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Yoneda 完备度量空间范畴的完备性和余完备性

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摘　要：本文探讨了 Yoneda 完备度量空间范畴的完备性和余完备性，证明：若态射是 Yoneda 连续映射或 Yoneda 连续的非扩张映射，则该范畴是完备且余完备的；若态射是 Yoneda 连续的 Lipschitz 映射，则该范畴是有限完备和有限余完备的，但既不完备也不余完备. 本文还证明了以实数值连续格为对象，Yoneda 连续的右伴为态射的范畴是完备的.

关键词：度量空间；Yoneda 完备；完备范畴；余完备范畴
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Completeness and cocompleteness of categories of Yoneda complete metric space

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Abstract: This paper investigates the completeness and cocompleteness of some categories of Yoneda complete metric space. It is shown that if the morphisms are chosen to be Yoneda continuous maps or Yoneda continuous nonexpansive maps, then the category is both complete and cocomplete; if the morphisms are chosen to be Yoneda continuous Lipschitz maps, then the category is finitely complete and finitely cocomplete, but neither complete nor cocomplete. It is also shown that the category of real-valued continuous lattice and Yoneda continuous right adjoints is complete.

Keywords: Metric space; Yoneda complete; Complete category; Cocomplete category
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1 Introduction

By a generalized metric on a set X we mean a map $d:X\times X\to[0,\infty]$ such that $d(x,x)=0$ for all $x\in X$ and $d(x,y)+d(y,z)\geqslant d(x,z)$ for all $x,y,z\in X$. The pair (X,d) is called a generalized metric space. Such a space is also called a quasi-pseudo-metric[1] or a hemi-metric[2]. In 1973, Lawvere[3] observed that a generalized metric space is precisely a category enriched over the symmetric monoidal closed category $([0,\infty]^{\mathrm{op}},+,0)$. In this note, following Lawvere, such a space is called a metric space instead of a generalized metric space.

Metric spaces can be thought of as real-valued (precisely, $[0,\infty]$-valued) ordered sets. Yoneda complete metric spaces are then a metric analogy of directed complete partially ordered sets, so, they are the core subject of Quantitative Domain Theory[1-2,4]. Quantitative domain theory is a new branch of domain theory and has undergone active research in the past three decades. The

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field concerns both the semantics of programming languages and the mathematical field of topology. A basic result in domain theory is that the category dcpo of directed complete partially ordered sets is both complete and cocomplete[5]. It is natural to ask whether this is still true for Yoneda complete metric spaces. The fact that there exist different choices of morphisms for Yoneda complete metric spaces makes the problem more complicated. In this paper, we investigate the completeness and cocompleteness of the following categories:

• $[0,\infty]$-dcpo: the category of Yoneda complete and separated metric spaces and Yoneda continuous non-expansive maps;

• YcMet: the category of Yoneda complete and separated metric spaces and Yoneda continuous maps;

• LipYcMet: the category of Yoneda complete and separated metric spaces and Yoneda continuous Lipschitz maps;

• $[0,\infty]$-CL: the category of $[0,\infty]$-valued continuous lattices and Yoneda continuous right adjoints.

It is shown that the categories $[0,\infty]$-dcpo and YcMet are both complete and cocomplete; the category LipYcMet is finitely complete and cocomplete, but neither complete nor cocomplete; the category $[0,\infty]$-CL is complete. These results are helpful in the development of quantitative domain theory.

2 Preliminaries

A metric on a set X is a map $d: X\times X\to[0,\infty]$ such that $d(x,x)=0$ for all $x\in X$ and $d(x,y)+d(y,z)\geqslant d(x,z)$ for all $x,y,z\in X$. The pair (X, d) is called a metric space. In this note, in order to simplify notations, we write X for the pair (X, d) and write $X(x,y)$ for $d(x,y)$. A metric space X is separated if for all $x,y\in X$, $X(x,y)=X(y,x)=0$ implies that $x=y$.

Example 2.1 For all $r,s\in[0,\infty]$, let

$$d_L(r,s)=\max\{s-r,0\},\ d_R(r,s)=\max\{r-s,0\}.$$

Then both $([0,\infty],d_L)$ and $([0,\infty],d_R)$ are separated metric spaces.

Given a metricspace X, the underlying order of X is the order $\leqslant$ defined by $x\leqslant y$ if $X(x,y)=0$. We write X_0 for the set X equipped with the underlying order.

Let $f: X\to Y$ be a map between metric spaces. We say $f: X\to Y$ is non-expansive if $X(x,y)\geqslant Y(f(x),f(y))$ for all $x,y\in X$, $f: X\to Y$ is Lipschitz if there is some $c>0$ such that $cX(x,y)\geqslant Y(f(x),f(y))$ for all $x,y\in X$. It is clear that a non-expansive map is precisely a 1-Lipschitz map.

Let $f: X\to Y$ and $g: Y\to X$ be non-expansive maps. We say that f is left adjoint to g, or g is right adjoint to f [6], if $Y(f(x),y)=X(x,g(y))$ for all $x\in X$ and $y\in Y$.

The argument of Ref. [7, Proposition 3.1] gives a proof of the following useful conclusion.

Proposition 2.2 Let $f: X\to Y$ and $g: Y\to X$ be non-expansive maps. Then f is left adjoint to g if and only if, as order-preserving maps, $f: X_0\to Y_0$ is left adjoint to $g: Y_0\to X_0$.

Let X be a metric space. A weight (a.k.a. a left module)[3] of X is a function $\varphi: X\to[0,\infty]$ such that $\varphi(x)\leqslant\varphi(y)+X(x,y)$ for all $x,y\in X$. A coweight of X is a function $\psi: X\to[0,\infty]$ such that $\psi(y)\leqslant\psi(x)+X(x,y)$ for all $x,y\in X$.

The weights of X can be thought of as lower fuzzy sets when X is viewed as a real-valued ordered set. Dually, coweights can be thought as an upper fuzzy sets[8-9].

The set of all weights of a metric space X is denoted by PX. For any $\varphi,\psi\in PX$, let

$$PX(\varphi,\psi)=\sup_{x\in X} d_L(\varphi(x),\psi(x)).$$

Then PX becomes a metric space.

The set of all coweights is denoted by $P^\dagger X$. For any $\varphi,\psi\in P^\dagger X$, let

$$P^\dagger X(\varphi,\psi)=\sup_{x\in X} d_R(\varphi(x),\psi(x)).$$

Then $P^\dagger X$ becomes metric space.

Definition 2.3[4,10] Given a metric space X and a weight φ of X, a colimit of φ is an element $\mathrm{colim}\varphi\in X$ such that for all $x\in X$,

$$X(\mathrm{colim}\varphi,x)=PX(\varphi,X(-,x)).$$

Dually, by a limit of a coweight ψ of X we mean an element $\lim\psi \in X$ such that for all $x \in X$,

$$X(x, \lim\psi) = P^{\dagger}X(X(x, -), \psi).$$

A metric space X is said to be cocomplete if and only if each weight of X has a colimit. X is said to be complete if and only if each coweight of X has a limit. It is known that X is complete if and only if it is cocomplete and the underlying order of a complete metric space is complete[11].

Definition 2.4[4,12] A net $\{x_i\}_{i\in I}$ in a metric space X is forward Cauchy if

$$\inf_i \sup_{k\geqslant j\geqslant i} X(x_j, x_k) = 0.$$

An element $x \in X$ is a Yoneda limit of a forward Cauchy net $\{x_i\}_{i\in I}$ if for all $y \in X$,

$$X(x, y) = \inf_i \sup_{j\geqslant i} X(x_j, y).$$

A metric space is Yoneda complete if every forward Cauchy net has a Yoneda limit.

It is known that the underlying order of a Yoneda complete metric space is directed complete, see Ref. [13, Proposition 4.5].

A weight of a metric space X is said to be flat[14] if $\varphi = \inf_i \sup_{j\geqslant i} X(-, x_j)$ for some forward Cauchy net $(x_i)_{i\in I}$ in X. Each weight of the form $X(-, x)$ is a flat weight, since it is generated by a constant net. The set of all flat weights is denoted by FX.

Proposition 2.5[15] For each forward Cauchy net $\{x_i\}_{i\in I}$ in a metric space X, x is a Yoneda limit of $\{x_i\}_{i\in I}$ if and only if x is a colimit of $\varphi = \inf_i \sup_{j\geqslant i} X(-, x_j)$.

Corollary 2.6 A metric space X is Yoneda complete if and only if the map

$$y: X \to FX, x \mapsto X(-, x)$$

has a left adjoint, denoted by colim: $FX \to X$.

The map $y: X \to FX$ is known as the Yoneda embedding.

Example 2.7[2] Both of the metric spaces $([0, \infty], d_L)$ and $([0, \infty], d_R)$ in Example 2.1 are Yoneda complete.

Yoneda complete metric spaces are a metric version of directed complete partially ordered sets. In this note we are concerned with the completeness and cocompleteness of some categories of such spaces with different kinds of morphisms.

Definition 2.8 A map $f: X \to Y$ between metric spaces is Yoneda continuous if for each forward Cauchy net $\{x_i\}_{i\in I}$ in X and each Yoneda limit x of $\{x_i\}_{i\in I}$, $\{f(x_i)\}_{i\in I}$ is a forward Cauchy net in Y with $f(x)$ as a Yoneda limit.

It is trivial that each Lipschitz map $f: X \to Y$ maps a forward Cauchy net in X to a forward Cauchy net in Y.

3 Main results

Proposition 3.1 The category $[0, \infty]$-dcpo of Yoneda complete and separated metric spaces and Yoneda continuous non-expansive maps is complete.

Proof It suffices to check that $[0, \infty]$-dcpo has products and equalizers.

Let $(X_j)_{j\in J}$ be a family of Yoneda complete metric spaces. Equip $X = \prod_{j\in J} X_j$ (the Cartesian product of X_j) with the metric

$$X(\overline{x}, \overline{y}) = \sup_{j\in J} X_j(x_j, y_j).$$

Then X is Yoneda complete and Yoneda limits in X are computed componentwise by Ref. [2, Lemma 7.4.13]. It is clear that X is a product of $(X_j)_{j\in J}$ in $[0, \infty]$-dcpo, hence $[0, \infty]$-dcpo has products.

Given a parallel pair of morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $[0, \infty]$-dcpo, the subspace

$$E = \{x \in X \mid f(x) = g(x)\}$$

of X with the embedding map $i: E \to X$ is easily verified to be an equalizer of f and g. So $[0, \infty]$-dcpo has equalizers.

Proposition 3.2 The category $[0, \infty]$-dcpo is cocomplete.

Proof Since $[0, \infty]$-dcpo is complete, then by Ref. [16, Theorem 23.14], it is sufficient to show that $[0, \infty]$-dcpo is well-powered and has a coseparator.

First, we show that $[0, \infty]$-dcpo is well-powered. It suffices to show that every monomorphism $f: X \to Y$ in $[0, \infty]$-dcpo is injective. Suppose that $f(a) = f(b)$. Let $*$ be a singleton metric space and $g, h: * \to X$ be given by $g(*) = a$ and $h(*) = b$. Since $fg(*) = fh(*)$, it follows that $a=g(*)=h(*)=b$, which shows that f is injective.

Second, we show that $[0,\infty]$-dcpo has a coseparator. We show that the metric space $([0,\infty],d_R)$ is a coseparator. Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be two different morphisms in $[0,\infty]$-dcpo. Then $f(x)\neq g(x)$ for some $x\in X$.

Without loss of generality, we assume that $Y(f(x),g(x))\neq 0$. Define $h:Y\to[0,\infty]$ by $h(y)=Y(y,g(x))$. Then $hf(x)\neq hg(x)$. It remains to show that h is non-expansive and Yoneda continuous. For all $y_1,y_2\in Y$,

$$d_R(h(y_1),h(y_2))=$$
$$Y(y_1,g(x))\ominus Y(y_2,g(x))\leqslant Y(y_1,y_2),$$

hence h is non-expansive. Assume that $\{y_i\}_{i\in I}$ is a forward Cauchy net in Y with y as a Yoneda limit. Then $Y(y,z)=\inf\limits_i \inf\sup\limits_{j\geqslant i} Y(y_j,z)$ for all $z\in Y$, so

$$d_R(h(y),r)=Y(y,g(x))\ominus r=$$
$$\inf_i\sup_{j\geqslant i} Y(y_j,g(x))\ominus r=$$
$$\inf_i\sup_{j\geqslant i}(Y(y_j,g(x))\ominus r)=$$
$$\inf_i\sup_{j\geqslant i} d_R(h(y_j),r).$$

So h is Yoneda continuous.

***Remark* 1** The requirement that the metric spaces being separated is not essential in the above proposition. A very minor improvement of the argument shows that the category of Yoneda complete maps is both complete and cocomplete. This remark also applies to other conclusions in this note.

Proposition 3.3 The category YcMet of Yoneda complete and separated metric spaces and Yoneda continuous maps is complete and cocomplete.

The proof is similar to that of Proposition 3.1 and Proposition 3.2, so it is omitted here.

Proposition 3.4 The category LipYcMet of Yoneda complete and separated metric spaces and Yoneda continuous Lipschitz maps is both finitely complete and finitely cocomplete.

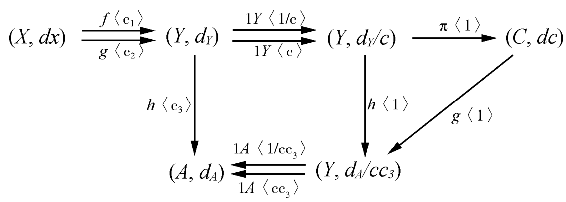
Proof Since the singleton metric space is a terminal object and the empty space is an initial object in LipYcMet, it suffices to check that LipYcMet has binary products, equalizers, binary coproducts and coequalizers. That LipYcMet has binary products and equalizers can be verified in a way similar to that of Proposition 3.1.

Let A, B be two Yoneda complete metric spaces. Equip $C=A\coprod B$ (the disjoint union of A and B) with the metric

$$C(x,y)=\begin{cases}A(x,y),\ x,y\in A,\\ B(x,y),\ x,y\in B,\\ \infty,\ \text{otherwise.}\end{cases}$$

Then, C is a Yoneda complete metric space by Ref. [2, Lemma 7.4.12]. It is easy to see that C is a coproduct of A and B.

Now, we show that LipYcMet has coequalizers. Let $(X,d_x)\overset{f}{\underset{g}{\rightrightarrows}}(Y,d_y)$ be two morphisms in LipYcMet. Assume that f is c_1-Lipschitz and g is c_2-Lipschitz. Let $c=\max\{c_1,c_2\}$. Consider the diagra2m



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where, for example, the symbol $f\langle c_1\rangle$ means f is a Lipschitz map with a Lipschitz constant c_1. Then f, g are both non-expansive and Yoneda continuous $(X,d_X)\to(Y,d_Y/c)$. By Proposition 3.2, there exists a Yoneda complete metric space (C,d_C) and a non-expansive Yoneda continuous map $\pi:(Y,d_Y/c)\to(C,d_C)$ such that C is a coequalizer of $(X,d_x)\overset{f}{\underset{g}{\rightrightarrows}}(Y,d_y/c)$ in $[0,\infty]$-dcpo. We claim that $\pi:(Y,d_Y)\to(C,d_C)$ is a coequalizer of f and g in LipYcMet. To see this, let (A,d_A) be a Yoneda complete metric space and h: $(Y,d_Y)\to(A,d_A)$ be a c_3-Lipschitz map.

Since $h:(Y,d_Y/c)\to(A,d_A/c\,c_3)$ is non-expansive and Yoneda continuous, there exists a unique non-expansive and Yoneda continuous map $s:(C,d_C)\to(A,d_A/c\,c_3)$ such that $h=s\pi$. It is trivial that $s:(C,d_C)\to(A,d_A)$ is the unique Lipschitz and Yoneda continuous map satisfying that $h=s\pi$.

Proposition 3.5 The category LipYcMet is neither complete nor cocomplete.

Proof Let $X_n=([0,\infty],d_L)$ for each $n\in\mathbf{N}$. We show that the family $(X_n)_{n\in\mathbf{N}}$ does not have a product in LipYcMet, hence LipYcMet is not complete.

Suppose on the contrary that $(X_n)_{n\in\mathbf{N}}$ has a product $(P\xrightarrow{f_n}X_n)_{n\in\mathbf{N}}$ with f_n being c_n-Lipschitz. Define a metric space $D=\{a,b\}$ by letting $D(a,b)=1$ and $D(b,a)=0$. Then the map $g_n:D\to X_n$, given by $g_n(a)=0$ and $g_n(b)=nc_n$, is nc_n-Lipschitz and Yoneda continuous. By the universal property of products, there is a unique Lipschitz and Yoneda continuous map $h:D\to P$ such that $g_n=f_nh$. Then $c_nP(h(a),h(b))\geqslant X_n(g_n(a),g_n(b))=nc_n$. Consequently, $P(h(a),h(b))=\infty$ contradicting that h is Lipschitz.

Next, we show that LipYcMet is not cocomplete. For each n, let X_n be the metric space D defined in the previous paragraph. We claim that $(X_n)_{n\in\mathbf{N}}$ does not have a coproduct. Suppose on the contrary that $(X_n\xrightarrow{f_n}C)_{n\in\mathbf{N}}$ is a coproduct with f_n being c_n-Lipschitz. For each n, define $g_n:X_n\to([0,\infty],d_L)$ by $g_n(a)=0, g_n(b)=nc_n$. Then, g_n is nc_n-Lipschitz and Yoneda continuous. By the universal property of coproduct, there is a unique Lipschitz and Yoneda continuous map h from C to $([0,\infty],d_L)$ such that $g_n=hf_n$. Assume that h is k-Lipschitz. Then

$$\begin{aligned}kc_n&=kc_nX_n(a,b)\geqslant kC(f_n(a),f_n(b))\geqslant\\&d_L(hf_n(a),hf_n(b))=\\&d_L(g_n(a),g_n(b))=nc_n,\end{aligned}$$

which shows that $k=\infty$, a contradiction.

Finally, we discuss the completeness of a subcategory of $[0,\infty]$-dcpo. This subcategory is a metric version of that of continuous lattices. Before introducing this subcategory, we need some preparation.

By Corollary 2.6, a metric space X is Yoneda complete if and only if the Yoneda embedding $y:X\to FX$ has a left adjoint.

Definition 3.6[17] A metric space X is said to be a $[0,\infty]$-domain (or real-valued domain) if it is Yoneda complete and is continuous in the sense that the left adjoint colim: $FX\to X$ of the Yoneda embedding $y:X\to FX$ has a left adjoint, which will be denoted by $\Downarrow:X\to FX$.

Proposition 3.7[18] Every retract of a $[0,\infty]$-domain in $[0,\infty]$-dcpo is a $[0,\infty]$-domain.

A separated and complete $[0,\infty]$-domain is said to be a real-valued continuous lattice (or, a $[0,\infty]$-continuous lattice). Write $[0,\infty]$-CL for the category having real-valued continuous lattices as objects and Yoneda continuous right adjoints as morphisms. It is clear that $[0,\infty]$-CL is the counterpart of the category CL[19] of continuous lattices in the metric setting.

Proposition3.8[20] Real-valued continuous lattices are exactly retracts of powers of the metric space $([0,\infty],d_L)$ in the category $[0,\infty]$-dcpo.

Proposition 3.9 The category $[0,\infty]$-CL is complete.

Proof It suffices to show that $[0,\infty]$-CL has equalizers and products. Let $(X_i)_{i\in I}$ be a family of real-valued continuous lattices. Since each X_i is a retract of some power of $([0,\infty],d_L)$ in the category $[0,\infty]$-dcpo, then so is the product $\prod_{i\in I}X_i$ of $(X_i)_{i\in I}$ in $[0,\infty]$-dcpo, hence $\prod_{i\in I}X_i$ is a $[0,\infty]$-valued continuous lattice.

For each $j\in I$, by help of Proposition 2.2, the projection $p_j:\prod_{i\in I}X_i\to X_j$ is easily verified to be a right adjoint. So $\prod_{i\in I}X_i$ is a product of $(X_i)_{i\in I}$ in $[0,\infty]$-CL.

Given a parallel pair of morphisms in $X\underset{g}{\overset{f}{\rightrightarrows}}Y$ $[0,\infty]$-CL, we claim that the subspace

$$E=\{x\in X \mid f(x)=g(x)\}$$

of X with the embedding map $i:E\to X$ is an equalizer of f,g. Since both f and g are right adjoints, they preserve limits[11]. So the subspace E is closed in X with respect to limits. Hence it is complete.

Since the embedding $i:E\to X$ preserves limits and E is complete, i has a left adjoint Ref. [11, Proposition 6. 8], say, $h:X\to E$. Then.

• i is Yoneda continuous, since both f and g are Yoneda continuous, hence E is closed with respect to Yoneda limits;

• h is Yoneda continuous, since every left adjoint is Yoneda continuous;

• $hi=id_E$, since $h:X_0\to E_0$ is left adjoint to $i:E_0\to X_0$, hence $h(x)$ is the meet of $\{e\in E \mid x\leqslant e\}$ in E_0, so $h\,i(e)=e$ for all $e\in E$.

Thus E is a retract of X in $[0,\infty]$-dcpo, hence a $[0,\infty]$-domain by Proposition 3. 7. Therefore E is an equalizer of f and g in $[0,\infty]$-CL.

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