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广义 Fock 空间之间的 Volterra 型积分算子与复合算子的乘积

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摘　要：本文利用 Berezin 变换等方法等价地刻画了从广义 Fock 空间 F_ϕ^p 到广义 Fock 空间 F_ϕ^q 的 Volterra 型积分算子与复合算子乘积 $V_{(g,\psi)}$ 的有界性，紧性及 Schatten-p 类性质，其中 $0<p,q<\infty$. 同时,本文还利用 Berezin 变换得到了这些算子本性范数的估计.
关键词：Volterra 型积分算子；复合算子；广义 Fock 空间；Berezin 变换
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Product of Volterra type integral operator and composition operators between generalized Fock spaces

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Abstract: In this paper, equivalent characterizations for the boundedness, compactness, and Schatten-p class properties of the product of a Volterra type integral operator and a composition operator between generalized Fock spaces F_ϕ^p and generalized Fock spaces F_ϕ^q are proposed in terms of certain Berezin integral transformations on the complex plane $\mathbf{C}$, where $0<p,q<\infty$. We also obtain some estimates on the essential norms of these operators.
Keywords: Volterra operator; Composition operator; Generalized Fock space; Berezin integral transformation
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1 Introduction

Let $\mathbf{C}$ be the complex plane and $\phi:[0,\infty)\to\mathbf{R}^+$ a twice continuously differentiable function. We extend ϕ to $\mathbf{C}$ by setting $\phi(z)=\phi(|z|)$, $z\in\mathbf{C}$ such that

$$c\omega_0\leqslant dd^c\phi\leqslant C\omega_0 \tag{1}$$

holds uniformly pointwise on $\mathbf{C}$ for some positive constants c and C (in the sense of positive (1, 1) forms), where $\omega_0=dd^c|\cdot|^2$ is the standard Euclidean Kähler form, $d^c=\frac{\mathrm{i}}{4}(\bar\partial-\partial)$ and d is the usual exterior derivative.

For $0<p<\infty$, the generalized Fock space F_ϕ^p consists of all entire functions f for which

$$\|f\|_{p,\phi}=\left(\int_{\mathbf{C}}|f|^p\mathrm{e}^{-p\phi}\mathrm{d}v\right)^{1/p}<\infty,$$

where $\mathrm{d}v$ is the Lebesgue measure on $\mathbf{C}$. It is

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clear that F_ϕ^p is a Banach space under the norm $\|\cdot\|_{p,\phi}$ if $1\leqslant p<\infty$. For $0<p<1$, F_ϕ^p is an F-space under $d(f,g)=\|f-g\|_{p,\phi}^p$. And also, F_ϕ^2 is a Hilbert space with Bergman kernel $K_{w,\phi}(z)$ and normalized kernel functions $k_{w,\phi}(z)=\frac{K_{w,\phi}(z)}{\sqrt{K_\phi(w,w)}}$. It's well known that, for example, the Fock-Sobolev space is a special case of the space F_ϕ^p with $\phi(z)=\frac{1}{2}|z|^2-m\log|z|$.

Note that we will write $A\lesssim B$ for two quantities A and B if there exists an unimportant constant C such that $A\leqslant CB$. Furthermore, $B\lesssim A$ is defined similarly and we will write $A\approx B$ if $A\lesssim B$ and $B\lesssim A$.

For $z\in\mathbf{C}$ and $r>0$, let $D(z,r)=\{w\in\mathbf{C}: |w-z|<r\}$. It follows from Schuster *et al*[1] that there exist positive constants θ and M, depending only on c, C such that for all $z,w\in\mathbf{C}$,

$$|K_\phi(z,w)|\mathrm{e}^{-\phi(z)}\mathrm{e}^{-\phi(w)}\leqslant M\mathrm{e}^{-\theta|z-w|}\leqslant M \tag{2}$$

and, in addition, there exists positive constant r_0 such that

$$|K_\phi(z,w)|\mathrm{e}^{-\phi(z)}\mathrm{e}^{-\phi(w)}\geqslant MK_\phi(z,z)\mathrm{e}^{-2\phi(z)}\geqslant M \tag{3}$$

for $z\in\mathbf{C}$ and $w\in D(z,r_0)$. With above results, we have

$$K_\phi(z,z)\approx\mathrm{e}^{2\phi(z)}, z\in\mathbf{C} \tag{4}$$

If H is a holomorphic function space on $\mathbf{C}$, we can define the Volterra type integral operator on H induced by a holomorphic symbol g as

$$V_gf(z)=\int_0^z f(w)g'(w)\mathrm{d}w.$$

Pommerenke[2] characterizes the boundedness, compactness, and other operator theoretic properties of V_g in terms of function theoretic conditions on g in 1977. There are a lot of interest following works about operator V_g, for instance, Aleman *et al*[3-4] on Hardy and Bergmann spaces. For more information, we refer to Refs. [5-7] and the references therein. The Volterra type integral operator V_g has an essential relationship with the multiplication operator $M_g(f)=gf$ by

$$M_g(f)=f(0)g(0)+V_g(f)+I_g(f),$$

where I_g is the Volterra companion integral operator given by

$$I_gf(z)=\int_0^z f'(w)g(w)\mathrm{d}w.$$

Let ψ be an entire function and $C_\psi f=f(\psi)$ be the composition operator on the space of analytic functions on $\mathbf{C}$ with symbol ψ. The induced product of Volterra type integral and composition operators is defined by

$$V_{(g,\psi)}f=\int_0^z f(\psi(w))g'(w)\mathrm{d}w.$$

If $\psi(z)=z$, then these operators are just the usual Volterra type integral operators V_g. As will be seen later, the study of $V_{(g,\psi)}$ reduces to studying the composition operator C_ψ when $|g'(z)/(1+|z|)|$ behaves like a constant for all z. Many operator experts have obtained rich results for this kind of operators, referred to Refs. [8-12]. The boundedness and compactness of weighted composition operators between different weighted Bergman spaces and different Hardy spaces expressed in terms of the generalized Berezin transform are characterized by Čučković *et al* [13-14]. The equivalent characterization of boundedness and compactness of composition operators on Bloch-Orlicz type spaces of the unit ball can be referred to Ref. [15]. Similar results were also obtained in Ref. [16] for the same operator acting on the classical Fock space F^2.

In this paper, we obtain some equivalent characterizations for the boundedness, compactness, and Schatten-p class properties of the product of Volterra type integral and composition operators between generalized Fock spaces in terms of certain Berezin transforms on the complex plane $\mathbf{C}$. By modifying all the results stated for $V_{(g,\psi)}$, one could also obtain similar results for the Volterra type composition operators

$$V_g\circ C_\psi f(z)=\int_0^z f(\psi(w))(g(\psi(w)))'\mathrm{d}w.$$

For $0<p<\infty$, if we choose the function ϕ such that

$$\int_0^\infty s\mathrm{e}^{-p\phi(s)}\mathrm{d}s<\infty,$$

the *p-distortion* function of ϕ is defined by

$$\varphi_{p,\phi}(r)=\frac{\int_r^{\infty} s\,\mathrm{e}^{-p\phi(s)}\mathrm{d}s}{(1+r)\mathrm{e}^{-p\phi(r)}},\ 0\leqslant r<\infty.$$

For $p>0$, the p-Berezin integral transform of g on generalized Fock space is defined to be

$$B_{(\psi,\phi)}(|g|^p)(w)=\int_{\mathbf{C}}\left|\frac{K_{w,\phi}(\psi(z))}{\sqrt{K_\phi(w,w)}\sqrt{K_\phi(z,z)}}\right|^p\cdot |g'(z)\varphi_{p,\phi}(|z|)|^p\mathrm{d}v(z).$$

2 Boundedness and compactness of $V_{(g,\psi)}$ on generalized Fock space

One of the main tools in proving our results is the following theorem which comes from the Corollary 11 in Constantin *et al*.'s paper[17].

A direct computation shows that

$$c\leqslant\Delta\phi(z)=\phi''(|z|)+\frac{\phi'(|z|)}{|z|}\leqslant C \tag{5}$$

for positive constants c, C. If the weight function satisfies

$$\lim_{r\to\infty}\sup\frac{\phi(r)}{r^2}=0,$$

then by L'Hospital's rule, we get

$$\lim_{r\to\infty}\sup\frac{\phi'(r)}{r}=0,\quad \lim_{r\to\infty}\sup\phi''(r)=0.$$

By the same way, if we choose that function such that

$$\lim_{r\to\infty}\inf\frac{\phi(r)}{r^2}=+\infty,$$

then

$$\lim_{r\to\infty}\inf\frac{\phi'(r)}{r}=+\infty,\quad \lim_{r\to\infty}\inf\phi''(r)=+\infty.$$

Obviously the above discussion contradicts the condition (5). So we can see that the weight function has a property that

$$\lim_{r\to+\infty}r\phi'(r)=+\infty.$$

Moreover, $\phi(z)$ does not grow faster than $|z|^2$ and decay more slowly than $|z|^2$ at infinity. At this time, it is easy to check that $\phi(r)$ satisfies the so called K_p-condition

$$\frac{\frac{\mathrm{d}}{\mathrm{d}r}(r\mathrm{e}^{-p\phi(r)})\int_r^{\infty}s\,\mathrm{e}^{-p\phi(s)}\mathrm{d}s}{r^2\ e^{-2p\phi(r)}}\leqslant K$$

for constant $K>0$. Then the following theorem holds according to Ref. [17].

Theorem 2.1 Assume that $0<p<\infty$ and ϕ is a function satisfying the K_p-condition, then

$$\int_{\mathbf{C}}|f(z)|^p\mathrm{e}^{-p\phi(z)}\mathrm{d}v(z)\approx |f(0)|^p+\int_C|f'(z)|^p\,|\varphi_{p,\phi}(|z|)|^p\mathrm{e}^{-p\phi(z)}\mathrm{d}v(z)$$

for any entire function f.

The following estimate is important to our main results.

Lemma 2.2 For each $p>0$, let $\mu_{(p,\phi)}$ be the positive pull-back measure on $\mathbf{C}$ defined by

$$\mu_{(p,\phi)}(E)=\int_{\psi^{-1}(E)}|g'(z)\varphi_{p,\phi}(|z|)|^p\mathrm{e}^{-p\phi(z)}\mathrm{d}v(z)$$

for every Borel subset E of $\mathbf{C}$. Then

$$\int_{D(w,1)}\mathrm{e}^{p\phi(z)}\mathrm{d}\mu_{(p,\phi)}(z)\lesssim B_{(\psi,\phi)}(|g|^p)(w),$$

where $D(w,1)$ is the disc with center w and radius 1.

Proof For each $z\in D(w,1)$, by (2), (3) and (4), we have

$$|k_{(w,\phi)}(z)|^p=\left|\frac{K_\phi(z,w)}{\sqrt{K_\phi(w,w)}}\right|^p\approx \left|\sqrt{K_\phi(z,z)}\right|^p\approx\mathrm{e}^{p\phi(z)}.$$

This shows that

$$\int_{D(w,1)}\mathrm{e}^{p\phi(z)}\mathrm{d}\mu_{(p,\phi)}(z)\lesssim \int_{D(w,1)}|k_{w,\phi}(z)|^p\mathrm{d}\mu_{(p,\phi)}(z)\leqslant \int_{\mathbf{C}}|k_{w,\phi}(z)|^p\mathrm{d}\mu_{(p,\phi)}(z).$$

The definition of the measure $\mu_{(p,\phi)}$ and the integral transform $B_{(\psi,\phi)}(|g|^p)$ give that

$$\int_{\mathbf{C}}|k_{(w,\phi)}(z)|^p\mathrm{d}\mu_{(p,\phi)}(z)\approx \int_{\mathbf{C}}|k_{(w,\phi)}(\psi(z))|^p\mathrm{e}^{-p\phi(z)} |g'(z)\varphi_{p,\phi}(|z|)|^p\mathrm{d}v(z)\approx B_{(\psi,\phi)}(|g|^p)(w).$$

The proof is finished.

Now we state our first main result.

Theorem 2.3 If $0<p\leqslant q<\infty$ and ψ is an entire function. Then

(i) The operator $V_{(g,\psi)}:F_\phi^p\to F_\phi^q$ is bounded if and only if $B_{(\psi,\phi)}(|g|^q)(w)\in L^\infty(\mathbf{C},\mathrm{d}v)$. Moreover

$$\|V_{(g,\psi)}\|\approx(\sup_{w\in\mathbf{C}}B_{(\psi,\phi)}(|g|^q)(w))^{1/q} \tag{6}$$

(ii) The operator $V_{(g,\psi)}: F_\phi^p \to F_\phi^q$ is compact if and only if

$$\lim_{|w|\to\infty} B_{(\psi,\phi)}(|g|^q)(w)=0.$$

Proof (i) Suppose that the operator $V_{(g,\psi)}$ is bounded. For $0<p<\infty$, we can get from (2), (3) and (4) that

$$\|k_{w,\phi}\|_{p,\phi}^p \approx \int_{\mathbf{C}} |K_\phi(z,w)|^p e^{-p\phi(z)-p\phi(w)} dv(z) \lesssim \int_{\mathbf{C}} e^{-p\theta|z-w|} dv(z) \lesssim 1.$$

On the other side,

$$\|k_{w,\phi}\|_{p,\phi} \gtrsim \left(\int_{D(w,r)} |K_\phi(z,w)|^p e^{-p\phi(z)} e^{-p\phi(w)} dv(z)\right)^{\frac{1}{p}} \approx 1.$$

This means $\|k_{w,\phi}\|_{p,\phi} \approx 1$. Thus, by applying $V_{(g,\psi)}$ on the normalized kernel functions along with Theorem 2.1,

$$B_{(\psi,\phi)}(|g|^q)(w) \lesssim \|V_{(g,\psi)} k_{w,\phi}\|_{q,\phi}^q \lesssim 1.$$

Then we get the necessity part.

To prove the sufficient part, we extend the techniques used in Refs. [6,16]. Combining the definition of the measure $\mu_{(q,\phi)}$, Theorem 2.1 and Theorem 2.1 in Ref. [18], we can get

$$\|V_{(g,\psi)} f\|_{q,\phi}^q \approx \int_{\mathbf{C}} |f(w)|^q d\mu_{(q,\phi)}(w) \lesssim \int_{\mathbf{C}} e^{q\phi(w)} d\mu_{(q,\phi)}(w) \int_{D(w,1)} |f(z)|^q e^{-q\phi(z)} dv(z).$$

Lemma 2.2 and Fubini's theorem show that

$$\|V_{(g,\psi)} f\|_{q,\phi}^q \lesssim \int_{\mathbf{C}} e^{q\phi(w)} d\mu_{(q,\phi)}(w) \int_{D(w,1)} |f(z)|^q e^{-q\phi(z)} dv(z) = \int_{\mathbf{C}} |f(z)|^q e^{-q\phi(z)} dv(z) \int_{D(z,1)} e^{q\phi(w)} d\mu_{(q,\phi)}(w) \lesssim \int_{\mathbf{C}} |f(z)|^q e^{-q\phi(z)} B_{(\psi,\phi)}(|g|^q)(z) dv(z) \lesssim \sup_{z\in\mathbf{C}} B_{(\psi,\phi)}(|g|^q)(z) \|f\|_{p,\phi}^q,$$

where the last inequality follows from the inclusion $F_\phi^p \subset F_\phi^q (p\leqslant q)$. Then it deduces that $V_{(g,\psi)}$ is bounded and the formula (6) holds.

(ii) Note that $k_{w,\phi}\to 0$ as $|w|\to\infty$ uniformly on any compact subset of $\mathbf{C}$ and $\|k_{w,\phi}\| \lesssim 1$ for all w, then $k_{w,\phi}\to 0$ as $|w|\to\infty$ then weakly in F_ϕ^p for $0<p\leqslant q$. If $V_{(g,\psi)}$ is compact,

$$0=\lim_{|w|\to\infty} \|V_{(g,\psi)} k_{w,\phi}\|_{q,\phi}^q \approx \lim_{|w|\to\infty} B_{(\psi,\phi)}(|g|^q)(w).$$

It means that necessity follows.

Now we prove the sufficiency of the condition. To this end, let f_n be a sequence of entire functions such that $f_n\to 0$ weakly in F_ϕ^p for $0<p\leqslant q$ as $n\to\infty$, it means that $f_n\to 0$ as $n\to\infty$ uniformly on any compact subset of $\mathbf{C}$ and $\|f_n\|_{p,\phi} \lesssim 1$ for all n. Then, as proved in (i) of this theorem, we obtain

$$\|V_{(g,\psi)} f_n\|_{q,\phi}^q \lesssim \int_{\mathbf{C}} |f_n(z)|^q e^{-q\phi(z)} B_{(\psi,\phi)}(|g|^q)(z) dv(z) = I_n.$$

For a fixed $R>0$, we set

$$I_n = \int_{\mathbf{C}} |f_n(z)|^q e^{-q\phi(z)} B_{(\psi,\phi)}(|g|^q)(z) dv(z) = \left(\int_{|z|\leqslant R} + \int_{|z|>R}\right) |f_n(z)|^q e^{-q\phi(z)} B_{(\psi,\phi)}(|g|^q)(z) dv(z) = I_{n1} + I_{n2}.$$

Firstly, we estimate I_{n1}. Noting that $\sup_{z\in\mathbf{C}} B_{(\psi,\phi)}(|g|^q)(z)<\infty$ and $f_n\to 0$ as $n\to\infty$ uniformly on any compact subset of $\mathbf{C}$, we can see that

$$\lim_{n\to\infty} \sup I_{n1} = \lim_{n\to\infty} \sup \int_{|z|\leqslant R} |f_n(z)|^q e^{-q\phi(z)} B_{(\psi,\phi)}(|g|^q)(z) dv(z) \lesssim$$

$$\limsup_{n\to\infty}\sup_{|z|\leqslant R}|f_n(z)|^q\int_{|z|\leqslant R}e^{-q\phi(z)}B_{(\psi,\phi)}(|g|^q)(z)dv(z)\lesssim\limsup_{n\to\infty}\sup_{|z|\leqslant R}|f_n(z)|^q\to 0.$$

Secondly, we prove the similar conclusion for the second piece of the integral I_{n2}. It follows from the conditions $\|f_n\|_{p,\phi}\lesssim 1$ for all n and $B_{(\psi,\phi)}(|g|^q)(z)\to 0$ as $|z|\to 0$ that

$$\limsup_{n\to\infty}I_{n2}=\limsup_{n\to\infty}\int_{|z|>R}|f_n(z)|^q e^{-q\phi(z)}B_{(\psi,\phi)}(|g|^q)(z)dv(z)\lesssim \sup_{|z|>R}B_{(\psi,\phi)}(|g|^q)(z)\limsup_{n\to\infty}\|f_n\|_{p,\phi}^q.$$

When $R\to\infty$, we see that the last expression in the right hand above converges to zero and so $V_{(g,\psi)}f_n\to 0$ in F_ϕ^q as $n\to\infty$.

The conditions in both (i) and (ii) are independent of the exponent p. However, p must be less than q. Furthermore, $V_{(g,\psi)}$ is bounded (compact) from F_ϕ^p to F_ϕ^q for some $p>0$, then it is also bounded (compact) for any $p\leqslant q$.

In general, it is difficult to characterize the boundedness, compactness, or Schatten class membership of a concrete operator with proper conditions. For reproducing kernel Hilbert spaces, the Berezin type transforms often are useful conditions, partly because we can know the effect of their action on the kernel functions, this is called the operator theoretic method. However, nobody can completely interpret why, how and when these conditions are effective. The boundedness and compactness of $V_{(g,\psi)}$ are respectively equivalent to

$$\sup_{z\in\mathbf{C}}\|V_{(g,\psi)}k_{z,\phi}\|_{q,\phi}<\infty,$$

$$\lim_{|z|\to\infty}\|V_{(g,\psi)}k_{z,\phi}\|_{q,\phi}=0.$$

A natural question is whether there exists an interplay between the two symbols g and ψ inducing bounded and compact operators $V_{(g,\psi)}$. We first observe that if $g'\neq 0$, then by the classical Liouville's theorem the function g cannot decay in any way. This forces that

$$B_{(\psi,\phi)}(|g|^q)(w)\approx\int_{\mathbf{C}}|K_{w,\phi}(\psi(z))|^p e^{-p\phi(z)-p\phi(w)}|g'(z)\varphi_{p,\phi}(|z|)|^p dv(z)\lesssim \int_{\mathbf{C}}e^{-\varepsilon p|\psi(z)-w|}e^{-p\phi(\psi(z))-p\phi(z)}|g'(z)\varphi_{p,\phi}(|z|)|^p dv(z)$$

is bounded only when $\psi(z)=az+b$ with $|a|\leqslant 1$. Moreover, if $|a|=1$ then $b=0$, and compactness is achieved when $|a|<1$. Then the following corollaries hold.

Corollary 2.4 Let $0<p\leqslant q<\infty$, $g'\neq 0$, and ψ be an entire function. If $V_{(g,\psi)}:F_\phi^p\to F_\phi^q$ is bounded, then $\psi(z)=az+b$ with $|a|\leqslant 1$, Moreover, if $V_{(g,\psi)}$ is compact, then $|a|<1$ and $b=0$.

In general, the boundedness of operator $V_{(g,\psi)}$ does not necessarily imply that the Volterra type integral operator V_g is bounded. This is because that the boundedness of the former allows g to be any entire function that grows more slowly than the exponential part of the integrand in $\{B_{(\psi,\phi)}(|g|^p)(w)\}$, while the boundedness of the latter forces g to grow as a power function of at most degree 2, as can be seen below. By setting $\psi(z)=z$ in the theorem, we immediately get the following result.

Corollary 2.5 Let $0<p\leqslant q<\infty$. Then V_g: $F_\phi^p\to F_\phi^q$ is

(i) bounded if and only if $g(z)=az^2+bz+c$, $a,b,c\in\mathbf{C}$;

(ii) compact if and only if $g(z)=az+b$, $a,b\in\mathbf{C}$.

Proof According to Theorem 2.3 and condition (5), we know that $\varphi_{p,\phi}(|z|)$ does not decay slowly tha n $\frac{1}{|z|}$ at infinity. Then $|2az+b|\phi_{p,\varphi}(|z|)$ is bounded on $\mathbf{C}$. It follows that $\sup_{w\in\mathbf{C}} B_{(\psi,\phi)}(|g|^p)(w)<\infty$. The sufficiency of the conditions in (i) are proved. By the same discussion, the sufficiency of the conditions in (ii) are immediate. We shall sketch the necessity. By formulae (2), (3) and (4)

$$B_{(\psi,\phi)}(|g|^q)(w)\gtrsim \int_{D(w,1)}|g'(z)\varphi_{p,\phi}(|z|)|^q dv(z)\gtrsim |g'(w)\varphi_{p,\phi}(|w|)|^q \quad (7)$$

where

$$D(w,1)=\{z\in\mathbf{C}: |z-w|<1\}.$$

The boundedness of $V_{(g,\psi)}$ implies $|g'(w)\varphi_{p,\phi}(|w|)|\lesssim 1$ for all $w\in\mathbf{C}$. In terms of the growth of $\varphi_{p,\phi}(|z|)$ at infinity and g is entire function, the desired expression for g follows.

On the other hand, if V_g is compact, then, since $k_{w,\phi}\to 0$ weakly in $F^p_\phi(0<p<\infty)$ as $|w|\to\infty$, we see from relation (7) that

$$|g'(z)\varphi_{p,\phi}(|z|)|\to 0,\ |z|\to\infty.$$

This can happen only when g is a polynomial of degree at most 1.

If ψ as $\psi(z)=\beta z$ with $|\beta|<1$, then g can have loose condition. More precisely, we get the following corollary.

Corollary 2.6 Let $\psi(z)=\beta z$ with $|\beta|<1$ and $0<p\leqslant q<\infty$. Then $V_{(g,\psi)}: F^p_\phi\to F^q_\phi$ is bounded if $|g(z)|\lesssim e^{p\phi(z)-p\phi(\beta z)}$, $z\in\mathbf{C}$.

For the case $0<q<p<\infty$, we get the following stronger conditions.

Theorem 2.7 Let $0<q<p<\infty$ and ψ be an entire function. Then the following statements are equivalent:

(i) $V_{(g,\psi)}: F^p_\phi\to F^q_\phi$ is bounded;

(ii) $V_{(g,\psi)}: F^p_\phi\to F^q_\phi$ is compact;

(iii) $B_{(\psi,\phi)}(|g|^q)(w)\in L^{\frac{p}{p-q}}(\mathbf{C},dv)$. Moreover

$$\|V_{(g,\psi)}\|\approx\left(\|B_{(\psi,\phi)}(|g|^q)\|_{L^{\frac{p}{p-q}}(\mathbf{C},dv)}\right)^{1/q} \quad (8)$$

Proof It's obvious that (ii) implies (i), then we just need to show that (iii)⇒ (ii) and (i) ⇒ (iii).

Firstly, if we assume that

$$B_{(\psi,\phi)}(|g|^q)(w)\in L^{\frac{p}{p-q}}(\mathbf{C},dv),$$

then we shall show that $V_{(g,\psi)}: F^p_\phi\to F^q_\phi$ is compact. Let f_n be a sequence of functions in F^p_ϕ satisfying that $f_n\to 0$ weakly. This means that f_n converges to zero uniformly on compact subsets of $\mathbf{C}$ and $\|f_n\|_{p,\phi}<\infty$ for all n. Then we proceed as in the proof of Theorem 2.3 until we get the equation that $I_n=I_{n1}+I_{n2}$. Since f_n converges to zero uniformly on compact subsets of $\mathbf{C}$ and

$$\int_{|z|\leqslant R} e^{-q\phi(z)}B_{(\psi,\phi)}(|g|^q)(z)dv(z)\lesssim \|B_{(\psi,\phi)}(|g|^q)\|_{L^{\frac{p}{p-q}}(\mathbf{C},dv)},$$

we can see that

$$\limsup_{n\to\infty} I_{n1}=\limsup_{n\to\infty}\int_{|z|\leqslant R}|f_n(z)|^q e^{-q\phi(z)}B_{(\psi,\phi)}(|g|^q)(z)dv(z)\lesssim$$
$$\limsup_{n\to\infty}\sup_{|z|\leqslant R}|f_n(z)|^q\int_{|z|\leqslant R}e^{-q\phi(z)}B_{(\psi,\phi)}(|g|^q)(z)dv(z)\lesssim$$
$$\limsup_{n\to\infty}\sup_{|z|\leqslant R}|f_n(z)|^q\ \|B_{(\psi,\phi)}(|g|^q)\|_{L^{\frac{p}{p-q}}(\mathbf{C},dv)}=0.$$

In views of $B_{(\psi,\phi)}(|g|^q)(w)\in L^{\frac{p}{p-q}}(\mathbf{C},dv)$ and Hölder's inequality, we obtain

$$\limsup_{n\to\infty} I_{n2}=\limsup_{n\to\infty}\int_{|z|>R}|f_n(z)|^q e^{-q\phi(z)}B_{(\psi,\phi)}(|g|^q)(z)dv(z)\lesssim$$
$$\limsup_{n\to\infty}\|f_n\|^q_{p,\phi}\left(\int_{|z|>R}(B_{(\psi,\phi)}(|g|^q)(z))^{\frac{p}{p-q}}dv(z)\right)^{1-\frac{q}{p}}\to 0,$$

when $R\to\infty$. This shows that $V_{(g,\psi)}$ is compact.

Now our proof will be complete once we show that (iii) follows from (i). To this end, note that $V_{(g,\psi)}$ is bounded if and only if

$$\int_{\mathbf{C}} |V_{(g,\psi)} f(z)|^q e^{-q\phi(z)} dv(z) \approx \int_{\mathbf{C}} |f(z)|^q d\mu_{(q,\phi)}(z).$$

However, it's easy to see that

$$\int_{\mathbf{C}} |f(z)|^q d\mu_{(q,\phi)}(z) = \int_{\mathbf{C}} |f(z)|^q e^{-q\phi(z)} d\lambda_{(q,\phi)}(z) \lesssim \| f \|_{p,\phi}^q,$$

where $d\lambda_{(q,\phi)}(z) = e^{q\phi(z)} d\mu_{(q,\phi)}(z)$. The above inequality means that $d\lambda_{(q,\phi)}$ is a (p,q) Fock-Carleson measure. By Theorem 2.8 in Ref. [19], this holds if and only if

$$\widetilde{\lambda_{(q,\phi)}}(w) = \int_{\mathbf{C}} \left| \frac{K_\phi(z,w)}{\sqrt{K_\phi(z,z)K_\phi(w,w)}} \right|^q d\lambda_{(q,\phi)}(z) \in L^{\frac{p}{p-q}}(\mathbf{C}, dv).$$

Substituting back $d\lambda_{(q,\phi)}$ and $d\mu_{(q,\phi)}$, we obtain from (2), (3) and (4) that

$$\widetilde{\lambda_{(q,\phi)}}(w) = \int_{\mathbf{C}} \left| \frac{K_\phi(z,w)}{\sqrt{K_\phi(z,z)K_\phi(w,w)}} \right|^q e^{p\phi(z)} d\mu_{(q,\phi)}(z) \approx B_{(\psi,\phi)}(|g|^q)(w).$$

It remains to prove the estimate (8). Since $\lambda_{(q,\phi)}$ is a (p,q) Fock-Carleson measure, the series of norm estimates in Theorem 2.8 in Ref. [19] yields

$$\| V_{(g,\psi)} \| \approx \left(\| \widetilde{\lambda_{(q,\phi)}} \|_{L^{\frac{p}{p-q}}(\mathbf{C}, dv)} \right)^{\frac{1}{q}} \approx \left(\| B_{(\psi,\phi)}(|g|^q) \|_{L^{\frac{p}{p-q}}(\mathbf{C}, dv)} \right)^{\frac{1}{q}}.$$

This ends the proof.

It is interesting to note that unlike condition (i) of Theorem 2.3, where we map smaller spaces into bigger ones, condition (iii) above is expressed in terms of both exponents p and q. When $\psi(z) = z$, the theorem simplifies to saying that V_g (for non-constant g) is bounded or compact if and only if g' is a constant $q > 2p/(p+2)$ and $g' = 0$ for $q < 2p/(p+2)$. This is because by subharmonicity, we have

$$\int_{\mathbf{C}} (B_{(\psi,\phi)}(|g|^q)(w))^{\frac{p}{p-q}} dv(w) \gtrsim \int_{\mathbf{C}} \left(\int_{D(w,1)} \left| \frac{K_\phi(\psi(z),w)}{\sqrt{K_\phi(z,z)K_\phi(w,w)}} \right|^q |g'(z)\varphi_{p,\phi}(|z|)|^q dv(z) \right)^{\frac{p}{p-q}} dv(w) \gtrsim \int_{\mathbf{C}} |g'(w)\varphi_{q,\phi}(|w|)|^{\frac{pq}{p-q}} dv(w),$$

from which the desired restrictions on g, p and q follow once we assume that the left-hand side of above inequality is finite.

3 Essential norm of $V_{(g,\psi)}$ on generalized Fock space

The essential norm $\| T \|_e$ of a bounded operator T on a Banach space B is defined as the distance from T to the space of compact operators on H. We refer to Refs. [13-14, 16, 20-21] for estimation of such norms for different operators on Hardy space, Bergman space, L^p, and some Fock spaces. Here we estimate the essential norm of $V_{(g,\psi)}$ as following.

Theorem 3.1 Let $1 < p \leqslant q < \infty$ and ψ an entire function. If $V_{(g,\psi)}: F_\phi^p \to F_\phi^q$ is bounded, then

$$\| V_{(g,\psi)} \|_e \approx (\limsup_{|w| \to \infty} B_{(\psi,\phi)}(|g|^q)(w))^{1/q}.$$

Before proving Theorem 3.1, we introduce some useful lemmas. Recall that each entire function f can be expressed as $f(z) = \sum_{k=0}^{\infty} a_k z^k$, let $K_n f(z) = \sum_{k=0}^{n} a_k z^k$ be the projection onto the subspace which is produced by $\{1, z, z^2, \cdots, z^n\}$ and $R_n = I - K_n$, where $If(z) = f$ is the identity map. Hence

$$R_n f(z) = \sum_{k=n+1}^{\infty} a_k z^k.$$

Then we have

Lemma 3.2 Let $1 < p < \infty$, Then

$$\lim_{n\to\infty}\|R_n f\|_{p,\phi}=0$$

and $|R_n f(w)|\to 0$ uniformly on any compact subset of $\mathbf{C}$.

Proof By Corollary 3 in Ref. [22], we know that there exists constant M such that

$$\int_0^{2\pi}|K_n f(re^{it})|^p dt\leqslant M\int_0^{2\pi}|f(re^{it})|^p dt.$$

for $f(re^{it})\in H^p(1<p<\infty)$, the Hardy space and $n\geqslant 1$. It follows from the use of polar coordinates that for any function f in generalized Fock space F_ϕ^p,

$$\|K_n f\|_{p,\phi}^p=\int_{\mathbf{C}}|K_n f(z)|^p e^{-p\phi(z)}dv(z)=\int_0^{+\infty}e^{-p\phi(r)}r dr\int_0^{2\pi}|K_n f(re^{it})|^p dt\leqslant M\|f\|_{p,\phi}^p$$

for any $n\in\mathbf{Z}^+$. Then Proposition 1 in Ref. [22] shows that

$$\lim_{n\to\infty}\|R_n f\|_{p,\phi}=\lim_{n\to\infty}\|f-K_n f\|_{p,\phi}=0.$$

It is obvious that $|R_n f(w)|\to 0$ uniformly on any compact subset of $\mathbf{C}$ from the properties of Taylor series of entire function.

By Lemma 3.2 and the principle of uniform boundness, we know that $\sup_n\|R_n\|<\infty$. We need the following lemma in proving Theorem 3.1.

Lemma 3.3 Let $1<p\leqslant q<\infty$ and ψ an entire function. If $V_{(g,\psi)}:F_\phi^p\to F_\phi^q$ is bounded, then

$$\|V_{(g,\psi)}\|_e\leqslant\liminf_{n\to\infty}\|V_{(g,\psi)}R_n\|.$$

Proof Let K be a compact operator on F_ϕ^p. Noting that

$$V_{(g,\psi)}=V_{(g,\psi)}(R_n+K_n),$$

we have

$$\|V_{(g,\psi)}-K\|\leqslant\|V_{(g,\psi)}R_n\|+\|V_{(g,\psi)}K_n-K\|\tag{9}$$

for all $n\geqslant 1$. It is easy to know that K_n is compact on F_ϕ^p, then $V_{(g,\psi)}K_n$ is compact. It follows that $\|V_{(g,\psi)}K_n\|_e=0$ for all $n\in\mathbf{Z}^+$. Taking the infimum over compact operators K and letting $n\to\infty$ in (9), we obtain the desired inequality.

Proof of Theorem 3.1 If taking a compact operator Q on F_ϕ^p and noting that $\|k_{w,\phi}\|_{p,\phi}=1$ and $k_{w,\phi}$ converges to zero uniformly on compact subsets of $\mathbf{C}$ as $|w|\to\infty$, we see that

$$\|V_{(g,\psi)}-Q\|\geqslant\limsup_{|w|\to\infty}\|V_{(g,\psi)}k_{w,\phi}-Qk_{w,\phi}\|_{q,\phi}\geqslant\limsup_{|w|\to\infty}(\|V_{(g,\psi)}k_{w,\phi}\|_{q,\phi}-\|Qk_{w,\phi}\|_{q,\phi})=\limsup_{|w|\to\infty}\|V_{(g,\psi)}k_{w,\phi}\|_{q,\phi}\approx\limsup_{|w|\to\infty}(B_{(\psi,\phi)}(|g|^q))^{\frac{1}{q}},$$

where the equality comes from compactness of Q. This shows the lower estimate in the theorem.

Now we turn to prove the upper inequality. For each unit vector f in F_ϕ^p, it follows from the proof in Theorem 2.3 that

$$\|V_{(g,\psi)}R_n f\|_{q,\phi}^q\approx\int_{\mathbf{C}}|R_n f(z)|^q d\mu_{(q,\phi)}(z)\lesssim\int_{\mathbf{C}}e^{q\phi(z)}d\mu_{(q,\phi)}(z)\limsup_{|z|\to\infty}\int_{D(z,r)}|R_n f(w)|^q e^{-q\phi(w)}dv(w)\lesssim\int_{\mathbf{C}}|R_n f(w)|^q e^{-q\phi(w)}B_{(\psi,\phi)}(|g|^q)(w)dv(w)=\left(\int_{|w|>R}+\int_{|w|\leqslant R}\right)|R_n f(w)|^q e^{-q\phi(w)}B_{(\psi,\phi)}(|g|^q)(w)dv(w)=I_{n1}+I_{n2},$$

where R is fixed positive number. We first estimate I_{n1} as follows. Firstly, we have

$$I_{n1}=\int_{|w|>R}|R_n f(w)|^q e^{-q\phi(w)}B_{(\psi,\phi)}(|g|^q)(w)dv(w)\lesssim\sup_{|w|>R}B_{(\psi,\phi)}(|g|^q)(w),$$

since $\sup_n\|R_n\|<\infty$. It remains to estimate I_{n2}. By Lemma 3.2, we obtain

$$I_{n2}=\int_{|w|\leqslant R}|R_n f(w)|^q e^{-q\phi(w)}B_{(\psi,\phi)}(|g|^q)(w)dv(w)\lesssim\sup_{|w|\leqslant R}|R_n f(w)|^q\sup_{w\in\mathbf{C}}B_{(\psi,\phi)}(|g|^q)(w)\int_{|w|\leqslant R}e^{-q\phi(w)}dv(w).$$

It follows from Lemma 3.2 that $\sup_{|w|\leqslant R}|R_nf(w)|^q\to 0$, $n\to\infty$. By Theorem 2.3, it follows that $I_{n2}\to 0$ as $n\to\infty$. Therefore,

$$\lim_{n\to\infty}\sup_{\|f\|_{p,\phi}\leqslant 1}\|V_{(g,\psi)}R_nf\|_{q,\phi}^q\lesssim \sup_{|w|>R}B_{(\psi,\phi)}(|g|^q)(w).$$

By Lemma 3.3, we get

$$\|V_{g,\psi)}R_nf\|_e^q\lesssim\lim_{R\to\infty}\sup_{|w|>R}B_{(\psi,\phi)}(|g|^q)(w)\approx \lim_{|w|\to\infty}\sup B_{(\psi,\phi)}(|g|^q)(w).$$

This completes the proof.

4 Schatten p-class operator $V_{(g,\psi)}$ on generalized Fock space

Let us now characterize the operator $V_{(g,\psi)}$ in the Schatten p-class membership for $0<p<\infty$.

A positive operator T on F_ϕ^2 is called the trace class operator if

$$\sum_{k=1}^{\infty}\langle Te_k,e_k\rangle_\phi=\sum_{k=1}^{\infty}\int_{\mathbf{C}}Te_k(z)\overline{e_k(z)}\mathrm{e}^{-2\phi(z)}\mathrm{d}v(z)<\infty,$$

where $\{e_k\}_{k=1}^{\infty}$ is some orthonormal basis of F_ϕ^2. We write the trace of T as $Tr(T)$. If $0<p<\infty$, a bounded operator T on F_ϕ^2 belongs to the Schatten class S_p if the positive operator $(T^*T)^{p/2}$ is in the trace class. We denote the S_p norm of T by $\|T\|_{S_p}$. For more information of Schatten class, we refer to Refs. [23-24].

Proposition 4.1 Let A be a Hilbert space and T be a bounded operator from F_ϕ^2 to A.

(i) If $p\geqslant 2$ and $T\in S_p$, then

$$\int_{\mathbf{C}}\|Tk_{w,\phi}\|_A^p\mathrm{d}v(w)<\infty \tag{10}$$

(ii) If $0<p\leqslant 2$ and (10) holds, then $T\in S_p$.

In general, the inverses of the two statements above can not hold, for instance, Hankel operators on the Hardy space H^2 in Ref. [25]. We are interested in whether the inverses still hold for the product of Volterra type integral and composition operators on F_ϕ^2. We will show the inverses is indeed the case (see Theorem 4.2). In particular, T belongs to the Hilbert-Schmidt class if and only if for any orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in F_ϕ^2,

$$\|T\|_{S_2}^2=\sum_{k=1}^{\infty}\int_{\mathbf{C}}|T^*e_k(z)|^2\mathrm{e}^{-2\phi(z)}\mathrm{d}v(z)=\int_{\mathbf{C}}\sum_{k=1}^{\infty}|\langle TK_{z,\phi},e_k\rangle|^2\mathrm{e}^{-2\phi(z)}\mathrm{d}v(z)=\int_{\mathbf{C}}\|Tk_{w,\phi}\|_{2,\phi}^2\mathrm{d}v(w)<\infty \tag{11}$$

If T is any positive operator in the trace class of F_ϕ^2, then

$$Tr(T)=\|T^{1/2}\|_{S_2}^2=\int_{\mathbf{C}}\|T^{1/2}k_{w,\phi}\|_{2,\phi}^2\mathrm{d}v(w)=\int_{\mathbf{C}}\langle Tk_{w,\phi},k_{w,\phi}\rangle_\phi\mathrm{d}v(w).$$

We know that T belongs to the Schatten class S_p if and only if $(T^*T)^{p/2}$ is in the trace class. Thus

$$Tr((T^*T)^{p/2})=\int_{\mathbf{C}}\langle(T^*T)^{p/2}k_{w,\phi},k_{w,\phi}\rangle_\phi\mathrm{d}v(w)\gtrsim\int_{\mathbf{C}}\|Tk_{w,\phi}\|_{2,\phi}^p\mathrm{d}v(w)$$

for $2\leqslant p<\infty$, and

$$Tr((T^*T)^{p/2})\lesssim\int_C\|Tk_{w,\phi}\|_{2,\phi}^p\mathrm{d}v(w)$$

for $0<p\leqslant 2$. In particular, when $T=V_{(g,\phi)}$, we have

$$\int_{\mathbf{C}}\|V_{(g,\phi)}k_{w,\phi}\|_{2,\phi}^p\mathrm{d}v(w)\approx\int_{\mathbf{C}}(B_{(\psi,\phi)}(|g|^2)(w))^{\frac{p}{2}}\mathrm{d}v(w),$$

which gives the proofs of the necessity for $p\geqslant 2$ and the sufficiency for $0<p\leqslant 2$ of our next theorem.

Theorem 4.2 If $0<p<\infty$ and ψ be an entire function. Then the bounded operator $V_{(g,\phi)}:F_\phi^2\to F_\phi^2$ belongs to S_p if and only if $B_{(\psi,\phi)}(|g|^2)\in L^{p/2}(\mathbf{C},\mathrm{d}v)$.

Proof The crucial step in proving the theorem is to introduce a Toeplitz operator on F_ϕ^2. Let μ be a finite positive Borel measure on $\mathbf{C}$ satisfying the admissibility condition

$$\int_{\mathbf{C}}|K_{w,\phi}(z)|^2\mathrm{e}^{-2\phi(z)}\mathrm{d}\mu(z)<\infty \tag{12}$$

for all $z\in\mathbf{C}$. Then we define the following Toeplitz operator by

$$T_\mu f(z)=\int_{\mathbf{C}}K_{w,\phi}(z)f(w)\mathrm{e}^{-2\phi(w)}\mathrm{d}\mu(w)$$

for each $z\in\mathbf{C}$. Since the kernel functions are dense in F^2_ϕ, it follow by the admissibility condition and Hölder's inequality that T_μ is well defined. We observe that by Theorem 2.1, the inner product

$$\langle f,h\rangle_\phi=f(0)\overline{h(0)}+\int_{\mathbf{C}}f'(z)\overline{h'(z)}(\varphi_{2,\phi}(|z|))^2\mathrm{e}^{-2\phi(z)}\mathrm{d}v(z)$$

defines a norm which is equivalent to the usual norm on F^2_ϕ. We prefer to use this norm since this alternative approach has the advantage that it permits us to easily associate the product of Volterra type integral and composition operators with Toeplitz operators.

For any $f\in F^2_\phi$, we calculate that

$$\begin{aligned}V^*_{(g,\psi)}V_{(g,\psi)}f(z)&=\langle V^*_{(g,\psi)}V_{(g,\psi)}f,K_{z,\phi}\rangle_\phi=\langle V_{(g,\psi)}f,V_{(g,\psi)}K_{z,\phi}\rangle_\phi=\\&\int_{\mathbf{C}}f(\psi(w))\overline{K_{z,\phi}(\psi(w))}|g'(w)|^2\ |\varphi(|w|)|^2\mathrm{e}^{-2\phi(w)}\mathrm{d}v(w)=\\&\int_{\mathbf{C}}f(\psi(w))K_{\psi(w),\phi}(z)|g'(w)|^2\ |\varphi(|w|)|^2\mathrm{e}^{-2\phi(w)}\mathrm{d}v(w)=\\&\int_{\mathbf{C}}f(\eta)K_{\eta,\phi}(z)\mathrm{e}^{-2\phi(\eta)}\mathrm{d}\mu(\eta)=T_\mu f(z).\end{aligned}$$

Therefore, if $V_{(g,\psi)}$ is a bounded operator on F^2_ϕ, then we claim that $V^*_{(g,\psi)}V_{(g,\psi)}=T_\mu$, where T_μ is the Toeplitz operator induced by the measure

$$\mathrm{d}\mu(z)=\zeta\circ\psi^{-1}(z)\mathrm{d}v(z),$$

where

$$\zeta(z)=|g'(z)|^2\ |\varphi_{2,\phi}(|z|)|^2\mathrm{e}^{2\varphi(\psi(z))-2\varphi(z)}.$$

For such particular measure μ, the admissibility condition (12) holds whenever $V_{(g,\psi)}$ is bounded on F^2_ϕ. Denote the associated Berezin symbol $\tilde{\mu}$ of μ by

$$\tilde{\mu}(z)=\langle T_\mu k_{z,\phi},k_{z,\phi}\rangle_\phi.$$

Then the results from Ref. [26] show that the Toeplitz operator T_μ belongs S_p if and only if $\tilde{\mu}$ belongs to $L^p(\mathbf{C},\mathrm{d}v)$ for each $0<p<\infty$.

On the other hand, $V_{(g,\psi)}$ belongs to S_p if and only if $V^*_{(g,\psi)}V_{(g,\psi)}$ belongs to $S_{p/2}$ (see Ref. [23]), and this holds if and only if $\tilde{\mu}(z)=\|V_{(g,\psi)}k_{z,\phi}\|^2_{2,\phi}$ belongs to $L^{p/2}(\mathbf{C},\mathrm{d}v)$. It is easily seen that

$$\|V_{(g,\psi)}k_{z,\phi}\|^2_{2,\phi}\approx B_{(\psi,\phi)}(|g|^2)(z).$$

Thus the result of this theorem holds.

Corollary 4.3 Let $p>2$ and V_g be a compact operator on F^2_ϕ. If $\varphi_{2,\phi}\in L^p(\mathbf{C},\mathrm{d}v)$, then V_g belongs to S_p for all $p>2$.

Proof If V_g is a compact operator, then by Corollary 2.5, $g'=C$, a constant. By (2) and Hölder's inequality, we have

$$\begin{aligned}&\int_{\mathbf{C}}(B_{(\psi,\phi)}(|g|^2)(z))^{\frac{p}{2}}\mathrm{d}v(z)\lesssim\\&\int_{\mathbf{C}}\mathrm{d}v(z)\left(\int_{\mathbf{C}}C^2\mathrm{e}^{-2\theta|z-w|}\varphi^2_{2,\phi}(|w|)\mathrm{d}v(w)\right)^{\frac{p}{2}}\lesssim\\&\int_{\mathbf{C}}\varphi^p_{2,\phi}(|w|)\mathrm{d}v(w)<\infty.\end{aligned}$$

If V_g belongs to S_p, then the above integrals should converge for all $p>2$.

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