$\frac{\partial^2 F}{\partial u_n \partial \delta_n}\Big|_{(u_n,\delta_n)=(0,0)} = \frac{c_3}{a_{12}(\omega_+ + 1)} = l_1,$$

$$\left(\frac{1}{2}\left(\frac{\partial^2 F}{\partial u_n^2}\right)^2 + \frac{1}{3}\frac{\partial^3 F}{\partial u_n^3}\right)\Big|_{(u_n,\delta_n)=(0,0)} =$$

$$\frac{2c_2}{a_{12}(\omega_+ + 1)} + \frac{2c_1^2}{a_{12}^2(\omega_+ + 1)^2} = \frac{2l_2}{a_{12}^2(\omega_+ + 1)^2},$$

where l_1, l_2 are the same as which are given in the statement of this theorem. By Ref. [11], system (6) undergoes a flip bifurcation at E_2 if $l_1 l_2 \neq 0$, so does the equivalent system (4). In addition, by the expression of $F(u_n,\delta_n)$, we get the normal form of (4) confined on its center manifold as

$$u_{n+1} = -\{1 - l_1(h - h_b) + o(h - h_b)\}u_n + \mathrm{sgn}(l_2)u_n^3 + o(u_n^3),$$

which implies the bifurcation direction of h as given in the statement of this theorem.

The bifurcation analysis for $(h,A,d,\sigma,r,\lambda,p)$ crossing through the non-hyperbolic surface F_2 is similar to the non-hyperbolic surface F_1. Thus, we omit it here. The proof is end.

In the following we discuss the bifurcation phenomena of system (4) when $(h,A,d,\sigma,r,\lambda,p)$ crosses the non-hyperbolic hypersurface NS.

Theorem 3.2　Assume that $(h_a,A,d,\sigma,r,\lambda,p)\in NS$. If

$$\{d + pd(R_0^{1/p} - 1)\}^2 \neq jpd(d+\sigma+r)(R_0^{1/p} - 1),\ j = 2,3 \qquad (8)$$

and $\zeta \neq 0$, system (4) goes through a Neimark-Sacker bifurcation at E_2, i.e., a unique attracting invariant closed curve appears when $h = h_a + \varepsilon$ if $\zeta < 0$ or a unique repelling invariant closed curve appears when $h = h_a - \varepsilon$ if $\zeta > 0$, where $0 < \varepsilon \ll 1$ and

$$\zeta := -\mathrm{Re}\left(\frac{(1-2\omega_+)\omega_-^2}{1-\omega_+}\gamma_{20}\gamma_{11}\right) - \frac{1}{2}|\gamma_{11}|^2 - |\gamma_{02}|^2 + \mathrm{Re}(\gamma_{21}\omega_-).$$

Here

$$\omega_\pm = 1 - h_a\{1 + pd(R_0^{1/p} - 1) \mp \mathrm{i}\sqrt{-\Delta}\}/2,$$

$$\rho := h_a\{d + pd(R_0^{1/p} - 1)\}/2,$$

$$\gamma_{20} := \frac{1}{8\sqrt{-\Delta}}\{2h_a(\rho - (d+\sigma+r)h_a)(p-1)(A\lambda^{\frac{1}{p}} - d(d+\sigma+r)^{\frac{1}{p}}) - (4\rho^2 - 4(d+\sigma+r)h_a\rho - h_a^2\Delta)(d+\sigma+r)^{\frac{1}{p}}\}p\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{p-2}{p}} + \frac{\mathrm{i}}{8}\{2(d+\sigma+r)^{\frac{p+1}{p}} - (p-1)(A\lambda^{\frac{1}{p}} - d(d+\sigma+r)^{\frac{1}{p}})\}p\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{p-2}{p}}h_a^2,$$

$$\gamma_{11} := p\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{p-2}{p}}\{(p-1)(A\lambda^{\frac{1}{p}} - d(d+\sigma+r)^{\frac{1}{p}})h_a - 2\rho(d+\sigma+r)^{\frac{1}{p}}\}\left\{\frac{1}{2\sqrt{-\Delta}}((d+\sigma+r)h_a - \rho) + \frac{\mathrm{i}}{4}h_a\right\},$$

$$\gamma_{02} := \frac{1}{2\sqrt{-\Delta}}\{2h_a(\rho - (d+\sigma+r)h_a)(p-1)(A\lambda^{\frac{1}{p}} - d(d+\sigma+r)^{\frac{1}{p}}) - (4\rho^2 - 4(d+\sigma+r)h_a\rho - h_a^2\Delta)(d+\sigma+r)^{\frac{1}{p}}\}p\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{p-2}{p}} + \frac{\mathrm{i}}{8}\{2(2\rho - (d+\sigma+r)h_a)(d+\sigma+r)^{\frac{1}{p}} - h_a(p-1)(A\lambda^{\frac{1}{p}} - d(d+\sigma+r)^{\frac{1}{p}})\}p\lambda^{\frac{1}{p}}(d+\sigma+r)^{\frac{p-2}{p}}h_a^2,$$

$$\gamma_{21} := \frac{1}{16}\{(2\rho + (d+\sigma+r)h_a)(d+\sigma+r)^{\frac{1}{p}} - h_a(p-2)(A\lambda^{\frac{1}{p}} - d(d+\sigma+r)^{\frac{1}{p}})\}p(p-1)\lambda^{\frac{2}{p}}(d+\sigma+r)^{\frac{2p-3}{p}}h_a^2 + \frac{\mathrm{i}}{32\sqrt{-\Delta}}\{4(\rho - (d+\sigma+r)h_a)(p-2)(A\lambda^{\frac{1}{p}} - d(d+\sigma+r)^{\frac{1}{p}}) + (12\rho^2 - 12(d+\sigma+r)h_a\rho - h_a^2\Delta)(d+\sigma+r)^{\frac{p+1}{p}}\}p(p-1)\lambda^{\frac{2}{p}}(d+\sigma+r)^{\frac{p-3}{p}}h_a,$$

h_a, Δ are defined in section 2.

Proof Since $(h_a, A, d, \sigma, r, \lambda, p) \in NS$, there are two complex eigenvalues $\omega_\pm$ of the Jacobian matrix at E_2 which satisfy $|\omega_\pm| = 1$. Let $\xi_n := S_n - S^*$, $\eta_n := I_n - I^*$, $\mu := h - h_a$. System (4) can be rewrote as

$$\begin{cases} \xi_{n+1} = a_1\xi_n + a_2\eta_n + a_3\xi_n^2 + a_4\xi_n\eta_n + \\ \quad a_5\xi_n^3 + a_6\xi_n^2\eta_n + o((|\xi_n| + |\eta_n|)^3), \\ \eta_{n+1} = k_1\xi_n + k_2\eta_n + k_3\xi_n^2 + k_4\xi_n\eta_n + k_5\xi_n^3 + \\ \quad k_6\xi_n^2\eta_n + o((|\xi_n| + |\eta_n|)^3) \end{cases} \tag{9}$$

where

$$a_1 = 1 - (h_a + \mu)(d + pd(R_0^{1/p} - 1)),$$
$$a_2 = -(h_a + \mu)(d + + \sigma + r),$$
$$a_3 = -\frac{1}{2}(h_a + \mu)p\,(p-1)\lambda^{1/p}(d + \sigma + r)^{-2/p}(A\lambda^{1/p} - d\,(d + \sigma + r)^{1/p}),$$
$$a_4 = -(h_a + \mu)p\,\lambda^{1/p}(d + \sigma + r)^{(p-1)/p},$$
$$a_5 = -\frac{1}{6}(h_a + \mu)p\,(p-1)(p-2)\lambda^{2/p} \cdot (d + \sigma + r)^{-3/p}(A\lambda^{1/p} - d\,(d + \sigma + r)^{1/p}),$$
$$a_6 = -\frac{1}{2}(h_a + \mu)p\,(p-1)\lambda^{2/p}(d + \sigma + r)^{(p-2)/p},$$
$$k_1 = (h_a + \mu)pd\,(R_0^{1/p} - 1),$$
$$k_2 = 1, k_3 = -a_3,$$
$$k_4 = -a_4, k_5 = -a_5, k_6 = -a_6.$$

The Jacobian matrix of system (9) at $(0,0)$ has a pair of eigenvalues

$$\omega_\pm(\mu) = 1 - (h_a + \mu)\{d + pd(R_0^{1/p} - 1) \mp \mathrm{i}\sqrt{-\Delta}\}/2.$$

By Table 1, $|\omega_\pm(0)| = 1$. Let $\hat{\alpha} \pm i\hat{\beta}$ represent $\omega_\pm(0)$. We obtain $0 \leqslant |\hat{\alpha}| < 1$ and $0 < |\hat{\beta}| \leqslant 1$ as $\Delta < 0$, which imply $\omega_\pm^k(0) \neq 1$ for all $k = 1, 2$. Additionally, $\omega_\pm^3(0) \neq 1$ if and only if $\hat{\alpha} \neq 1, -1/2$; By straight calculation, we can get that $\omega_\pm^3(0) \neq 1$ if and only if (8) holds for $j = 3$ and $\omega_\pm^4(0) \neq 1$ if and only if (8) holds for $j = 2$. Then, $\omega_\pm^k(0) \neq 1$ for all $k = 1, \cdots, 4$, i. e., condition (SH1) of Ref. [5, Theorem 3. 5. 2] holds when (8) holds.

On the other hand,

$$d|\omega_\pm(0)|/d\mu = d + pd(R_0^{1/p} - 1) > 0.$$

Thus, condition (SH2) of Ref. [5, Theorem 3. 5. 2] holds. Through transformation

$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = \begin{pmatrix} 0 & -(d + \sigma + r)h_a \\ \beta & \rho \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

where ρ is exactly as the definition in this theorem, (9) is transformed into

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \hat{\alpha} & -\hat{\beta} \\ \hat{\beta} & \hat{\alpha} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1(u_n, v_n) \\ f_2(u_n, v_n) \end{pmatrix},$$

where $f_1(u_n, v_n)$ and $f_2(u_n, v_n)$ consist of all high order terms higher than 2 order. Let $\omega_\pm(0)$ as $\omega_\pm$. By straight calculation, it is easy to check that $\gamma_{20}, \gamma_{11}, \gamma_{02}, \gamma_{21}$ given in this theorem are exactly as

$$\{f_{1u_nu_n} - f_{1v_nv_n} + 2f_{2u_nv_n} + \mathrm{i}(f_{2u_nu_n} - f_{2v_nv_n} - 2f_{1u_nv_n})\}/8,$$
$$\{f_{1u_nu_n} + f_{1v_nv_n} + \mathrm{i}(f_{2u_nu_n} + f_{2v_nv_n})\}/4,$$
$$\{f_{1u_nu_n} - f_{1v_nv_n} - 2f_{2u_nv_n} + \mathrm{i}(f_{2u_nu_n} - f_{2v_nv_n} + 2f_{1u_nv_n})\}/8,$$
$$\{f_{1u_nu_nu_n} + f_{1u_nv_nv_n} + f_{2u_nu_nv_n} + f_{2v_nv_nv_n} + \mathrm{i}(f_{2u_nu_nu_n} + f_{2u_nv_nv_n} - f_{1u_nu_nv_n} - f_{1v_nv_nv_n})\}/16.$$

System (9) goes through a Neimark-Sacker bifurcation at $(0,0)$ if $\zeta \neq 0$ by Ref. [5, Theorems 3. 5. 2], then system (4) appear same bifurcation at E_2, where ζ is as the definition in this theorem. We get the bifurcation direction and the attractivity of the closed invariant curve by the sign of ζ as presented in this theorem. The proof is end.

The results of flip and Neimark-Sacker bifurcations reveal that the susceptible and infective individuals can coexist in stable period-n orbits and cycles. It is very difficult to consider that case that $(h, A, d, \sigma, r, \lambda, p)$ crosses non-hyperbolic surface R because 1:2 resonance happens. On the other hand, we do not discuss the dynamical behavior near E_1 in this paper. Bifurcation phenomena maybe complicated near E_1 because of its complicated topological type and non-hyperbolic situation. From the perspective of biology, the results of flip and Neimark-Sacker bifurcations reveal that the susceptible and infective individuals can coexist in stable.

References：

[1] Capasso V, Serio G. A generalization of the Kermack-McKendrick deterministic epidemic model [J]. Math Biosci, 1978, 42: 43.

[2] Carr J. Applications of centre manifold theory [M]. New York: Springer-Verlag, 1981.

[3] Van den Driessche P, Watmough J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission [J]. Math Biosci, 2002, 180: 29.

[4] Du W, Zhang J, Qin S *et al*. Bifurcation analysis in a discrete SIR epidemic model with the saturated contact rate and vertical transmission [J]. J Nonlinear Sci Appl, 2016, 9: 4976.

[5] Guckenheimer J, Holmes P. Nonlinear oscillations, dynamical systems and bifurcations of vector fields [M]. New York: Springer-Verlag, 1983.

[6] Grote K, Meyer-Spasche R. Euler-like discrete models of the Logistic differential equation [J]. Comput Math Appl, 1998, 36: 211.

[7] Hu Z Y, Teng Z D, Zhang L. Stability and bifurcation analysis in a discrete SIR epidemic model [J]. Math Comput Simulat, 2014, 97: 80.

[8] Kermack W, McKendrick A. A Contribution to the mathematical theory of epidemics [J]. P Roy Soc A-Math Phy, 1927, 115: 700.

[9] Kuznetsov Y A. Elements of applied bifurcation theory [M]. New York: Springer-Verlag, 1998.