非线性弹性问题的增强混合有限元方法

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摘　要：为研究非线性弹性问题的数值解，本文介绍了一种具有强对称应力张量的全增强混合有限元方法. 该方法除了包括通常线弹性问题中的应力张量和位移外 还将应变张量作为辅助未知量. 通过引入 Galerkin 最小二乘项，本文得到了两层鞍点算子方程，并以此作为问题的弱方程. 为得到离散增强方程的适定性，本文以分片常量多项式来逼近应变张量，以分片线性多项式逼近应力张量和位移，得到了最优阶误差估计. 数值算例验证了方法的有效性.

关键词：非线性弹性问题；两层鞍点方程；增强混合有限元方法
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An augmented mixed finite element method for nonlinear elasticity problems

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Abstract: This paper introduces and analyzes a full augmented mixed finite element method for the nonlinear elasticity problems with strongly imposed symmetry stress tensor. The mixed method includes the strain tensor as an auxiliary unknown, which combines with the usual stress-displacement approach adopted in linear elasticity. By introducing different stability terms, we obtain an augmented mixed finite element variation formulation and a full augmented mixed finite element variation formulation. In order to obtain the well-posed of the two schemes, we adopt different finite element spaces to approximate the unknowns and derive the optimal error estimate. Finally, a numerical example is presented to confirm the theoretical analysis.

Keywords: Nonlinear elasticity problem; Two-fold saddle point formulation; Augmented-mixed finite element method
(2010 MSC 65L60)

1　Introduction

Stable mixed finite elements for linear elasticity, such as PEERS of order 0, also leads to well-posed Galerkin schemes for the nonlinear problem. In Ref. [1], the authors extend the result from Ref. [2] and show that the well-posedness of a partially augmented Galerkin scheme is ensured by any finite element subspace for the strain tensor together with the PEERS space of order $k>0$ for the remaining unknowns. However, the stress tentor in the two papers is asymmetrical, which leads to introduce a further unknown named rotation. So they both have four

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variables, that is, the strain tensor and the usual stress-displacement-rotation.

In this paper, we attempt to deal with strongly imposed symmetry strain in the nonlinear elasticity problem, which makes the unknowns less than that of Refs. [2] and [3] and lets the expression of equations much briefer. Moreover, we employ the $k-1$ degree piecewise polynomial finite element space to approximate the strain tensor t and the k degree piecewise polynomial finite element space to approximate the stress tensor σ and the displacement u. An advantage of this method is that these elements can keep the S space same accuracy with R-T spaces of order k for the stress tensor while substantially decrease the number of degrees of freedom under the same convergence degree. Finally, we apply classical results on nonlinear functional analysis to prove the well-posedness of the resulting continuous and utilize the usual $Ce'a$ estimate to derive the optimal order error estimation.

2 Problem statement

Here we introduce some notations and function spaces. Firstly, let $\mathbf{R}^{2\times2}$ be the space of square matrices of 2×2 order with real entries, $I:=(\delta_{ij})$ be the identity matrix of $\mathbf{R}^{2\times2}$, S be the space of symmetric tensors of $\mathbf{R}^{2\times2}$. In addition, given $\tau:=(\tau_{ij})$, $\zeta:=(\zeta_{ij})$, we use

$$\mathrm{tr}(\tau):=\sum_{i=1}^{2}\tau_{ii},\ \tau^{d}=\tau-\frac{1}{2}\mathrm{tr}(\tau)\mathrm{I},$$

$$\tau^{t}:=(\tau_{ji}),\ \tau:\zeta=\sum_{i,j=1}^{2}\tau_{ii}\zeta_{jj}$$

to denote the transpose, the trace, the deviator of a tensor τ and the tonsorial product between τ and ξ. Then, we use $H^m(\Omega)$ to denote the Hilbert spaces on Ω, $\|\cdot\|_m$ to be its norms. When $m=0$, we write $L^2(\Omega)$, $\|\cdot\|_0$ instead of $H^m(\Omega)$ and $\|\cdot\|_m$. We introduce the divergence spaces:

$$H(\mathrm{div};\Omega):=\{w\in L^2(\Omega):\mathrm{div}\,w\in \mathrm{L}^2(\Omega)\},$$

$$H(\mathrm{div};\Omega;S):=\{\tau \text{ in } H(\mathrm{div};\Omega),\tau \text{ in } S\}.$$

The Hilbert norms of $H(\mathrm{div};\Omega)$, and $H(\mathrm{div};\Omega;S)$ are denoted by $\|\cdot\|_{\mathrm{div}}$ and $\|\cdot\|_{H(\mathrm{div};\Omega;S)}$.

Let Ω be a bounded and simply connected polygonal domain with Lipschitz-continuous boundary $\Gamma=\partial\Omega$. Given a body force $f\in L^2(\Omega)$ and $g\in H^{\frac{1}{2}}(\Gamma)$, we try to find the displacemeng u and the symmetric stress tensor σ, such that

$$\begin{gathered}\sigma=\tilde{\lambda}(\|\varepsilon(u)^d\|)(\mathrm{div}\,u)\mathbb{I}+\tilde{\mu}(\|\varepsilon(u)^d\|)\varepsilon(u)\\ \mathrm{div}\,\sigma=-f \text{ in } \Omega,\ u=g \text{ on } \Gamma\end{gathered}\tag{1}$$

where $\tilde{\lambda},\tilde{\mu}:\mathbf{R}^{+}\to\mathbf{R}$ are the Lam$'$e non-linear function, $\varepsilon(u):=\frac{1}{2}(\nabla u+(\nabla u)^t)$ is the strain tensor of small deformations, $\|\cdot\|$ is the euclidean norm. Then, we set

$$\lambda(r):=\tilde{\lambda}(\|r^d\|),\mu(r):=\tilde{\mu}(\|r^d\|)$$

for all $r\in L^2(\Omega)$ and introduce the new unknown $t=\varepsilon(u)\in L^2(\Omega)$. Problem (1) adopts the equivalent form

$$\begin{cases}t=\varepsilon(u) & \text{in } \Omega,\\ \sigma=\lambda(t)\mathrm{tr}(t)\mathrm{I}+\mu(t)t & \text{in } \Omega,\\ \mathrm{div}\,\sigma=-f & \text{in } \Omega,\ u=g \quad \text{on } \Gamma\end{cases}\tag{2}$$

Next, by integrating by parts, we consider the following problem:

Find $(t,\sigma,u)\in L^2(\Omega)\times H(\mathrm{div};\Omega;S)\times H^1(\Omega)$, such that

$$\begin{cases}\int_\Omega\{\lambda(t)\mathrm{tr}(t)tr(s)+\mu(t)t:s\}-\int\sigma:s=0,\\ -\int t:\tau-\int u\cdot\mathrm{div}\tau=-\langle\tau\,v,g\rangle,\\ -\int_\Omega v\cdot\mathrm{div}\,\sigma=\int_\Omega f\cdot v\end{cases}\tag{3}$$

for all $(s,\tau,v)\in L^2(\Omega)\times H(\mathrm{div};\Omega;S)\times H^1(\Omega)$.

Finally, in order to analysis our main results, we define the Hilbert spaces $X_1:=L^2(\Omega)$, $M_1:=H(\mathrm{div};\Omega;S)$ and $M:=H^1(\Omega)$. Besides, we defin the nonlinear operator $A_1:X_1\to X'$, the linear operators $B_1:X_1\to M_1{}'$, $B:M_1\to M'$, the bounded linear functional $H\in X_1{}'$, $G\in M_1{}'$, and $F\in M'$, where $X_1{}'$, $M_1{}'$, M' is the dual of X_1, M_1, M, respectively. Given each r, $s\in X_1$, τ, $\zeta\in M_1$, $v\in M$, we define

$$[A_1(r),s]:=\int_\Omega\{\lambda(r)\mathrm{tr}(r)\mathrm{tr}(s)+\mu(r)r:s\},$$

$$[B(\zeta),v]:=-\int v\cdot\mathrm{div}\,\zeta,$$

$$[B_1(r),\tau]:=-\int_\Omega r:\tau,[H,s]:=0,$$

$$[G,\tau]:=-\langle\tau v,g\rangle_\Gamma,\ [F,v]:=\int_\Omega f\cdot v\tag{4}$$

3 The full augmented variational formulation

In this section we adopt the following three steps to derive a full augmented formulation and a discrete scheme.

In the first step, we introduce the constitutive law relating σ and τ multiplied by the stabilization parameters $\kappa_0, \kappa_1, \kappa_2, \kappa_3 > 0$, to be chosen later, and add

$$\kappa_0 \int_\Omega (\sigma - \{\lambda(t) tr(t) \mathrm{I} + \mu(t) t\}) : \tau = 0,$$

$$\kappa_1 \int_\Omega (\mathrm{div}\sigma + f) \cdot \mathrm{div}\tau = 0,$$

$$\kappa_2 \int_\Omega (\varepsilon(u) - t) : \varepsilon(v) = 0,$$

$$\kappa_3 \int_\Gamma u \cdot v = \kappa_3 \int_\Gamma g \cdot v$$

to the first equation of (3).

In the second step, we subtract the second equation from the first equation of (3).

In the third step, we add the third equation of (3) to the first equation again.

In this way, we arrive at the following fully augmented variational formulation:

Find $(t, \sigma, u) \in X := L^2(\Omega) \times H(\mathrm{div}; \Omega; S) \times H^1(\Omega)$, such that

$$[A(t, \sigma, u), (s, t, v)] = [F, (s, \tau, v)] \tag{5}$$

for all $(s, t, v) \in \mathrm{X}$, where the nonlinear operator $A: X \to X'$ and the functional $F \in X'$ is defined by

$$[A(t,\sigma,u),(s,t,v)] := [A_1(t), s] + [B_1(s), \sigma] - [B_1(t), \tau] + \kappa_0 \int_\Omega (\sigma - \{\lambda(t) tr(t) \mathrm{I} + \mu(t) t\}) : \tau + \kappa_3 \int_\Gamma u \cdot v + \kappa_2 \int_\Omega (\varepsilon(u) - t) : \varepsilon(v) + [B_1(\tau, v)] - [B_1(\tau, u)] - [B_1(\tau, u)] + \kappa_1 \int \mathrm{div}\sigma \cdot \mathrm{div}\tau$$

$$[F, (s, \tau, v)] := \int_\Gamma \tau\mu \cdot g - \int f \cdot \mathrm{div}\tau + \kappa_3 \int_\Gamma g \cdot v - \int f \cdot \mathrm{div}\tau \tag{6}$$

Our next goal is to show the unique solvability of the variational formulation (5). We first recall the following theorem.

Theorem 3.1[1] Let X be a Hilbert space and $A: X \to X'$ be a nonlinear operator. Assume that A is Lipschitz-continuous and strongly monotone on X, that is, there exist constants $\tilde{\gamma}, \tilde{\alpha} > 0$, such that

$$\| A(x) - A(y) \|_{X'} \leqslant \tilde{\gamma} \| x - y \|_X, \quad \forall x, y \in X,$$

$$[A(x) - A(y), x - y] \geqslant \tilde{\alpha} \| x - y \|_X^2, \quad \forall x, y \in X.$$

Then, given $F \in X'$, there exists a unique $x \in X$ such that $[A(x), y] = [F, y]$, $x \in X$. Further more, the following estimate holds

$$\| x \|_X \leqslant \frac{1}{\tilde{\alpha}} \| F \|_{X'}.$$

In order to apply Theorem 3.1 to the fully augmented formulation (5), we need to prove first the required properties for our nonlinear operator. We begin with the Lipschitz-continuity.

Lemma 3.2[4] Let A be the nonlinear operator defined by (6). Then there exists a constant $\tilde{\gamma} > 0$, depending on $\| B \|$, and the parameters $\kappa_i, i \in \{0, 1, 2, 3\}$, such that

$$\| A(t, \sigma, u) - A(s, \tau, v) \|_{X'} \leqslant \tilde{\gamma} \| (t, \sigma, u) - (s, \tau, v) \|_X$$

for all $(t, \sigma, u), (s, \tau, v) \in X$.

In turn, the strong monotonicity of A makes use of a slight extension of the second Korn inequality, which establishes the existence of a constant $c_1 > 0$ such that

$$\| \varepsilon(v) \|_{0,\Omega}^2 + \| v \|_{0,\Gamma}^2 \geqslant c_1 \| v \|_{1,\Omega}^2, \quad \forall v \in H^1(\Omega) \tag{7}$$

The proof of (7) follows from a direct application of the Peetre-Tartar Lemma[5].

Lemma 3.3 Let A be the nonlinear operator defined by (6), the parameter $\kappa_0 \in (0, \frac{2 \times \alpha_1}{\gamma_1^2})$, where γ_1 and α_1 are positive constants. In addition, assume κ_1, κ_2, and κ_3 are chosen such that $0 < \kappa_1$, $0 < \kappa_2 < 2\alpha$ and $0 < \kappa_3$, where α is a constant, $\alpha := \min\{\alpha_1 - \frac{\kappa_0 \gamma_1^2}{2}, \frac{\kappa_0}{2}\}$, and $\alpha_3 := c_1 \min\{\kappa_2, 2\kappa_3\}$. Then there exists a constant $\tilde{\alpha} > 0$, depending on α, c_1, κ_1, κ_2 and κ_3, such that

$$[A(t, \sigma, u) - A(s, \tau, v), (t, \sigma, u) - (s, \tau, v)] \geqslant \tilde{\alpha} \| (t, \sigma, u) - (s, \tau, v) \|_X^2$$

for all $(t, \sigma, u), (s, \tau, v) \in X$.

Proof Given $(t,\sigma,u),(s,\tau,v)\in X$, we observe, according to the definition of A and the fact that the terms involving B cancell out, that

$$A(t,\sigma,u)-A(s,\tau,v),(t,\sigma,u)-(s,\tau,v)]=[A(t,\sigma)-A(s,\tau),(t,\sigma)-(s,\tau)]+\|\operatorname{div}(\sigma-\tau)\|_{0,\Omega}^2+\kappa_2\int_\Omega(\varepsilon(u-v)-(t-s)):\varepsilon(u-v)+\kappa_3\|u-v\|_{0,\Gamma}^2.$$

Then, the Cauchy-Schwarz inequality and the basic estimate $ab\leqslant\frac{1}{2}(a^2+b^2)$ yields

$$A(t,\sigma,u)-A(s,\tau,v),(t,\sigma,u)-(s,\tau,v)]\geqslant\alpha\{\|t-s\|_{0,\Omega}^2+\|\sigma-\tau\|_{0,\Omega}^2\}+\kappa_1\|\operatorname{div}(\sigma-\tau)\|_{0,\Omega}+\frac{\kappa_2}{2}\|\varepsilon(u-v)\|_{0,\Omega}^2-\frac{\kappa_2}{2}\|t-s\|_{0,\Omega}^2+\kappa_3\|\varepsilon u-v\|_{0,\Gamma}^2.$$

By using the the Korn inequality (7)

$$A(t,\sigma,u)-A(s,\tau,v),(t,\sigma,u)-(s,\tau,v)]\geqslant(\alpha-\frac{\kappa_2}{2})\|t-s\|_{0,\Omega}^2+\alpha_2\|\sigma-\tau\|_{\mathrm{div},\Omega}^2+\frac{\kappa_2}{2}\|(u-v)\|_{0,\Omega}^2+\kappa_3\|u-v\|_{0,\Gamma}^2\geqslant(\alpha-\frac{\kappa_2}{2})\|t-s\|_{0,\Omega}^2+\alpha_2\|\sigma-\tau\|_{\mathrm{div},\Omega}^2+\alpha_3\|u-v\|_{1,\Omega}^2\geqslant\tilde{\alpha}\|(t,\sigma,u)-(s,\tau,v)\|_X^2,$$

where $\alpha_2:=\min\{\alpha,\kappa_1\}$ $\alpha_3:=\min\{\frac{\kappa_2}{2},\kappa_3\}$. The proof is ended by $\tilde{\alpha}:=\min\{(\alpha-\frac{\kappa_2}{2}),\alpha_2,\alpha_3\}$.

The well-posedness of the fully augmented formulation (5) can be established from Ref. [2].

Theorem 3. 4 Assume that the parameters $\kappa_0,\kappa_1,\kappa_2$ and κ_3 are chosen as in Lemma 3. 2. Then there exists a unique $(t,\sigma,u)\in X$ solution of (5). Moreover, there exists $C>0$, depending on $\tilde{\alpha}$, such that

$$\|((t,\sigma),u)\|_X\leqslant C\{\|f\|_{0,\Omega}+\|g\|_{\frac{1}{2},\Gamma}\}.$$

Proof By Lemmas 3. 2 and Lemma 3. 3, the proof is a direct application of Theorem 3. 1.

Let $X_{1,h},M_{1,h}$, and M_h be finite dimensional subspaces of $L^2(\Omega),H(\operatorname{div};\Omega;S)$ and $H^1(\Omega)$. Define $X_h:=X_{1,h}\times M_{1,h}\times M_h$. We are interested in the following discrete scheme:

Find $(t_h,\sigma_h,u_h)\in X_h$, such that

$$[A(t_h,\sigma_h,u_h),(s,\tau,v)]=[F,(s,\tau,v)]\quad(8)$$

for all $(s,\tau,v)\in X_h$.

The following theorem establishe the well-posedness and convergence properties.

Theorem 3. 5 Assume that the parameters κ_0, κ_1, κ_2, and κ_3 are chosen as in Lemma 3. 2. Let $X_{1,h},M_{1,h}$ and M_h be arbitrary finite dimensional subspaces of $L^2(\Omega)$, $H(\operatorname{div};\Omega;S)$, and $H^1(\Omega)$, respectively. Then there exists a unique $(t_h,\sigma_h,u_h)\in X_h$, solution of (8). Moreover, there exist C_1, $C_2>0$, independent of h, such that

$$\|(t_h,\sigma_h,u_h)\|_X\leqslant C_1\{\|f\|_{0,\Omega}+\|g\|_{1/2,\Gamma}\}\quad(9)$$

$$\|(t,\sigma,u)-(t_h,\sigma_h,u_h)\|_X\leqslant C_2\inf_{(s_h,\tau_h,v_h)\in X_h}\|(t,\sigma,u)-(s_h,\tau_h,v_h)\|_X\quad(10)$$

Proof It is clear that the Lipschitz-continuity and strong monotonicity of A are certainly valid on $X_h\times M_h$, with the same constants $\tilde{\gamma}$ and $\tilde{\alpha}$, respectively. Therefore, the unique resolvability of (8) and the estimate (9) are again consequence of Theorem 3. 1. In turn, the *Ce′a* estimate (10) follows from standard arguments, similarly as for linear problems. We omit further details.

In order to develop the rate of convergence of the Galerkin solution provided by Theorem 3. 5, we need the approximation properties of the finite element subspace involved. Therefore, we define

$$X_{1,h}:=\{s_h\in L^2(\Omega):s_h|_T\in P_k(T)\ \forall T\in T_h\}\quad(11)$$

$$M_{1,h}:=\{\tau_h\in H(\operatorname{div};\Omega;S)\cap C(\Omega)^{2\times2}:\tau_h|_T\in \mathrm{P}_{k+1}(T)\quad\forall T\in T_h\}\quad(12)$$

$$M_h:=\{v_h\in H^1(\Omega):v_h|_T\in P_{k+1}(T)\ \forall T\in T_h\}\quad(13)$$

The following theorem provides the corresponding rate of convergence.

Theorem 3. 6 Assume that the parameters $\kappa_0,\kappa_1,\kappa_2$, and κ_3 are chosen as in Lemma 3. 2. Given an integer $k\geqslant0$, let $X_{1,h}$, $M_{1,h}$ and M_h be the finite element subspatces defined by (11) ~ (13), $(t,\sigma,u)\in X$ and $(t_h,\sigma_h,u_h)\in X_h$ be the unique solutions of the continuous and discrete formulations (5) and (8), respectively. Suppose that $t\in H^{\delta}(\Omega)$, $\sigma\in H^{\delta}(\Omega)$, and $u\in H^{\delta}(\Omega)$ for some $\delta\in(0,\kappa+1]$. Then there exists $C>0$, independent of h, such that

$$\|(t,\sigma,u)-(t_h,\sigma_h,u_h)\|_X\leqslant Ch^{\delta}\{\|t\|_{\delta,\Omega}+\|\sigma\|_{\delta,\Omega}+\|u\|_{\delta,\Omega}\}.$$

Proof It follows from the Ce′a estimate (10) in Ref. [3].

4 Numerical example

In this section we present a numerical example illustrating the performance of the Galerkin schemes (8). We consider $k=0$. Let N stand for the total number of degrees of freedom (unknowns), h and h' denote two consecutive meshsizes with corresponding error ε and ε'. The total errors are given by

$$\varepsilon(t)=\|t-t_h\|_{0,\Omega},\quad \varepsilon(\sigma)=\|\sigma-\sigma_h\|_{0,\Omega},$$
$$\varepsilon_0(u)=\|u-u_h\|_{0,\Omega},$$

In addition, we introduce the experimental rates of convergence

$$r(t):=\frac{\log(\varepsilon(t)/\varepsilon'(t))}{\log(h/h')},$$
$$r(\sigma):=\frac{\log(\varepsilon(\sigma)/\varepsilon'(\sigma))}{\log(h/h')},$$
$$r_0(u):=\frac{\log(\varepsilon_0(u)/\varepsilon_0'(u))}{\log(h/h')}.$$

In the example, we set $\Omega=]0,1[\times]0,1[$ and choose the data f and g, so that the exact solution is given by

$$u(x_1,x_2)=\begin{pmatrix}\sin x_1\cos x_2\exp(x_1x_2)\\ \cos x_1\sin x_2\exp(-x_1x_2)\end{pmatrix}$$

for all $(x_1,x_2)^t\in\Omega$.

In Tab. 1, we summarize the convergence history of the example. We observe that the $O(h)$ predicted by Theorems 3. 5 (with $\delta=1$) is obtained by all the unknowns.

Tab. 1 The convergence of the unknowns in the fully-augmented formalation

N	h	$e(t)$	$r(t)$	$e(\sigma)$	$r(\sigma)$	$e_0(u)$	$r_0(u)$
10 205	1/30	1.027 2e-001	1.001 4	2.063 8e-001	1.003 2	1.001 1e-003	1.001 9
258 29	1/48	6.417 2e-002	1.000 9	1.289 5e-001	1.000 7	3.903 1e-004	1.000 5
48 581	1/66	4.666 0e-002	1.000 7	9.377 9e-002	1.000 0	2.062 4e-004	1.000 2
78 461	1/84	3.665 7e-002	1.000 6	7.368 6e-002	0.999 9	1.272 4e-004	1.000 1
115 469	1/102	3.018 5e-002	1.000 5	6.068 5e-002	0.999 8	8.625 3e-005	1.000 1
159 605	1/120	2.565 5e-002	1.000 4	5.158 4e-002	0.999 8	6.229 6e-005	1.000 1
210 869	1/138	2.230 8e-002	1.000 4	4.485 7e-002	0.999 8	4.709 2e-005	1.000 0
269 261	1/156	1.973 3e-002	1.000 4	3.968 2e-002	1.000 0	3.684 3e-005	1.000 0

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