doi：103969/j. issn. 0490-6756. 2018. 06. 005

具临界指数的强阻尼波方程的时间依赖全局吸引子

孟凤娟[1]，曹凤雪[2]
(1. 江苏理工学院数理学院，常州 213001；2. 江苏理工学院计算机工程学院，常州 213001)

摘 要：本文考虑如下具有临界增长指数的强阻尼波方程

$$\varepsilon(t)\, u_{tt} - \Delta u_t - \Delta u + \varphi(u) = f$$

解的长时间行为.本文首先得到过程的耗散性,然后利用过程分解技巧得到过程的渐近紧性,最后给出了方程依赖于时间的吸引子的存在性.

关键词：时间依赖全局吸引子;强阻尼波方程;临界指数
中图分类号：O175. 29 **文献标识码**：A **文章编号**：0490-6756(2018)06-1155-07

Time-dependent global attractors for strongly damped wave equation with critical exponent

MENG Feng-Juan[1], *CAO Feng-Xue*[2]
(1. School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, China;
2. School of Computer Engineering, Jiangsu University of Technology, Changzhou 213001, China)

Abstract: In this paper, we are concerned with the longtime behavior of the non-autonomous strongly damped wave equation

$$\varepsilon(t) u_{tt} - \Delta u_t - \Delta u + \varphi(u) = f.$$

We first obtain the dissipativity of the process associated with the equation. Then, by using the skill of operator decomposition, we prove the asymptotic compactness of the process. Finally, we prove the existence of the time-dependent attractors for the equation.

Keywords: Time-dependent global attractor; Strongly damped wave equation; Critical exponent
(2010 MSC 35L05, 35L30)

1 Introduction

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For any $\tau \in \mathbf{R}$, we consider the following nonautonomous strongly damped wave equation:

$$\begin{cases} \varepsilon u_{tt} - \Delta u_t - \Delta u + \varphi(u) = f, \ t > \tau, \\ u\,|_{\partial\Omega} = 0, u(x,\tau) = u_0, u_t(x,\tau) = u_1 \end{cases} \tag{1}$$

where the unknown variable $u = u(x,t): \Omega \times [\tau, \infty) \to \mathbf{R}$ and $u_0, u_1: \Omega \to \mathbf{R}$ are assigned data, $f \in L^2(\Omega)$ is independent of time.

Equation (1) arises as an evolutionary mathematical model in various systems for the relevant physical application. For example, it describes the variation of the configuration at rest of a homogeneous and isotropic linearly viscoelastic solid with short memory. For more details, see Ref. [1] and the references therein.

We impose the following assumptions on ε and the nonlinear term φ.

收稿日期：2018-01-28
基金项目：国家自然科学基金(11701230,11526100);江苏省青蓝工程和江苏省高校境外研修计划
作者简介：孟凤娟(1982—)，女，博士，副教授，主要研究方向为无穷维动力系统. E-mail：fjmeng@jsut. edu. cn

(C_1): $\varepsilon=\varepsilon(t)$ is a function of t, $\varepsilon\in C^1(\mathbf{R})$ is a decreasing bounded function and satisfies

$$\lim_{t\to+\infty}\varepsilon(t)=0 \quad (2)$$

In particular, there exists $L>0$ such that

$$\sup_{t\in\mathbf{R}}[\,|\varepsilon(t)|+|\varepsilon'(t)|\,]\leqslant L \quad (3)$$

(C_2): Let $\varphi\in C^1(\mathbf{R})$ be such that

$$|\varphi(r)|\leqslant c_1(1+|r|^5),\ \forall r\in\mathbf{R} \quad (4)$$

$$\liminf_{|r|\to\infty}\varphi'(r)>-\lambda_1 \quad (5)$$

where $\lambda_1>0$ is the first eigenvalue of the strictly positive Dirichlet operator $A=-\Delta$ with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$.

***Remark* 1** From (4), the more general condition

$$|\varphi(r)-\varphi(s)|\leqslant c\,|r-s|\,(1+r^4+s^4),\ \forall r,s\in\mathbf{R} \quad (6)$$

is hold naturally, where $c=c(c_1)$.

Set

$$\Phi(u)=\int_\Omega\Big(\int_0^{u(x)}\varphi(y)\mathrm{d}y\Big)\mathrm{d}x.$$

from (5) we can obtain that

$$\Phi(u)\geq-\frac{\nu}{2}\|u\|^2-c_1 \quad (7)$$

$$\langle\varphi(u),u\rangle\geq\Phi(u)-\frac{\nu}{2}\|u\|^2-c_1 \quad (8)$$

for some $\nu<\lambda_1$.

When ε is a positive constant, system (1) is autonomous and the problem is completely understood within the framework of semigroups. It is known[2,3] that under the conditions (4), (5), equation (1) generates a C^0- semigroup $S(t)$ in the natural energy phase space $H=H_0^1(\Omega)\times L^2(\Omega)$, the asymptotic behavior of solutions to equation (1) has been investigated quite extensively by several authors in recent years[4-10].

On the other hand, when ε is a positive constant and f is depends on time, system (1) is non-autonomous, the asymptotic behavior of nonautonomous strongly damped wave equation has been considered[11]. The pullback attractor for the strongly damped wave equation has also been considered[12].

When ε is not a constant, but a positive decreasing function of time $\varepsilon(t)$ vanishing at infinity, the natural energy associated to the system is

$$E(t)=\varepsilon(t)\int_\Omega|u_t(x,t)|^2\mathrm{d}x+\int_\Omega|\nabla u(x,t)|^2.$$

It is easy to see that the vanishing character of ε at infinity prevents the existence of absorbing set or pullback absorbing set in the usual sense.

To circumvent these issues, in Ref. [13], the authors made a essential progress by adopting a new point of view on pullback dissipativity. The authors provided a suitable modification of the notion of pullback attractor and established a new theory of pullback flavor for dynamical systems. To this end, the authors described the solution operators as a family of maps

$$S(t,\tau):X_\tau\to X_t,\ t\geq\tau\in\mathbf{R}$$

acting on a time-dependent family of spaces X_t.

In Ref. [14], the authors recovered and improved the results in Ref. [13] by giving new insights on attractors on time-dependent spaces. Moreover, the authors established a new framework to study the longterm behavior of weakly damped wave equation.

In Ref. [15], the authors established a sufficient and necessary condition for the existence of attractors on time-dependent spaces, which is equivalent to that provided by Conti *et al*.[14]. Furthermore, a technical method for verifying compactness of the process via contractive functions was given. Finally, the existence of time-dependent global attractor for the wave equation with nonlinear damping was proved by using the new framework.

In Ref. [16], by using the method of Ref. [15], the authors considered the longtime behavior of some nonlinear evolution equation.

Since the wave equations with different kind of damping arise from different evolutionary mathematical model, it is meaningful to consider the longtime behavior of the strongly damped wave equation under the new framework. The aim of the present paper is to study the longtime behavior of the solutions to (1) with ε depending on time, according to the abstract framework developed in Ref. [14]. We obtain time-dependent absorbing set and time-dependent global attractor for (1). Since (1) contains the strong damping term Δu_t, compared to the weakly damped wave equation, the critical nonlinearity exponent from 3 to 5 in 3 dimensional space, see Refs. [2,3,5,7~11], etc. In order to obtain the asymptotic compactness of the process, we apply the techniques introduced in Ref. [7] for the autonomous case to overcome the difficulty due to the critical nonlinearity. Moreover, we need to tackle the terms caused by time dependent term $\varepsilon(t)$.

2 Preliminaries

Throughout the paper, C denotes any positive constant which may be different from line to line even in the same line (sometimes for special differentiation, we also denote the different positive constants by C_i.

We denote the inner product and the norm on $L^2(\Omega)$ by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. For $0\leqslant\sigma\leqslant 2$, we define the hierarchy of (compactly) nested Hilbert spaces

$$H_\sigma=D(A^{\frac{\sigma}{2}}),\langle u,v\rangle_\sigma=\langle A^{\frac{\sigma}{2}}u,A^{\frac{\sigma}{2}}v\rangle,$$

$$\|u\|_\sigma=\|A^{\frac{\sigma}{2}}u\|.$$

Then, for $t\in\mathbf{R}$ and $0\leqslant\sigma\leqslant 2$, we introduce the time-dependent spaces

$$H_t^\sigma=H_{\sigma+1}\times H_\sigma$$

endowed with the time-dependent product norms

$$\|\{u_0,u_1\}\|_{H_t^\sigma}^2=\|u_0\|_{\sigma+1}^2+\varepsilon(t)\|u_1\|_\sigma^2.$$

In particular, the symbol σ is always omitted whenever zero, that is $H_t=H_1\times H$ with

$$\|\{u_0,u_1\}\|_{H_t}^2=\|\nabla u_0\|^2+\varepsilon(t)\|u_1\|^2.$$

Then we have the compact embedding

$$H_t^\sigma\hookrightarrow H_t,\ 0<\sigma\leqslant 2$$

with the injection constants independent of $t\in\mathbf{R}$. Note that the spaces H_t are all the same as linear spaces and the norms $\|\cdot\|_{H_t}^2$ and $\|\cdot\|_{H_\tau}^2$ are equivalent for any fixed $t,\tau\in\mathbf{R}$. However, this equivalence blows up as we let $t,\tau\to\pm\infty$.

For every $t\in\mathbf{R}$, let X_t be a family of normed spaces, we introduce the R-ball of X_t

$$B_t(R)=\{z\in X_t:\|z\|_{X_t}\leqslant R\}.$$

For any given $\varepsilon>0$, the ε- neighborhood of a set $B\subset X_t$ is defined as

$$\mathcal{O}_t^\varepsilon(B)=\bigcup_{x\in B}\{y\in X_t:\|x-y\|_{X_t}<\varepsilon\}=\bigcup_{x\in B}\{x+B_t(\varepsilon)\}.$$

We denote the Hausdorff semidistance of two (nonempty) sets $B,C\subset X_t$ by

$$\delta_t(B,C)=\sup_{x\in B}\inf_{y\in C}\|x-y\|_{X_t}.$$

Finally, given any set $B\subset X_t$, the symbol $\overline{B}$ stands for the closure of B in X_t.

Definition 2.1 For $t\in\mathbf{R}$, let X_t be a family of normed spaces. A process is a two-parameter family of mappings $\{S(t,\tau):X_\tau\to X_t,t\geq\tau,\tau\in\mathbf{R}\}$ with properties

(i) $S(\tau,\tau)=\mathrm{Id}$ is the identity operator on X_τ, $\tau\in\mathbf{R}$;

(ii) $S(t,s)S(s,\tau)=S(t,\tau)$, $\forall t\geq s\geq\tau,\tau\in\mathbf{R}$.

Definition 2.2 A family $\mathfrak{B}=\{B_t\}_{t\in\mathbf{R}}$ of bounded sets $B_t\subset X_t$ is called uniformly bounded if there exists $R>0$ such that

$$B_t\subset B_t(R),\ \forall t\in\mathbf{R}.$$

Definition 2.3 A family $\mathfrak{C}=\{C_t\}_{t\in\mathbf{R}}$ is called pullback absorbing if it is uniformly bounded and for every $R>0$, there exists $t_0=t_0(t,R)\leqslant t$ such that

$$\tau\leqslant t_0\Rightarrow S(t,\tau)B_\tau(R)\subset B_t.$$

The process $S(t,\tau)$ is called dissipative whenever it admits a pullback absorbing family.

Definition 2.4 A time-dependent absorbing set for the process $S(t,\tau)$ is a uniformly bounded family $\mathfrak{B}=\{B_t\}_{t\in\mathbf{R}}$ with the following property: for every $T\geq 0$ there exists $t_0=t_0(T)\geq 0$ such that $\tau\leqslant t-t_0\Rightarrow S(t,\tau)B_\tau(R)\subset B_t$.

Definition 2.5 A (uniformly bounded) family $\mathfrak{K}=\{K_t\}_{t\in\mathbb{R}}$ is called pullback attracting if for all $\varepsilon>0$ the family $\{\mathcal{O}_t^\varepsilon(K_t)\}_{t\in\mathbb{R}}$ is pullback absorbing.

Consider the collection $K=\{\mathfrak{K}=\{K_t\}_{t\in\mathbf{R}}:K_t\subset X_t$ compact, $\mathfrak{K}$ pullback attracting$\}$. When $K\neq\varnothing$, we say that the process is asymptotically compact.

Definition 2.6 We call a time-dependent global attractor the smallest element of K, i. e. the family $\mathfrak{A}=\{A_t\}_{t\in R}\in K$ such that $A_t\subset K_t$, $\forall t\in\mathbf{R}$ for any element $\mathfrak{K}=\{K_t\}_{t\in\mathbf{R}}\in K$.

Recall that for any pair of fixed times $t\geq\tau$, the map $S(t,\tau):X_\tau\to X_t$ is said to be closed if $x_n\to x$, in X_τ, $S(t,\tau)x_n\to y$ in X_t, $\Rightarrow S(t,\tau)x=y$.

Definition 2.7 The process $S(t,\tau)$ is called

(i) closed if $S(t,\tau)$ is a closed map for any pair of fixed times $t\geq\tau$;

(ii) T- closed for some $T>0$ if $S(t,t-T)$ is a closed map for all t.

Note that if the process $S(t,\tau)$ is a continuous (or even norm-to-weak continuous) map for all $t\geq\tau$, then the process is closed. Of course, if the process $S(t,\tau)$ is closed it is T- closed, for any $T>0$.

Theorem 2.8 If $S(t,\tau)$ is asymptotically compact, then there exists a unique time-dependent attractor $\mathfrak{A}$. Furthermore, if $S(t,\tau)$ is a T-closed process for some $T>0$, then $\mathfrak{A}$ is invariant.

3 Well-posedness

In this section, we state the results about the well-posedness of problem (1) which can be seen in Refs. [2,3,7].

Definition 3.1 Under the conditions (4) and (5), for any initial data $(u_0,u_1)\in H_\tau$ on any interval $[\tau,t]$ with $t>\tau$, there exists a unique solution $u\in C([\tau,t],H_0^1(\Omega))$, $u_t\in C([\tau,t],L^2(\Omega))\cap L^2([\tau,t],H_0^1(\Omega))$, which continuously depend on the initial data. That is, problem (1) generates a strongly continuous process $S(t,\tau):H_\tau\to H_t, t\geq\tau\in\mathbf{R}$, where $S(t,\tau):H_\tau\to H_t$ acting as $S(t,\tau)z=\{u(t),u_t(t)$.

Remark 2 In Refs. [2,3], based on the theory of analytic semigroups, the authors established the well-posedness of the strongly damped wave equation. We can also obtain the existence of the solution according to the standard Fatou-Galerkin method, which is based on Lemma 4.2 below.

Moreover, we state the continuous dependence estimate for $S(t,\tau)$ on H_τ, which can be to verify the uniqueness of the solution.

Theorem 3.2 Given $R>0$, for every pair of initial data $z_i=\{u_{0i},u_{1i}\}\in H_\tau$ such that $\|z_i\|_{H_\tau}\leq R$, $i=1,2$, the difference of the corresponding solutions satisfies

$$\|S(t,\tau)z_1-S(t,\tau)z_2\|_{H_t}\leqslant e^{C(t-\tau)}\|z_1-z_2\|_{H_\tau},\ \forall t\geq\tau \tag{9}$$

for some constant $C=C(R)\geq 0$.

Proof Given two different initial data $z_1,z_2\in H_\tau$ such that $\|z_i\|_{H_\tau}\leqslant R$, $i=1,2$. By Lemma 4.2 below we know that

$$\|S(t,\tau)z_i\|_{H_\tau}\leqslant C \tag{10}$$

Let $\{u_i(t),\partial_t u_i(t)\}=S(t,\tau)z_i$, by (1), the difference $\bar{z}(t)=\{\bar{u}(t),\bar{u}_t(t)\}=S(t,\tau)z_1-S(t,\tau)z_2$ satisfies

$$\varepsilon\bar{u}_{tt}-\Delta\bar{u}_t-\Delta\bar{u}+\varphi(u_1)-\varphi(u_2)=0.$$

Multiplying above equality by $2\bar{u}_t$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{z}\|_{H_t}^2+2\|\nabla\bar{u}_t\|^2-\varepsilon'\|\bar{u}_t\|^2=-2\langle\varphi(u_1)-\varphi(u_2),\bar{u}_t\rangle.$$

Considering the uniform estimates (10) for the solutions and according to (6) we have

$$\begin{aligned}&-2\langle\varphi(u_1)-\varphi(u_2),\bar{u}_t\rangle\leqslant\\&C(1+\|u_1\|_{L^6}^4+\|u_2\|_{L^6}^4)\|\bar{u}\|_{L^6}\|\bar{u}_t\|_{L^6}\leqslant\\&C(1+\|\nabla u_1\|^4+\\&\|\nabla u_2\|^4)\|\nabla\bar{u}\|\,\|\nabla\bar{u}_t\|\leqslant\\&C\|\nabla\bar{u}\|^2+2\|\nabla\bar{u}_t\|^2.\end{aligned}$$

Then we can obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{z}(t)\|_{H_t}^2\leqslant C\|\bar{z}(t)\|_{H_t}^2.$$

Applying the Gronwall lemma on $[\tau,t]$, the proof is completed.

4 Dissipativity

In this section, we study the dissipative feature of the process $S(t,\tau)$ associated with (1).

Theorem 4.1 There exists $R_0>0$ such that the family $\mathfrak{B}=\{B_t(R_0)\}_{t\in\mathbf{R}}$ is a time-dependent absorbing set for $S(t,\tau)$. Moreover, we have

$$\sup_{z\in B_\tau(R_0)}\left[\|S(t,\tau)z\|_{H_t}+\int_\tau^\infty\|\nabla u_t(y)\|^2\mathrm{d}y\right]\leqslant M,\ \forall\tau\in\mathbf{R} \tag{11}$$

for some $M>R_0$.

In order to prove Theorem 4.1, we need the following lemma.

Lemma 4.2 Let $t\geq\tau$, for $z\in H_\tau$, let $S(t,\tau)z$ be the solution of (1). Then there exists an increasing positive function Q such that

$$\|S(t,\tau)z\|_{H_t} \leqslant Q(\|z\|_{H_\tau})e^{-\delta(t-\tau)} + C_1, \forall \tau \leqslant t.$$

Proof Multiplying (1) by u_t, we have

$$\frac{d}{dt}(\|\nabla u\|^2 + \varepsilon\|u_t\|^2 + 2\Phi(u) - 2\langle f,u\rangle) - \varepsilon'\|u_t\|^2 + 2\|\nabla u_t\|^2 = 0.$$

A further multiplication by δu (where $\delta > 0$ is small) yields

$$\frac{d}{dt}(\delta\|\nabla u\|^2 + 2\delta\varepsilon\langle u_t,u\rangle) + 2\delta\|\nabla u\|^2 - 2\delta\varepsilon\|u_t\|^2 + 2\delta\langle\varphi(u),u\rangle - 2\delta\langle f,u\rangle = 2\delta\varepsilon'\langle u_t,u\rangle,$$

and estimating

$$2\delta|\varepsilon' u_t,u| \leqslant 2\delta L\|u_t\|\ \|u\| \leqslant \frac{\lambda_1}{2}\|u_t\|^2 + \frac{2\delta^2L^2}{\lambda_1}\|u\|^2 \leqslant \frac{\lambda_1}{2}\|u_t\|^2 + \frac{2\delta^2L^2}{\lambda_1^2}\|\nabla u\|^2.$$

Introducing the functional

$$E = (1+\delta)\|\nabla u\|^2 + \varepsilon\|u_t\|^2 + 2\Phi(u) + 2\delta\varepsilon\langle u_t,u\rangle - 2\langle f,u\rangle$$

and noticing that $u_t \in L^2(\tau,t;H_0^1(\Omega))$, we use Poincaré inequality $\frac{\|\nabla u_t\|^2}{\|u_t\|^2} \geq \lambda_1$ together with (7) and (8), and we obtain

$$\frac{d}{dt}E + \delta E + \|\nabla u_t\|^2 + \Gamma \leqslant 2c_1\delta,$$

where

$$\Gamma = (\delta - \delta^2 - \frac{2\delta^2L^2}{\lambda_1^2})\|\nabla u\|^2 - \delta\nu\|u\|^2 + (\frac{\lambda_1}{2} - \varepsilon' - 3\delta\varepsilon)\|u_t\|^2 - 2\varepsilon\delta^2\langle u_t,u\rangle.$$

Thus, by setting δ small enough such that $\Gamma \geq 0$, we have

$$\frac{d}{dt}E + \delta E + \|\nabla u_t\|^2 \leqslant c_1\delta \tag{12}$$

Applying the Gronwall lemma, we have

$$E(t) \leqslant E(\tau)e^{-\delta(t-\tau)} + c_1.$$

Denote $E(t) = \|S(t,\tau)z\|_{H_t}^2$. Due to (7), (8) for $0 < \rho < 1, 0 < \rho$, it is apparent that

$$\rho E(t) - C \leqslant E(t) \leqslant \rho E(t) + C \tag{13}$$

Combining with (12) and (13), we prove Lemma 4.2.

Proof of Theorem 4.1 The proof is essentially established in Ref. [14], we only need to make a few minor changes for our problem. For convenience of the reader, here we restate it.

Let $R_0 = 1 + 2C_1$. An application of Lemma 4.2 for $z \in B_\tau(R)$ yields

$$\|S(t,\tau)z\|_{H_t} \leqslant Q(R)e^{-\delta(t-\tau)} + 2C_1 = R_0,$$

provided that $t - \tau \geq t_0$, where

$$t_0 = \max\{0, \frac{\log\frac{Q(R)}{1+C_1}}{\delta}\}.$$

This concludes the proof of the existence of the time-dependent absorbing set.

In order to prove (11), it is enough to integrate (12) with $\delta = 0$ on $[\tau,\infty)$.

5 Existence of time-dependent global attractor

The aim of this section is to prove the existence of the time-dependent global attractor for (1). To this end, by Theorem 2.8. we will verify the asymptotic compactness of the corresponding process. We will take advantage of the same techniques used in Ref. [7] also in Ref. [14] that consists in finding a suitable decomposition of the process in the sum of a decaying part and of a compact one.

In light of Refs. [7, 17], owing to the assumptions (4) and (5), we can decompose φ as follows. Set $= \varphi_0 + \varphi_1$, where $\varphi_0 \in C(\mathbf{R}), \varphi_1 \in C(\mathbf{R})$ satisfying

$$|\varphi_0(r)| \leqslant c(1+|r|^5), \forall r \in \mathbf{R} \tag{14}$$

$$\varphi_0(r)r \geq 0, \forall r \in \mathbf{R} \tag{15}$$

$$|\varphi_1(r)| \leqslant c(1+|r|^\gamma), \gamma < 5, \forall r \in \mathbf{R} \tag{16}$$

$$\liminf_{|r|\to\infty} \frac{\varphi_1(r)}{r} > -\lambda_1 \tag{17}$$

Let $\mathfrak{B} = \{B_t(R_0)\}_{t\in\mathbf{R}}$ be a time-dependent absorbing set and $\tau \in \mathbf{R}$ be fixed. Then, for any $\tau \in B_t(R_0)$, we decompose the solution $S(t,\tau)z$ into the sum

$$S(t,\tau)z = \{u(t),u_t(t)\} = S_1(t,\tau)z + S_2(t,\tau)z,$$

where $S_1(t,\tau)z = \{v(t),v_t(t)\}$ and $S_2(t,\tau)z = \{w(t),w_t(t)\}$ are the solutions to the problem

$$\begin{cases}\varepsilon v_{tt} - \Delta u_t - \Delta u + \varphi_0(v) = 0, \\ S_1(\tau,\tau) = z\end{cases} \tag{18}$$

and

$$\begin{cases}\varepsilon w_{tt}-\Delta w_t-\Delta w+\varphi(u)-\varphi_0(v)=f,\\ S_2(\tau,\tau)=0\end{cases} \tag{19}$$

respectively.

In a first step, we show that $S_1(t,\tau)$ has an exponential decay in H_t.

Lemma 5.1 There exists $\delta=\delta(\mathfrak{B})>0$ such that

$$\|S_1(t,\tau)z\|_{H_t}\leqslant C\,\mathrm{e}^{-\delta(t-\tau)},\ \forall t\geq\tau.$$

Proof Repeating word by word the proof of Lemma 4.2. that applies to the present case with $S_1(t,\tau)$ in place of $S(t,\tau)$ (with the further simplification that $\varphi_1=0$ and $f=0$), we get the bound

$$\|S_1(t,\tau)z\|_{H_t}\leqslant C \tag{20}$$

Then we denote

$$E_1=(1+\delta)\|\nabla u\|^2+\varepsilon\|u_t\|^2+2\Phi_0(u)+2\delta\varepsilon\langle u_t,u\rangle,$$

where

$$\Phi_0(u)=\int_\Omega\Big(\int_0^{u(x)}\varphi_0(y)\mathrm{d}y\Big)\mathrm{d}x,$$

we multiply (18) by $2v_t+2\delta v$. In view of (15) and since $f=0$, similar to (12) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1+\delta E_1\leqslant 0.$$

Applying the Gronwall lemma and using subsequently the estimates for E_1 that similar to (13), the proof is completed.

Lemma 5.2 There exists $M=M(\mathfrak{B})>0$ such that $\sup_{t\geq\tau}\|S_2(t,\tau)z\|_{H_t^\sigma}\leqslant M$, where $\sigma=\min\{\frac{1}{4},\frac{5-\gamma}{2}\}$.

Proof The idea is inspired by Refs. [7,11,14]. Let

$$E_2(t)=\varepsilon\|w_t\|_\sigma^2+(1+\delta)\|w\|_{1+\sigma}^2+2\delta\varepsilon\langle w_t,A^\sigma w\rangle.$$

In view of (3), we can estimate

$$2\delta|\varepsilon\langle w_t,A^\sigma w\rangle|\leqslant\frac{\varepsilon}{2}\|w_t\|_\sigma+C\delta\|w\|_\sigma \tag{21}$$

Choose $\delta>0$ small enough and $C_1>0$ large enough, then we have

$$\frac{1}{2}\|S_2(t,\tau)z\|_{H_t^\sigma}^2\leqslant E_2(t)\leqslant 2\|S_2(t,\tau)z\|_{H_t^\sigma}^2+C_1 \tag{22}$$

Multiplying (19) with $2A_\sigma w_t+2\delta A_\sigma w$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t)+(-\varepsilon'-2\delta\varepsilon)\|w_t\|_\sigma^2+2\|w_t\|_{\sigma+1}^2+2\delta\|w\|_{\sigma+1}^2+2\langle\varphi(u)-\varphi_0(v),A^\sigma w\rangle=2\delta\varepsilon'\langle w_t,A^\sigma w\rangle-\langle\varphi(u)-\varphi_0(v),2A^\sigma w_t\rangle+\langle f,2A^\sigma w_t+2\delta A^\sigma w\rangle \tag{23}$$

In the following, we will deal with some terms in (23) one by one. Due to Lemma 4.2 and Lemma 5.1, we can obtain

$$\|\nabla u\|+\|\nabla v\|\leqslant C \tag{24}$$

Due to the decomposition of φ, we can write

$$\varphi(u)-\varphi_0(v)=\varphi(u)-\varphi(v)+\varphi_1(v).$$

Combining with (24) and by virtue of (6) we get

$$-\langle\varphi(u)-\varphi(v),2A^\sigma w_t\rangle\leqslant C(1+\|u\|_{L^6}^4+\|v\|_{L^6}^4)\|w\|_{L^{\frac{6}{1-2\sigma}}}\|A^\sigma w_t\|_{L^{\frac{6}{1+2\sigma}}}\leqslant C(1+\|\nabla u\|^4+\|\nabla v\|^4)\|A^{\frac{1+\sigma}{2}}w\|\|A^{\frac{1+\sigma}{2}}w_t\|\leqslant C\|w\|_{\sigma+1}\|w_t\|_{\sigma+1}\leqslant C\|w\|_{\sigma+1}^2+\frac{1}{3}\|w_t\|_{\sigma+1}^2 \tag{25}$$

Similar to (25), we have

$$-\langle\varphi(u)-\varphi(v),2\delta A^\sigma w\rangle\leqslant C\delta(1+\|u\|_{L^6}^4+\|v\|_{L^6}^4)\|w\|_{L^{\frac{6}{1-2\sigma}}}\|A^\sigma w\|_{L^{\frac{6}{1+2\sigma}}}\leqslant C\delta\|w\|_{\sigma+1}^2 \tag{26}$$

where $C=C(c_1)$. Moreover, since $\frac{\gamma}{5-2\sigma}\leqslant 1$, by (16) we deduce that

$$-\langle\varphi_1(v),2A^\sigma w_t\rangle\leqslant C(1+\|v\|_{L^{\frac{6\gamma}{5-2\sigma}}}^\gamma)\|A^\sigma w_t\|_{L^{\frac{6}{1+2\sigma}}}\leqslant C(1+\|\nabla v\|^\gamma)\|w_t\|_{1+\sigma}\leqslant C+\frac{1}{3}\|w_t\|_{1+\sigma}^2 \tag{27}$$

Similar to (27), we have

$$-\langle\varphi_1(v),2\delta A^\sigma w\rangle\leqslant C+C\delta\|w\|_{1+\sigma}^2 \tag{28}$$

At the same time,

$$2\delta\varepsilon'\langle w_t,A^\sigma w\rangle\leqslant 2\delta L\|w_t\|_\sigma\|w\|_\sigma\leqslant 2C\|w_t\|_{\sigma+1}+C\delta\|w\|_{\sigma+1} \tag{29}$$

Finally, using the interpolation inequality, similar to that in Ref. [11], we have

$$\langle f,2A^\sigma w_t+2\delta A^\sigma w\rangle\leqslant C+\delta\|w_t\|_{\sigma+1}+\delta^3\|w\|_{\sigma+1} \tag{30}$$

Combining (25)~(30), choose δ small enough and proper C, we deduce from (23) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t)+\delta E_2(t)\leqslant C$$

and the Gronwall lemma entails

$$E_2(t)\leqslant C(\tau)e^{-\delta(t-\tau)}+C.$$

In turn, (22) yields the boundedness of $S_2(t,\tau)z$ in H_t^σ.

Lemma 5.3 Under the conditions (2)～(5), the process $S(t,\tau):H_\tau\to H_t$ generated by problem (1) is asymptotically compact.

Proof Consider the family $M=\{M_t\}_{t\in\mathbf{R}}$, where

$$M_t=\{z\in H_t^\sigma\leqslant M\}.$$

By the compact embedding $H_t^\sigma\to H_t$, M_t is compact. Since the injection constants are independent of t, M is uniformly bounded. Collecting Theorem 4.1, Lemma 5.1 and Lemma 5.2, we can obtain that M is pullback attracting, that is

$$\delta_t(S(t,\tau)B_\tau(R_0),M_t)\leqslant C\,e^{-\delta(t-\tau)},\ \forall\, t\geq\tau.$$

Thus the process $S(t,\tau)$ is asymptotically compact.

Theorem 5.4 Under the conditions (2)～(5), the process $S(t,\tau):H_\tau\to H_t$ generated by problem (1) has a invariant time-dependent global attractor $\mathfrak{A}=\{A_t\}_{t\in\mathbf{R}}$.

Proof From Lemma 5.3, by Theorem 2.8, there exists a unique time-dependent global attractor $\mathfrak{A}=\{M_t\}_{t\in\mathbf{R}}$. Furthermore, due to the strong continuity of the process stated in Theorem 4.1, we can obtain that $\mathfrak{A}$ is invariant by Theorem 2.8.

References:

[1] Pata V, Zelik S. Smooth attractors for strongly damped wave equations [J]. Nonlinearity, 2006, 19: 1495.

[2] Carvalho A N, Cholewa J W. Local well posedness for strongly damped wave equations with critical nonlinearities [J]. B Aust Math Soc, 2002, 66: 443.

[3] Carvalho A N, Cholewa J W. Attractors for strongly damped wave equations with critical nonlinearities [J]. Pac J Math, 2002, 207: 287.

[4] Chueshov I, Lasiecka I. Long-time behavior of second order evolution equations with nonlinear damping [J]. Mem Am Math Soc, 2008, 195: 1.

[5] Khanmamedov A K. Global attractors for strongly damped wave equations with displacement dependent damping and nonlinear source term of critical exponent [J]. Discrete Cont Dyn-A, 2011, 31: 119.

[6] Kalantarov V, Zelik S. Finite-dimensional attractors for the quasi-linear strongly-damped wave equation [J]. J Differ Equations, 2009, 247: 1120.

[7] Pata V, Squassina M, On the strongly damped wave equation [J]. Commun Math Phys, 2005, 253: 511.

[8] Yang M H, Sun C Y. Attractors for strongly damped wave equations [J]. Nonlinear Anal-Real, 2009, 10: 1097.

[9] Yang M H, Sun C Y. Dynamics of strongly damped wave equations in locally uniform spaces: Attractors and asymptotic regularity [J]. T Amer Math Soc, 2009, 361: 1069.

[10] Yang M H, Sun C Y. Exponential attractors for the strongly damped wave equations [J]. Nonlinear Anal-Real, 2010, 11: 913.

[11] Sun C Y, Cao D M, Duan J Q. Non-autonomous wave dynamics with memory asymptotic regularity and uniform attractor [J]. Discrete Cont Dyn-B, 2008, 9: 743.

[12] Caraballo T, Carvalho A N, Langa J A. Felipe Rivero, A non-autonomous strongly damped wave equation: Existence and continuity of the pullback attractor [J]. Nonlinear Anal-Theor, 2011, 74: 2272.

[13] Plinio F Di, Duane G S, Temam R. Time dependent attractor for the oscillon equation [J]. Discrete Cont Dyn-A, 2011, 29: 141.

[14] Conti M, Pata V, Temam R. Attractors for processes on time-dependent space, application to wave equation [J]. J Differ Equ, 2013, 255: 1254.

[15] Meng F J, Yang M H, Zhong C K. Attractor for wave equations with nonlinear damping on time-dependent space [J]. Discrete Cont Dyn-B, 2016, 1: 205.

[16] Liu T T, Ma Q Z. Time dependent global attractor of nonlinear evolution equation with nonlinear damping [J]. J Sichuan Univ: Nat Sci Ed, 2017, 5: 917.

[17] Arrieta J, Carvalho A N, Hale J K. A damped hyperbolic equation with critical exponent [J]. Commun Part Diff Eq, 1992, 17: 841.

引用本文格式：

中　文：孟凤娟，曹凤雪. 具临界指数的强阻尼波方程的时间依赖全局吸引子[J]. 四川大学学报：自然科学版，2018，55：1155.

英　文：Meng F J, Cao F X. Time-dependent global attractor for strongly damped wave equation with critical exponent [J]. J Sichuan Univ: Nat Sci Ed, 2018, 55: 1155.