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一类非线性分数阶微分方程边值问题正解的存在性

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摘　要：本文运用 Schauder 不动点定理和 Krasnoselskii's 不动点定理获得了非线性分数阶微分方程边值问题 ${}^{C}D_{0^+}^{\alpha}u(t)=f(t,u(t),u'(t),u''(t))$，$t\in(0,1)$，$u'(0)+u''(0)=0$，$u'(1)+u''(1)=0$，$u(0)=0$ 正解的存在性，其中 $2<\alpha\leqslant 3$，${}^{C}D_{0^+}^{\alpha}$ 是 Caputo 分数阶导数.

关键词：正解；存在性；Schauder 不动点定理；Krasnoselskii's 不动点定理
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Existence of positive solutions for a class of nonlinear fractional differential equations with boundary values

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Abstract: In this paper, by using the Schauder fixed point theorem and Krasnoselskii's fixed point theorem, the existence of positive solutions for the boundary value problem of the nonlinear fractional differential equation ${}^{C}D_{0^+}^{\alpha}u(t)=f(t,u(t),u'(t),u''(t))$, $t\in(0,1)$, $u'(0)+u''(0)=0$, $u'(1)+u''(1)=0$, $u(0)=0$ *is obtained*, where $2<\alpha\leqslant 3$, ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative.

Keywords: Positive solution; Existence; Schauder fixed point theorem; Krasnoselskii's fixed point theorem
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1　引　言

分数阶微分方程出现在很多领域中，如生物物理，地球物理，血液动物学，空气动力学等.目前，对于非线性项不含一阶导数和二阶导数的分数阶边值问题正解的存在性已有很多结果[1-14].例如，梁等[12]运用序集上的不动点定理获得了分数阶三点边值问题${}^{C}D_{0^+}^{\alpha}u(t)+f(t,u(t))=0$，$t\in(0,1)$，$u(0)=u'(0)=u''(0)=0$，$u''(1)=\beta u''(\eta)$解的存在性，其中 $3<\alpha\leqslant 4$，${}^{C}D_{0^+}^{\alpha}$ 是 Riemann-Liouville 分数阶导数.张[13]运用锥上的不动点定理研究了分数阶微分方程边值问题${}^{C}D_{0^+}^{\alpha}u(t)=f(t,u(t))$，$t\in(0,1)$，$u(0)+u'(0)=0$，$u(1)+u'(1)=0$ 正解的存在性与多重性，其中常数 $1<\alpha\leqslant 2$，${}^{C}D_{0^+}^{\alpha}$ 是 Caputo 分数阶导数，且 $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ 连续.赵[6]运用迭代法研究了分数阶四点边值问题${}^{C}D_{0^+}^{\alpha}u(t)+f(t,u(t))=0$，$t\in(0,1)$，$u'(0)-\beta u(\xi)=0$，$u'(1)+\gamma u(\eta)=0$ 正解的存在性与多重性，其中 $1<\alpha\leqslant 2$，$0\leqslant\xi\leqslant\eta\leqslant 1$，$0\leqslant\beta,\gamma\leqslant 1$.

受以上结果的启发，本文用 Schauder 不动点定理和 Krasnoselskii's 不动点定理研究了分数阶边值问题

$${}^{C}D_{0^+}^{\alpha}u(t)=f(t,u(t),u'(t),u''(t)),t\in(0,1),$$

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$$u'(0)+u''(0)=0,u'(1)+u''(1)=0,u(0)=0 \tag{1}$$

正解的存在性. 其中 $2<\alpha\leqslant 3$. 与已有文献相比，本文所用方法不同且非线性项 f 含一阶导数项 u' 和二阶导数项 u''.

2　预备知识

记 $X=C^2[0,1]$，其中 $\|u\|=[\|u\|_\infty^2+\|u'\|_\infty^2+\|u''\|_\infty^2]^{\frac{1}{2}}$，$\|u\|_\infty=\max\limits_{0\leqslant t\leqslant 1}|u(t)|$. 记锥

$$P=\{u\mid u\in X,u(t)\geqslant 0\}.$$

定义 2.1[15]　连续函数 $u:(0,+\infty)\to\mathbf{R}$ 的 $\alpha>0$ 阶 Riemann-Liouville 分数阶积分定义为：

$$I_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\mathrm{d}s.$$

引理 2.2[15]　设 $\alpha>0$. 则 ${}^{\mathrm{C}}D_{0^+}^{\alpha}u(t)=0$ 有解

$$u(t)=c_0+c_1t+\cdots+c_nt^{n-1},c_i\in\mathbf{R},$$
$$i=0,1,2,\cdots,n,n=[\alpha]+1.$$

引理 2.3[15]　设 $\alpha>0$. 则 ${}^{\mathrm{C}}D_{0^+}^{\alpha}u(t)=y(t)$ 有解

$$u(t)=y(t)+c_0+c_1t+\cdots+c_nt^{n-1},$$
$$c_i\in\mathbf{R},i=0,1,2,\cdots,n,n=[\alpha]+1.$$

引理 2.4[2]　令 $g(t)\in C[0,1]$. 则边值问题

$$\begin{cases}{}^{\mathrm{C}}D_{0^+}^{\alpha}u(t)=g(t),t\in(0,1),\\ u'(0)+u''(0)=0,u'(1)+u''(1)=0,u(0)=0\end{cases} \tag{2}$$

有解

$$u(t)=\int_0^1G(t,s)g(s)\mathrm{d}s,$$

其中

$$G(t,s)=\begin{cases}\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-2}(t-\frac{t^2}{2})}{\Gamma(\alpha-1)}+\dfrac{(1-s)^{\alpha-3}(t-\frac{t^2}{2})}{\Gamma(\alpha-2)},0\leqslant s\leqslant t\leqslant 1,\\ \dfrac{(1-s)^{\alpha-2}(t-\frac{t^2}{2})}{\Gamma(\alpha-1)}+\dfrac{(1-s)^{\alpha-3}(t-\frac{t^2}{2})}{\Gamma(\alpha-2)},0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

由引理 2.4，边值问题(1)有解

$$u(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),u'(s),u''(s))\mathrm{d}s+$$
$$\int_0^1\left[\frac{(1-s)^{\alpha-2}(t-\frac{t^2}{2})}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}(t-\frac{t^2}{2})}{\Gamma(\alpha-2)}\right]f(s,u(s),u'(s),u''(s))\mathrm{d}s.$$

定义算子 $T:X\to X$：

$$(Tu)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),u'(s),u''(s))\mathrm{d}s+$$
$$\int_0^1\left[\frac{(1-s)^{\alpha-2}(t-\frac{t^2}{2})}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}(t-\frac{t^2}{2})}{\Gamma(\alpha-2)}\right]f(s,u(s),u'(s),u''(s))\mathrm{d}s.$$

引理 2.5[2]　$G(t,s)$ 满足下列条件：

(i)　$G(t,s)\in C((0,1]\times[0,1))$ 且对任意的 $t,s\in(0,1]\times[0,1)$，都有 $G(t,s)>0$；

(ii) 存在非负函数 $\gamma(s)\in C[0,1)$，使得

$$\min_{\frac{1}{3}\leqslant t\leqslant\frac{2}{3}}G(t,s)\geqslant\gamma(s)M(s)$$
$$\max_{0\leqslant t\leqslant 1}G(t,s)\leqslant M(s),$$
$$M(s)=\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)},$$

$$\gamma(s)=\frac{5(\alpha-1)(1-s)+5(\alpha-1)(\alpha-2)}{18(1-s)^{\alpha-2}+9(\alpha-1)(1-s)+9(\alpha-1)(\alpha-2)}.$$

设 X 是 Banach 空间，$P\subset X$ 是一个锥，$\alpha,\beta,\delta:X\to\mathbf{R}^+$ 的连续凸函数满足 $\alpha(\lambda u)=|\lambda|\alpha(u)$，$\beta(\lambda u)=|\lambda|\beta(u)$，$\delta(\lambda u)=|\lambda|\delta(u)$，$u\in X$，$\lambda\in\mathbf{R}$，且 $\|u\|\leqslant m\max\{\alpha(u),\beta(u)\}$，$u\in X$，$\alpha(u_1)\leqslant\alpha(u_2)$，

$u_1,u_2\in P$, $u_1\leqslant u_2$，其中 $m>0$ 是常数. 对任意 $u\in X$，定义泛函

$$\alpha(u)=\max_{0\leqslant t\leqslant 1}|u(t)|,\beta(u)=\max_{0\leqslant t\leqslant 1}|u'(t)|,\delta(u)=\max_{0\leqslant t\leqslant 1}|u''(t)|.$$

引理 2.6[1] 设 $r_2>r_1>0$, $H>L>0$ 为常数，且 $\Omega_i=\{u\in X:\alpha(u)<r_i,\beta(u)<L,\delta(u)<H\}$ 是有界开集，$i=1,2$. 设 $D_i=\{u\in X,\alpha(u)=r_i\}$，$T:P\to P$ 全连续且满足：

(C_1) $\alpha(Tu)<r_1,u\in D_1\cap P$；$\alpha(Tu)>r_2,u\in D_2\cap P$；

(C_2) $\beta(Tu)<L,u\in P$；$\delta(Tu)<H,u\in P$；

(C_3) 存在 $P\in(\overline{\Omega_2}\cap P)\backslash\{0\}$，使得 $\alpha(p)\neq 0$，且 $\alpha(u+\lambda p)\geqslant\alpha(u),u\in p,\lambda\geqslant 0$，

则 T 在 $(\Omega_2\backslash\overline{\Omega_1})\cap P$ 中至少有一个不动点.

3 主要结果

引理 3.1 设 $f\in C([0,1]\times[0,+\infty)\times\mathbf{R}^2\to[0,+\infty))$，则 $T:P\to P$ 是全连续的.

证明 由引理 2.5 及算子的定义知 $Tu\in X$ 且 $(Tu)(t)>0$. 因此，$T(P)\subset P$.

下证 T 连续. 因为 $f\in C([0,1]\times[0,+\infty)\times\mathbf{R}^2,[0,+\infty))$，且 $G(t,s)\in C((0,1]\times[0,1))$，则 T 连续.

最后证 T 是紧的. 令 $\Omega\subset P$ 是有界的，即对任意 $u\in\Omega$，存在实数 $r>0$，使得 $\|u\|\leqslant r$. 由范数的定义可知，$|u|\leqslant r,|u'|\leqslant r,|u''|\leqslant r$. 令

$$\eta=\max_{0\leqslant t\leqslant 1,0\leqslant u\leqslant r,0\leqslant u'\leqslant r,0\leqslant u''\leqslant r}|f(t,u,u',u'')|+1.$$

则

$$|(Tu)(t)|\leqslant\int_0^1 G(t,s)|f(s,u(s),u'(s),u''(s))|\mathrm{d}s\leqslant\eta\int_0^1 M(s)\mathrm{d}s\leqslant\frac{3\eta}{\Gamma(\alpha-1)},$$

$$\begin{aligned}|(Tu)'(t)|\leqslant&\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s),u''(s))|\mathrm{d}s+\\&\int_0^1\Big[\frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}(1-t)}{\Gamma(\alpha-2)}\Big]|f(s,u(s),u'(s),u''(s))|\mathrm{d}s\leqslant\\&\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\eta\mathrm{d}s+\int_0^1\Big[\frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}(1-t)}{\Gamma(\alpha-2)}\Big]\eta\mathrm{d}s\leqslant\frac{3\eta}{\Gamma(\alpha-1)},\end{aligned}$$

$$\begin{aligned}|(Tu)''(t)|\leqslant&\int_0^t\frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}|f(s,u(s),u'(s),u''(s))|\mathrm{d}s+\\&\int_0^1\Big[-\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\Big]|f(s,u(s),u'(s),u''(s))|\mathrm{d}s\leqslant\\&\int_0^t\frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}\eta\mathrm{d}s+\int_0^1\Big[-\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\Big]\eta\mathrm{d}s\leqslant\frac{3\eta}{\Gamma(\alpha-1)}.\end{aligned}$$

从而 $T(\Omega)$ 一致有界. 另一方面，对任意 $\varepsilon>0$，取 $\delta=\frac{\varepsilon\Gamma(\alpha-1)}{3\eta}$. 则对任意 $u\in T(\Omega)$，$t_1,t_2\in[0,1]$，当 $|t_2-t_1|<\delta$ 时有

$$\begin{aligned}|(Tu)(t_2)-(Tu)(t_1)|\leqslant&\int_0^{t_1}|G(t_2,s)-G(t_1,s)|f(s,u(s),u'(s),u''(s))\mathrm{d}s+\\&\int_{t_1}^{t_2}|G(t_2,s)-G(t_1,s)|f(s,u(s),u'(s),u''(s))\mathrm{d}s+\\&\int_{t_2}^{1}|G(t_2,s)-G(t_1,s)|f(s,u(s),u'(s),u''(s))\mathrm{d}s\leqslant\\&\int_0^{t_1}\Big[\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t_2-t_1)(1-\frac{t_2}{2}-\frac{t_1}{2})(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\end{aligned}$$

$$\frac{(t_2-t_1)(1-\frac{t_2}{2}-\frac{t_1}{2})(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}]\eta\mathrm{d}s+$$

$$\int_{t_1}^{t_2}[\frac{(t_2-t_1)(1-\frac{t_2}{2}-\frac{t_1}{2})(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}+$$

$$\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t_2-t_1)(1-\frac{t_2}{2}-\frac{t_1}{2})(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}]\eta\mathrm{d}s+$$

$$\int_{t_2}^{1}[\frac{(t_2-t_1)(1-\frac{t_2}{2}-\frac{t_1}{2})(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}+$$

$$\frac{(t_2-t_1)(1-\frac{t_2}{2}-\frac{t_1}{2})(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}]\eta\mathrm{d}s\leqslant$$

$$\left[\frac{t_2^\alpha-t_1^\alpha}{\Gamma(\alpha+1)}+\frac{2(t_2-t_1)}{\Gamma(\alpha)}\right]\eta\leqslant\frac{3(t_2-t_1)}{\Gamma(\alpha-1)}\eta,$$

$$|(Tu)'(t_2)-(Tu)'(t_1)|\leqslant\int_0^{t_1}[\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{(t_2-t_1)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}+$$

$$\frac{(t_2-t_1)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}]\eta\mathrm{d}s+\int_{t_1}^{t_2}[\frac{(t_2-t_1)(1-s)^{\alpha-2}+(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}+$$

$$\frac{(t_2-t_1)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}]\eta\mathrm{d}s+$$

$$\int_{t_2}^{1}[\frac{(t_2-t_1)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{(t_2-t_1)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}]\eta\mathrm{d}s\leqslant$$

$$\left[\frac{t_2^\alpha-t_1^\alpha}{\Gamma(\alpha+1)}+\frac{2(t_2-t_1)}{\Gamma(\alpha)}\right]\eta\leqslant\frac{3(t_2-t_1)}{\Gamma(\alpha-1)}\eta,$$

$$|(Tu)''(t_2)-(Tu)''(t_1)|\leqslant\frac{3(t_2-t_1)}{\Gamma(\alpha-1)}\eta.$$

因此 $T(\Omega)$等度连续. 由 Arzela-Ascoli 定理可知，$\overline{T(\Omega)}$是相对紧的. 于是 $T:P\to P$ 是全连续的. 证毕.

记

$$A=\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-2)},$$

$$\tau=\max_{0\leqslant t\leqslant 1}\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}a(s)\mathrm{d}s+$$

$$\int_0^1\left[\frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}(1-t)}{\Gamma(\alpha-2)}\right]a(s)\mathrm{d}s+$$

$$\int_0^1 a(s)\mathrm{d}s.$$

定理 3.2　设 $f\in C([0,1]\times[0,+\infty)\times\mathbf{R}^2,[0,+\infty))$，且存在非负函数 $a(t)\in C[0,1]$，使得

$$|f(t,u,v,w)\leqslant m_1|u|^{\sigma_1}+m_2|v|^{\sigma_2}+m_3|w|^{\sigma_3}+a(t),m_i>0,0<\sigma_i<1,i=1,2,3.$$

则边值问题(1)有一个正解.

证明　令

$$\overline{P_a}=\{u|u\in P,\ \|u\|\leqslant a\},$$

$$a\geqslant\max\{(4m_1A)^{\frac{1}{1-\sigma_1}},(4m_2A)^{\frac{1}{1-\sigma_2}},(4m_3A)^{\frac{1}{1-\sigma_3}},4\tau\}.$$

下证 $T:\overline{P_a}\to\overline{P_a}$. 若 $u\in\overline{P_a}$，有

$$0\leqslant u(t)\leqslant\max_{0\leqslant t\leqslant 1}|u(t)|\leqslant\|u\|\leqslant a,$$

$$0\leqslant u'(t)\leqslant\max_{0\leqslant t\leqslant 1}|u'(t)|\leqslant\|u'\|\leqslant a,$$

$$0\leqslant u''(t)\leqslant\max_{0\leqslant t\leqslant 1}|u''(t)|\leqslant\|u''\|\leqslant a.$$

则

$$f(t,u,v,w)\leqslant m_1|a|^{\sigma_1}+m_2|a|^{\sigma_2}+m_3|a|^{\sigma_3}+a(t),m_i>0,0<\sigma_i<1,i=1,2,3.$$

又因为

$$|(Tu)(t)| = \left|\int_0^1 G(t,s)f(s,u(s),u'(s),u''(s))\mathrm{d}s\right| \leqslant$$

$$(m_1|a|^{\sigma_1}+m_2|a|^{\sigma_2}+m_3|a|^{\sigma_3})\int_0^1 M(s)\mathrm{d}s+\int_0^1 a(s)M(s)\mathrm{d}s \leqslant$$

$$(m_1|a|^{\sigma_1}+m_2|a|^{\sigma_2}+m_3|a|^{\sigma_3})\left[\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-2)}\right]+\int_0^1 a(s)M(s)\mathrm{d}s,$$

$$|(Tu)'(t)| \leqslant \int_0^1 \left|\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right| f(s,u(s),u'(s),u''(s))\mathrm{d}s+$$

$$\int_0^1 \left|\frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}(1-t)}{\Gamma(\alpha-2)}\right| f(s,u(s),u'(s),u''(s))\mathrm{d}s \leqslant$$

$$\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}a(s)ds+\int_0^1\left[\frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}(1-t)}{\Gamma(\alpha-2)}\right]a(s)\mathrm{d}s+$$

$$(m_1|a|^{\sigma_1}+m_2|a|^{\sigma_2}+m_3|a|^{\sigma_3})\left[\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-2)}\right],$$

$$|(Tu)''(t)| \leqslant \int_0^t \left|\frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}\right| f(s,u(s),u'(s),u''(s))\mathrm{d}s+$$

$$\int_0^1 \left|-\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\right| f(s,u(s),u'(s),u''(s))\mathrm{d}s \leqslant$$

$$\int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}a(s)ds+\int_0^1\left[-\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\right]a(s)\mathrm{d}s+$$

$$(m_1|a|^{\sigma_1}+m_2|a|^{\sigma_2}+m_3|a|^{\sigma_3})\left[\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-2)}\right],$$

$$\|Tu\| = [\|Tu\|_\infty^2+\|(Tu)'\|_\infty^2+\|(Tu)''\|_\infty^2]^{\frac{1}{2}} \leqslant$$

$$\max_{0\leqslant t\leqslant 1}\|(Tu)(t)\|+\max_{0\leqslant t\leqslant 1}\|(Tu)'(t)\|+\max_{0\leqslant t\leqslant 1}\|(Tu)''(t)\| \leqslant$$

$$\tau+(m_1|a|^{\sigma_1}+m_2|a|^{\sigma_2}+m_3|a|^{\sigma_3})A \leqslant$$

$$\frac{a}{4}+\frac{a}{4}+\frac{a}{4}+\frac{a}{4} \leqslant a,$$

从而得 $T:\overline{P_a}\to\overline{P_a}$. 由引理 3.1,$T:\overline{P_a}\to\overline{P_a}$是全连续的. 因此,由 Schauder 不动点定理得边值问题(1)存在一个正解.

记

$$I=\int_0^1 M(s)\mathrm{d}s,$$

$$J=\int_{\frac{1}{3}}^{\frac{2}{3}}\gamma(s)M(s)\mathrm{d}s,$$

$$Q=\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-2)}.$$

注 1 由引理 2.5 知 $\gamma(s)$ 在$[0,1)$上递增. 从而 $\gamma(s)\geqslant\gamma(0)$. 令 $\gamma_0=\gamma(0)$. 若存在非负函数 $g(s)$ 使得 $u(t)=\int_0^1 G(t,s)g(s)\mathrm{d}s$,则

$$\min_{\frac{1}{3}\leqslant t\leqslant\frac{2}{3}} u(t) = \min_{\frac{1}{3}\leqslant t\leqslant\frac{2}{3}}\int_0^1 G(t,s)g(s)\mathrm{d}s \geqslant$$

$$\gamma_0\int_0^1 G(t,s)g(s)\mathrm{d}s=\gamma_0 u(t).$$

下面我们将应用引理 2.6 研究边值问题(1)正解的存在性. 本文假设:存在 $H>L>b>\gamma_0 b>c>0$, 使得 $f(t,u,v,w)$ 满足

(H1) $f(t,u,v,w)<\dfrac{c}{I},(t,u,v,w)\in[0,1]\times[0,c]\times[-L,L]\times[-H,H]$;

(H2) $f(t,u,v,w)\geqslant\dfrac{b}{J},(t,u,v,w)\in[0,1]\times[\gamma_0 b,b]\times[-L,L]\times[-H,H]$;

(H3) $f(t,u,v,w)<\dfrac{L}{Q},(t,u,v,w)\in[0,1]\times[0,b]\times[-L,L]\times[-H,H]$.

为了应用锥上的不动点定理,需作以下辅助函数:

$$\tilde{f}(t,u,v,w)=\begin{cases} f(t,u,v,w),(t,u,v,w)\in[0,1]\times[0,b]\times(-\infty,+\infty)\times(-\infty,+\infty), \\ f(t,b,v,w),(t,u,v,w)\in[0,1]\times[b,+\infty]\times(-\infty,+\infty)\times(-\infty,+\infty); \end{cases}$$

$$f^*(t,u,v,w)=\begin{cases}\tilde{f}(t,u,v,w),(t,u,v,w)\in[0,1]\times[0,+\infty]\times(-L,L)\times(-\infty,+\infty),\\ \tilde{f}(t,u,-L,w),(t,u,v,w)\in[0,1]\times[0,+\infty]\times(-\infty,-L)\times(-\infty,+\infty),\\ \tilde{f}(t,u,L,w),(t,u,v,w)\in[0,1]\times[0,+\infty]\times(L,+\infty)\times(-\infty,+\infty);\end{cases}$$

$$\bar{f}(t,u,v,w)=\begin{cases}f^*(t,u,v,w),(t,u,v,w)\in[0,1]\times[0,+\infty]\times(-\infty,+\infty)\times[-H,H],\\ f^*(t,u,v,-H),(t,u,v,w)\in[0,1]\times[0,+\infty]\times(-\infty,+\infty)\times(-\infty,-H],\\ f^*(t,u,v,H),(t,u,v,w)\in[0,1]\times[0,+\infty]\times(-\infty,+\infty)\times(H,+\infty).\end{cases}$$

则对 $f\in C([0,1]\times[0,+\infty)\times\mathbf{R}^2\to[0,+\infty))$ 有

$$(\bar{T}u)(t)=\int_0^1 G(t,s)\bar{f}(s,u(s),u'(s),u''(s))ds.$$

定理 3.3 设 f 在 $[0,1]\times[0,+\infty)\times\mathbf{R}^2\to[0,+\infty)$ 连续且(H1)～(H3)成立. 则边值问题至少有一个正解 $u(t)$ 满足

$$c<\alpha(u)<b,\ |u'(t)|<L,\ |u''(t)|<H.$$

证明 令

$$\Omega_1=\{u\in X,\ |u(t)|<c,\ |u'(t)|<L,\ |u''(t)|<H\},$$

$$\Omega_2=\{u\in X,\ |u(t)|<b,\ |u'(t)|<L,\ |u''(t)|<H\},$$

且

$$D_1=\{u\in X,\alpha(u)=c\},$$

$$D_2=\{u\in X,\alpha(u)=b\}.$$

下面分别验证引理 2.6 中的(C1)～(C3)成立.

由(H1)及 $\alpha(u)=c,u\in D_1\cap P$ 有

$$\alpha(\bar{T}u)=\max_{0\leqslant t\leqslant 1}\left|\int_0^1 G(t,s)\bar{f}(s,u(s),u'(s),u''(s))\mathrm{d}s\right|\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)\frac{c}{I}\mathrm{d}s\leqslant\frac{c}{I}\int_0^1 G(t,s)\mathrm{d}s\leqslant c.$$

由(H2)，$\alpha(u)=b,u\in D_2\cap P$ 及注 1 有

$$\alpha(\bar{T}u)=\max_{0\leqslant t\leqslant 1}\left|\int_0^1 G(t,s)\bar{f}(s,u(s),u'(s),u''(s))\mathrm{d}s\right|> \max_{0\leqslant t\leqslant 1}\int_{\frac13}^{\frac23} G(t,s)\frac{b}{J}\mathrm{d}s>\frac{b}{J}\int_{\frac13}^{\frac23} G(t,s)\mathrm{d}s=b.$$

从而得(C1)成立.

由(H3)及 $u\in P\cap\Omega_2$，有

$$\beta(\bar{T}u)\leqslant\max_{0\leqslant t\leqslant 1}\int_0^t\left|\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right|\bar{f}(s,u(s),u'(s),u''(s))ds+\int_0^1\left|\frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}(1-t)}{\Gamma(\alpha-2)}\right|\bar{f}(s,u(s),u'(s),u''(s))\mathrm{d}s<\left[\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-1)}\right]\frac{L}{Q}=L.$$

$$\delta(\bar{T}u)\leqslant\max_{0\leqslant t\leqslant 1}\int_0^t\left|\frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}\right|\bar{f}(s,u(s),u'(s),u''(s))\mathrm{d}s-\int_0^1\left|\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\right|\bar{f}(s,u(s),u'(s),u''(s))\mathrm{d}s<\left[\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-1)}\right]\frac{H}{Q}=H.$$

从而得(C2)成立.

又由引理 3.1 知 $\bar{T}:P\to P$ 是全连续的. 则存在非负函数 $p\in(\Omega_2\cap P)\backslash\{0\}$，使得 $\alpha(u+\lambda p)\geqslant\alpha(u),u\in p,\lambda\geqslant 0$. 从而得(C3)成立.

于是，由引理 2.6 知，存在 $u\in(\Omega_2\backslash\overline{\Omega_1})\cap P$，使得 $u(t)=(\bar{T}u)(t)$. 因此，$u(t)$ 是边值问题(1)的一个正解，且满足：

$$c<\alpha(u)<b,\ |u'(t)|<L,\ |u''(t)|<H.$$

证毕.

参考文献：

[1] Yang X, Wei Z L. Existene of positive solutions for the boundary value problem of nonlinear fractional differential equations[J]. Commun Nonlinear Sci, 2012,17: 85.

[2] 岳俊瑞. 非线性分数阶微分方程边值问题的正解[D]. 兰州：兰州理工大学，2016.

[3] Ahmad B，Nieto J J. Sequential fractional differential equations with three-point boundary value problem [J]. Comp Math Appl，2012，64：3046.

[4] Ahmad B. Sharp estimates for the unique solution of two-point fractional-order boundary value problem [J]. Appl Math Lett，2017，65：77.

[5] Ai-Refai M，Ali-Hajji M. Monotone iterative sequences for nonlinear boundary value problem of fractional order [J]. Nonlinear Appl，2011，74：3531.

[6] Zhao X K，Ge W G. Positive solutions for fractional four-point boundary value problem [J]. Commun Nonlinear Sci，2011，16：3665.

[7] Abdulla A B，Al-Refai M. On the existece and uniqueness of solutions for a class of nonlinear fractional boundary value problem [J]. King Saud University-Sci，2016，28：103.

[8] Rehman M，Ali-Khan R，Asif N A. Three-point boundary value problem for nonlinear fractional differential equations [J]. Acta Math Sci，2011，31：1337.

[9] Bai Z B. Eigenvalue intervals for a class of fractional boundary value problem [J]. Comp Math Appl，2012，64：3253.

[10] Bai Z B，Sun W C. Existence and multiplicity of positive solutions for singular fractional boundary value problem [J]. Comp Appl Math，2012，63：1369.

[11] Zhao Y G，Sun S R，Han Z L. The existence of multiple positive soiutions for boundary value problem of nonlinear fractional differential equations [J]. Commun Nonlinear Sci，2011，16：2086.

[12] Liang S H，Zhang J H. Positive solutions for boundary value problem of nonlinear fractional differential equations [J]. Nonl-Theor Analinear，2009，71：5545.

[13] Zhang S Q. Positive solutions for boundary value problems of nonlinear fractional differential equatins [J]. Electron J Differ Eq，2006，36：1.

[14] Liang S H，Zhang J H. Existence and uniqueness of strictly nondecreasing and positive solutions for a fractional three-point boundary value problem [J]. Elec J Diff Eqns，2011，62：1333.

[15] Kilbas A A，Sriuastave H M，Trujillo J J. Theory and applications of fractional differentrial equations [M]. Amsterdarn：North-Holland，2006.

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