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一个三维系统的音叉分岔

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摘　要：本文研究了一个三维系统的音叉分岔，该系统中的每个方程都包含一个二次交叉乘积项.本文分析了当单个参数在临界值附近变化时系统的平衡点数量的改变，即单个参数的音叉分岔，以及系统发生音叉分岔时产生的平衡点的稳定性.

关键词：平衡点；音叉分岔；稳定性
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Pitchfork bifurcations of a three-dimensional system

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Abstract: This paper aims at the pitchfork bifurcations of a three-dimensional system, in which each equation in the system contains a single quadratic cross-product term. The change of the number of equilibria of the system as one parameter varies near a critical value, *i.e.*, the pitchfork bifurcations for one parameter, is analyzed. The stability of the equilibria generated by the pitchfork bifurcations is investigated as well.

Keywords: Equilibrium; Pitchfork bifurcation; Stability
(2010 MSC 37G10)

1 Introduction

Three-dimensional differential systems are investigated widely because of their plentiful dynamical phenomenon. In 1963 Lorenz[1] found the first chaotic attractor in a three dimensional system. From then on, various three-dimensional systems, such as Rössler system, Chen system, Lü system, Liu system, Bao system, Pehlivan system, Jafari system and Sampath system[2-9] *etc*, have been proposed.

In Ref. [10], Qi *et al*. considered a three-dimensional nonlinear system, in which each equation contains a single quadratic cross-product term, which is described as

$$\begin{cases} \dot{x}=a(y-x)+yz, \\ \dot{y}=cx-y-xz, \\ \dot{z}=xy-bz \end{cases} \tag{1}$$

where $(x,y,z)\in\mathbf{R}^3$, $(a,b,c)\in\mathbf{R}^3_+$ and $\dot{x}\coloneqq \mathrm{d}x/\mathrm{d}t$, $\dot{y}\coloneqq \mathrm{d}y/\mathrm{d}t$, $\dot{z}\coloneqq \mathrm{d}z/\mathrm{d}t$. Qi *et al*. numerically analyzed the basic properties of System (1) by Lyapunov exponents and bifurcation diagrams. In Ref. [11], the pitchfork bifurcation in System (1) was investigated by the classical center manifold method for the case that $a>1$ and c changed near 1. In Ref. [12] Qi and Liang investigated the mechanics of System (1) by comparing it with Kolmogorov system, Euler equation and Hamiltonian

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function. In Ref. [13], System (1) was transformed into the Kolmogorov type system to investigate the mechanics of the system. For System (1), the phase portrait and trajectories is shown in Ref. [14] for some initial conditions with colors.

In this paper, we continue to study the pitchfork bifurcations in System (1) and consider that c changes near 1 and no matter $a>1$ or $a\leqslant 1$. Our main method is to use the parameter partition determined by the number of equilibria, which is essentially different from the center manifold method used in Ref. [11].

2 The pitchfork bifurcations

Let

$\Lambda_1 := \{(a,b,c)\in \mathbf{R}^3_+ : -4a+a^2+2ac+c^2>0, d-e>0\}$,

$\Lambda_2 := \{(a,b,c)\in \mathbf{R}^3_+ : -4a+a^2+2ac+c^2>0, d-e\leqslant 0, d+e>0\}$,

$\Lambda_3 := \{(a,b,c)\in \mathbf{R}^3_+ : -4a+a^2+2ac+c^2>0, d-e\leqslant 0, d+e\leqslant 0\}$,

$\Lambda_4 := \{(a,b,c)\in \mathbf{R}^3_+ : -4a+a^2+2ac+c^2=0, c>a\}$,

$\Lambda_5 := \{(a,b,c)\in \mathbf{R}^3_+ : -4a+a^2+2ac+c^2=0, c\leqslant a\}$,

$\Lambda_6 := \{(a,b,c)\in \mathbf{R}^3_+ : -4a+a^2+2ac+c^2<0\}$.

Clearly, $\bigcup_{i=1}^{6}\Lambda_i=\mathbf{R}^3_+$.

Lemma 2.1 (i) System (1) has five equilibria E_0, E_1, E_2, E_3, E_4 if and only if $(a,b,c)\in\Lambda_1$;

(ii) System (1) has three equilibria E_0, E_1, E_2, if and only if $(a,b,c)\in\Lambda_2\cup\Lambda_4$;

(iii) System (1) has a unique equilibrium E_0 if and only if $(a,b,c)\in\Lambda_3\cup\Lambda_5\cup\Lambda_6$,

here $E_0, E_1, \ldots, E_4$ lie at $(0,0,0)$, (x_1,y_1,z_1), $(-x_1,-y_1,z_1)$, (x_2,y_2,z_2), $(-x_2,-y_2,z_2)$ respectively, and

$$x_1:=\sqrt{\frac{d+e}{2a}},\quad x_2:=\sqrt{\frac{d-e}{2a}},$$

$$y_1:=\sqrt{\frac{2d+2e}{a}}\frac{abc}{f+e},$$

$$y_2:=\sqrt{\frac{2d-2e}{a}}\frac{abc}{f-e},$$

$$z_1:=\frac{(d+e)c}{f+e},\ z_2:=\frac{(d-e)c}{f-e},$$

$$d:=-2ab+abc+bc^2,$$

$$e:=\sqrt{b^2c^2(-4a+a^2+2ac+c^2)}$$

$$f:=abc+bc^2 \tag{2}$$

Proof By solving

$$\begin{cases} a(y-x)+yz=0, \\ cx-y-xz=0, \\ xy-bz=0, \end{cases}$$

we get that

$$y=bcx/(x^2+b), z=cx^2/(x^2+b) \tag{3}$$

and x satisfies

$$ax^5+(2ab-abc-bc^2)x^3+(ab^2-ab^2c)x=0 \tag{4}$$

Obviously, $x=0$ is one root of (4) and all nonzero roots satisfy

$$ax^4+(2ab-abc-bc^2)x^2+ab^2-ab^2c=0 \tag{5}$$

Then $x=\pm\sqrt{\frac{d\pm e}{2a}}$, where d and e are given in (2).

In the following, all possible cases are considered.

(i) If $e>0, d-e>0$, i. e., $(a,b,c)\in\Lambda_1$, then (5) has four distinct nonzero real roots x_1, x_2, $-x_1$, $-x_2$, given in (2). Correspondingly, we get $y_1, y_2, -y_1, -y_2, z_1, z_2$ by (3). Therefore, there are five equilibria E_0, E_1, E_2, E_3, E_4.

(ii) If $e>0, d-e\leqslant 0, d+e>0$, i. e., $(a,b,c)\in\Lambda_2$, then (5) has two distinct nonzero real roots $x_1, -x_1$ given in (2). Correspondingly, we get $y_1, -y_1, z_1$ by (3). Therefore, there are three equilibria E_0, E_1, E_2.

(iii) If $e>0, d-e\leqslant 0, d+e\leqslant 0$, i. e., $(a,b,c)\in\Lambda_3$, then (5) has no nonzero real roots. Thus, there is a unique equilibrium E_0.

(iv) If $e=0, c>a$, i. e., $(a,b,c)\in\Lambda_4$, then (5) has two distinct nonzero real roots $x_1, -x_1$, given in (2). Correspondingly, we get $y_1, -y_1$, z_1 by (3). Therefore, there are three equilibria E_0, E_1, E_2.

(v) If $e=0, c\leqslant a$, i. e., $(a,b,c)\in\Lambda_5$, then (5) has no nonzero real roots. Thus, there is a unique equilibrium E_0.

(vi) If e is not real, i. e., $(a,b,c)\in\Lambda_6$, then

(5) has no nonzero real roots. Thus, there is a unique equilibrium E_0.

Now we study the bifurcations of equilibria when parameter c changes near 1.

Theorem 2.2 Let $0<\varepsilon\ll 1$. (i) Assume that $a\geqslant 1$. The pitchfork bifurcation happens when c changes from 1 to $1+\varepsilon$, and the number of equilibria of System (1) changes from 1 to 3.

(ii) Assume that $a<1$. The pitchfork bifurcation happens when c changes from 1 to $1-\varepsilon$, and the number of equilibria of System (1) changes from 3 to 5.

Proof When $a>1$ and $c=1$ in (2) we have $d+e=0, d-e<0$, *i.e.*, $(a,b,c)\in\Lambda_3$. When $a=1$ and $c=1$ in (2) we have $e=0, a=c$, *i.e.*, $(a,b,c)\in\Lambda_5$. Thus when $a\geqslant 1$ and $c=1$, System (1) has a unique equilibrium E_0 by Lemma 2.1. When $a\geqslant 1$ and $c=1+\varepsilon$, from (2) we get $e>0$ and

$$d+e=\begin{cases}2a^2b\,\varepsilon/(a-1)+O(\varepsilon^2)>0, & a>1,\\ 2b\,\varepsilon^{1/2}+O(\varepsilon)>0, & a=1,\end{cases}$$

$$d-e=\begin{cases}2(1-a)b+O(\varepsilon)<0, & a>1,\\ -2b\,\varepsilon^{1/2}+O(\varepsilon)<0, & a=1.\end{cases}$$

Thus $(a,b,c)\in\Lambda_2$, *i.e.*, System (1) has three equilibria E_0,E_1,E_2 fallows from Lemma 2.1. (i) is proved.

When $a<1$ and $c=1$, we have $d+e>0$, $d-e=0$. Thus $(a,b,c)\in\Lambda_2$, *i.e.*, System (1) has three equilibria E_0,E_1,E_2 follows from Lemma 2.1. When $a<1$ and $c=1-\varepsilon$, we get $e>0$ and

$$d-e=2a^2b\,\varepsilon/(1-a)+O(\varepsilon^2)>0.$$

Thus $(a,b,c)\in\Lambda_1$, *i.e.*, System (1) has five equilibria E_0,E_1,E_2,E_3,E_4 follows from Lemma 2.1. From the x-coordinates of E_3,E_4, we find that E_3,E_4 appear by the pitchfork bifurcation of E_0. (ii) is proved.

3 The stability of equilibria

In Theorem 2.2, the pitchfork bifurcation of E_0 happens when c changes near 1. In this section, we study the stability of those equilibria bifurcated from E_0.

Theorem 3.1 Equilibria E_1 and E_2, appearing by the pitchfork bifurcation of E_0 when $a\geqslant 1$ and c changes from 1 to $1+\varepsilon$, are locally asymptotically stable.

Proof Since System (1) is symmetric about the z-axis and E_1 is the corresponding symmetric equilibrium of E_2, we only need to consider E_1.

When $a>1$ and $c=1+\varepsilon$, we get

$$x_1=\sqrt{\frac{ab}{a-1}}\varepsilon^{\frac{1}{2}}+O(\varepsilon),$$

$$y_1=\sqrt{\frac{ab}{a-1}}\varepsilon^{\frac{1}{2}}+O(\varepsilon)$$

and

$$z_1=\frac{a}{a-1}\varepsilon+O(\varepsilon^{\frac{3}{2}})$$

by (2). The Jacobian matrix at E_1 is given by

$$J(E_1)=\begin{pmatrix}-a & a+\frac{a}{a-1}\varepsilon+O(\varepsilon^{\frac{3}{2}}) & \sqrt{\frac{ab}{a-1}}\varepsilon^{\frac{1}{2}}+O(\varepsilon)\\ 1+\frac{1}{1-a}\varepsilon+O(\varepsilon^{\frac{3}{2}}) & -1 & -\sqrt{\frac{ab}{a-1}}\varepsilon^{\frac{1}{2}}+O(\varepsilon)\\ \sqrt{\frac{ab}{a-1}}\varepsilon^{\frac{1}{2}}+O(\varepsilon) & \sqrt{\frac{ab}{a-1}}\varepsilon^{\frac{1}{2}}+O(\varepsilon) & -b\end{pmatrix}.$$

Thus the characteristic equation at E_1 is

$$\lambda^3+(a+b+1)\lambda^2+(ab+b+O(\varepsilon^{\frac{3}{2}}))\lambda+2ab\varepsilon+O(\varepsilon^{\frac{3}{2}})=0,$$

whose coefficients satisfy

$$\Delta_1:=a+b+1>0,$$

$$\Delta_2:=\begin{vmatrix}a+b+1 & 1\\ 2ab\varepsilon+O(\varepsilon^{\frac{3}{2}}) & ab+b+O(\varepsilon^{\frac{3}{2}})\end{vmatrix}=(a+b+1)(ab+b+O(\varepsilon^{\frac{3}{2}}))-(2ab\varepsilon+O(\varepsilon^{\frac{3}{2}}))>0,$$

$$\Delta_3:=\begin{vmatrix}a+b+1 & 1 & 0\\ 2ab\varepsilon+O(\varepsilon^{\frac{3}{2}}) & ab+b+O(\varepsilon^{\frac{3}{2}}) & a+b+1\\ 0 & 0 & 2ab\varepsilon+O(\varepsilon^{\frac{3}{2}})\end{vmatrix}=(2ab\varepsilon+O(\varepsilon^{\frac{3}{2}}))\Delta_2>0.$$

Then, all eigenvalues have negative real parts follows from the Routh-Hurwitz Theorem[15]. Thus E_1 is locally asymptotically stable, so does E_2.

When $a=1$ and $c=1+\varepsilon$, we get

$$x_1=\sqrt{b}\varepsilon^{\frac{1}{4}}+\frac{3}{4}\sqrt{b}\varepsilon^{\frac{3}{4}}+O(\varepsilon),$$

$$y_1=\sqrt{b}\varepsilon^{\frac{1}{4}}-\frac{\sqrt{b}}{4}\varepsilon^{\frac{3}{4}}+O(\varepsilon)$$

and

$$z_1=\varepsilon^{\frac{1}{2}}+\frac{1}{2}\varepsilon+O(\varepsilon^{\frac{5}{4}})$$

by (2). The Jacobian matrix at E_1 is

$$J(E_1)=\begin{pmatrix} -1 & 1+\varepsilon^{\frac{1}{2}}+\frac{1}{2}\varepsilon+O(\varepsilon^{\frac{5}{4}}) & \sqrt{b}\varepsilon^{\frac{1}{4}}-\frac{\sqrt{b}}{4}\varepsilon^{\frac{3}{4}}+O(\varepsilon) \\ 1-\varepsilon^{\frac{1}{2}}+\frac{1}{2}\varepsilon+O(\varepsilon^{\frac{5}{4}}) & -1 & -\sqrt{b}\varepsilon^{\frac{1}{4}}-\frac{3}{4}\sqrt{b}\varepsilon^{\frac{3}{4}}+O(\varepsilon) \\ \sqrt{b}\varepsilon^{\frac{1}{4}}-\frac{\sqrt{b}}{4}\varepsilon^{\frac{3}{4}}+O(\varepsilon) & \sqrt{b}\varepsilon^{\frac{1}{4}}+\frac{3}{4}\sqrt{b}\varepsilon^{\frac{3}{4}}+O(\varepsilon) & -b \end{pmatrix}.$$

Thus the characteristic equation at E_1 is

$$\lambda^3+(b+2)\lambda^2+(2b+O(\varepsilon))\lambda+4b\varepsilon+O(\varepsilon^{\frac{5}{4}})=0,$$

whose coefficients satisfy

$$\Delta_1:=b+2>0,$$

$$\Delta_2:=\begin{vmatrix} (b+2) & 1 \\ 4b\varepsilon+O(\varepsilon^{\frac{5}{4}}) & 2b+O(\varepsilon) \end{vmatrix}=(b+2)(2b+O(\varepsilon))-4b\varepsilon+O(\varepsilon^{\frac{5}{4}})>0,$$

$$\Delta_3:=\begin{vmatrix} (b+2) & 1 & 0 \\ 4b\varepsilon+O(\varepsilon^{\frac{5}{4}}) & 2b+O(\varepsilon) & b+2 \\ 0 & 0 & 4b\varepsilon+O(\varepsilon^{\frac{5}{4}}) \end{vmatrix}=(4b\varepsilon+O(\varepsilon^{\frac{5}{4}}))\Delta_2>0.$$

Then, all eigenvalues have negative real parts follows from the Routh-Hurwitz Theorem[15]. Thus E_1 is locally asymptotically stable, so does E_2.

Theorem 3.2 Equilibria E_3 and E_4, appearing by the pitchfork bifurcation of E_0 when $a<1$ and c changes from 1 to $1-\varepsilon$, are unstable.

Proof Since System (1) is symmetric about the z-axis and E_3 is the corresponding symmetric equilibrium of E_4, we only need to consider E_3.

When $a<1$ and $c=1-\varepsilon$, we get

$$x_2=\sqrt{\frac{ab}{1-a}}\varepsilon^{\frac{1}{2}}+O(\varepsilon),$$

$$y_2=\sqrt{\frac{ab}{1-a}}\varepsilon^{\frac{1}{2}}+O(\varepsilon)$$

and

$$z_2=\frac{a}{1-a}\varepsilon+O(\varepsilon^{\frac{3}{2}})$$

by (2). The characteristic equation at E_3 is

$$\lambda^3+(a+b+1)\lambda^2+(ab+b+O(\varepsilon^{\frac{3}{2}}))\lambda-2ab\varepsilon+O(\varepsilon^{\frac{3}{2}})=0 \tag{6}$$

whose coefficients satisfy

$$\Delta_1:=a+b+1>0,$$

$$\Delta_2:=\begin{vmatrix} a+b+1 & 1 \\ -2ab\varepsilon+O(\varepsilon^{\frac{3}{2}}) & ab+b+O(\varepsilon^{\frac{3}{2}}) \end{vmatrix}=(a+b+1)(ab+b+O(\varepsilon^{\frac{3}{2}}))+2ab\varepsilon+O(\varepsilon^{\frac{3}{2}})>0,$$

$$\Delta_3:=\begin{vmatrix} a+b+1 & 1 & 0 \\ -2ab\varepsilon+O(\varepsilon^{\frac{3}{2}}) & ab+b+O(\varepsilon^{\frac{3}{2}}) & a+b+1 \\ 0 & 0 & -2ab\varepsilon+O(\varepsilon^{\frac{3}{2}}) \end{vmatrix}=(-2ab\varepsilon+O(\varepsilon^{\frac{3}{2}}))\Delta_2<0.$$

Then, some eigenvalues have positive or zero real parts follows from the Routh-Hurwitz theorem[15]. Obviously, there is no zero root by the expression of (6). If (6) has a pair of pure imaginary roots $\pm i\omega$ ($\omega\neq 0$), substituting $\lambda=i\omega$ into (6) we get

$$-i\omega^3-(a+b+1)\omega^2+(ab+b+O(\varepsilon^{\frac{3}{2}}))i\omega-2ab\varepsilon+O(\varepsilon^{\frac{3}{2}})=0,$$

i.e.,

$$\begin{cases}\omega^3-(ab+b+O(\varepsilon^{\frac{3}{2}}))\omega=0,\\ (a+b+1)\omega^2+2ab\varepsilon+O(\varepsilon^{\frac{3}{2}})=0\end{cases} \tag{7}$$

Clearly, (7) has no solution for ω. Thus (6) has eigenvalues with positive real parts. Therefore, E_3 is unstable, neither does E_4.

References:

[1] Lorenz E N. Deterministic nonperiodic flow [J]. J Atmos Sci, 1963, 20: 130.

[2] Rösslor O E. An equation for continuous chaos [J]. Phys Lett A, 1976, 57: 397.
[3] Chen G R, Ueta T. Yet another chaotic attractor [J]. Int J Bifurcation Chaos, 1999, 9: 1465.
[4] Lü J H, Chen G R. A new chaotic attractor coined [J]. Int J Bifurcat Chaos, 2002, 12: 659.
[5] Liu C X, Liu T, Liu L, *et al*. A new chaotic attractor [J]. Chaos Soliton Fract, 2004, 22: 1031.
[6] Bao B C, Liu Z, Xu J P. New chaotic system and its hyperchaos generation [J]. J Syst Eng Electron, 2009, 20: 1179.
[7] Pehlivan I, Moroz I M, Vaidyanathan S. Analysis, synchronization and circuit design of a novel butterfly attractor [J]. J Sound Vib, 2014, 333: 5077.
[8] Jafari S, Sprott C. Simple chaotic flows with a line equilibrium [J]. Chaos Soliton Fract, 2013, 57: 79.
[9] Sampath S, Vaidyanathan S, Volos C K, *et al*. An eight-term novel four-scroll chaotic system with cubic nonlinearity and its circuit simulation [J]. J Eng Sci Technol Rev, 2014, 8: 1.
[10] Qi G Y, Chen G R, Du S Z, *et al*. Analysis of a new chaotic system [J]. Phys A, 2005, 352: 295.
[11] Wang F Z, Qi G Y, Chen Z Q, *et al*. Analysis, circuit implementation and synchronization of a new three-dimensional chaotic system [J]. Acta Phys-sin, 2006, 55: 4005.
[12] Qi G Y, Liang X Y, Force analysis of Qi chaotic system [J]. Int J Bifurca Chaos, 2016, 26: 13.
[13] Qi G Y, Zhang J F, Energy cycle and bound of Qi chaotic system [J]. Chaos Soliton Fract, 2017, 99: 7.
[14] Rodriguez-Licea M A, Perez-Pinal F J, Nunez-Perez J C, *et al*. On the n-dimensional phase portraits [J]. Appl Sci-Basel, 2019, 9: 872.
[15] Wiggins S. Introduction to applied nonlinear dynamical systems and chaos [M]. New York: Springer-Verlag, 2003.

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