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广义三项指数和的四次幂均值

唐　睿[1]，陈　龙[1,2]

(1. 四川大学数学学院，成都 610064；2. 攀枝花学院数学与计算机学院，攀枝花 617000)

摘　要：本文利用高斯和及特征和的性质研究了关于模 p(p 为奇素数) 的一类广义三项指数和的四次幂均值，给出其在不同条件下的精确计算公式和渐近公式

关键词：广义三项指数和；特征和；四次幂均值；渐进公式

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On the fourth power mean of the generalized three-term exponential sum

TANG Rui[1], *CHEN Long*[1,2]

(1. School of Mathematics, Sichuan University, Chengdu 610064, China;
2. School of Mathematics and Computer Science, Panzhihua University, Panzhihua 617000, China)

Abstract: This paper aims at the calculation of the fourth power mean of the generalized three-term exponential sums modulo p, an odd prime. By using some properties of the classical Gaussian sums and characteristical sums, we give some exact formulae and asymptotic formulae for it.

Keywords: Generalized three-term exponential sum; Characteristical sum; Fourth power mean; Asymptotic formula
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1 Introduction

Let $q \geqslant 3$ be an integer. For any integer m and n, the two-term exponential sum $G(k,h,m,n;q)$ is defined as

$$G(k,h,m,n;q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + na^h}{q}\right),$$

where as usual, $e(y) = \mathrm{e}^{2\pi i y}$, k and n are positive integers with $k > h$. This sum plays an essential role in the research of analytic number theory. Plenty of classical problems are closely related to it. When $k = p$ is an odd prime, it is closely related to Fourier analysis on finite fields. Many researchers had researched the properties of $G(k,h,m,n;q)$ and obtained many interesting results[1-11]. For instance, Zhang and Zhang[2] proved that for any odd prime p with $(p,n)=1$, we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2, \text{ if } 3 \nmid p-1, \\ 2p^3 - 7p^2, \text{ if } 3 \mid p-1. \end{cases}$$

They[4] also obtained that

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^5+na}{p}\right)\right|^4 = \begin{cases} 3p^3 - p^2\left(8 + 2\left(\frac{-1}{p}\right) + 4\left(\frac{-3}{p}\right)\right), \text{ if } 5 \nmid p-1, \\ 3p^3 + O(p^2), \text{ if } 5 \mid p-1, \end{cases}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol mod p.

In this paper, we mainly consider the compu-

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作者简介：唐睿(1988－)，男，四川成都人，硕士，主要研究方向为数论. E-mail：976040110@qq.com

tational problem of the fourth power mean of the generalized three-term exponential sum and Kloosterman sum

$$\sum_{m=1}^{p-1}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^k+a^2+a}{p}\right)\right|^4 \qquad (1)$$

where k is a positive integer, p is an odd prime. The main tool is some properties of the Gauss sum to compute (1) for $\gcd(k,p-1)=1,2,3,4$ when $k=3,4$ and give the exact calculating formulae and asymptotic formula when $\gcd(k,p-1)$ be an odd prime. We will prove the following results

Theorem 1.1 Let p be an odd prime with $\gcd(n,p-1)=1$. Then

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4 = p(p-1)^2(2p-3).$$

Theorem 1.2 Let p be an odd prime with $gcd(n,p-1)=2$. Then

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4 = p(p-1)(2p^2-7p+8).$$

Theorem 1.3 Let p be an odd prime with $\gcd(n,p-1)=3$, integer r with $r^3\equiv 1 \pmod p$ and $r\not\equiv 1 \pmod p$. Then we have the following identity

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4 = p(p-1)(2p^2-5p+15)-4p(p-1)\tau(\chi_2)\cdot \left(\frac{1-r}{p}\right)\left[e\left(\frac{4^{-1}(r^2-r)}{p}\right)+\left(\frac{-1}{p}\right)\cdot e\left(\frac{4^{-1}(r-r^2)}{p}\right)\right]-2p(p-1)\tau(\chi_2)\cdot \left(\frac{3}{p}\right)\left[e\left(\frac{-3\cdot 4^{-1}}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{3\cdot 4^{-1}}{p}\right)\right].$$

Theorem 1.4 Let p be an odd prime with $\gcd(n,p-1)=4$ and integer r with $r^2\equiv -1 \pmod p$. Then we have the following identity

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4 = p(p-1)(2p^2-7p+24)-2p(p-1)\cdot \sqrt{p}\left[e\left(\frac{4^{-1}}{p}\right)+e\left(\frac{-4^{-1}}{p}\right)+2\left(\frac{2}{p}\right)e\left(\frac{4^{-1}r}{p}\right)+2\left(\frac{2}{p}\right)e\left(\frac{-4^{-1}r}{p}\right)\right].$$

Theorem 1.5 Let p be an odd prime and $\gcd(n,p-1)=d$ far less than p. If $2\nmid d$, then

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4 = 2p^4-7p^3+O(p^{\frac{5}{2}}).$$

2 Preliminaries

At first, we introduce some properties of the Gauss sum modulo p. Let χ be a character defined modulo p, χ_0 is the principal character modulo p, then

$$\sum_{a=0}^{p-1}\chi(a)=\begin{cases}0, & \chi\neq\chi_0,\\ p-1, & \chi=\chi_0,\end{cases}$$

$$\sum_{\chi\bmod p}\chi(a)=\begin{cases}0, & a\not\equiv 1 \pmod p;\\ p-1, & a\equiv 1 \pmod p.\end{cases}$$

By the orthogonality relations for Dirichlet characters, we have

$$\sum_{\chi\bmod p}\chi(a)\overline{\chi(b)}=\sum_{\chi\bmod p}\chi(ab^{-1})=\begin{cases}0, & a\not\equiv b \pmod p,\\ p-1, & a\equiv b \pmod p.\end{cases}$$

The Gauss sum of a Dirichlet character modulo p is

$$\tau(\chi)=\sum_{a=1}^{p}\chi(a)e\left(\frac{a}{p}\right).$$

Let χ be a primitive character mod p, we have

$$\sum_{a=1}^{p}\chi(a)e\left(\frac{na}{p}\right)=\sum_{a=1}^{p}\chi(n^{-1})\chi(na)e\left(\frac{na}{p}\right)=\chi(n^{-1})\tau(\chi).$$

If χ is a real primitive character mod p, then

$$\tau(\chi)=\begin{cases}\sqrt{p}, & \chi(-1)=1,\\ \mathrm{i}\sqrt{p}, & \chi(-1)=-1.\end{cases}$$

If χ is a n-th character mod p, then for any integer m with $(m,p)=1$ we have

$$\sum_{a=0}^{p-1}e\left(\frac{ma^n}{p}\right)=\sum_{a=0}^{p-1}(1+\chi(a)+\chi^2(a)+\cdots+\chi^{n-1}(a))e\left(\frac{ma}{p}\right).$$

Particularly,

$$\sum_{a=0}^{p-1}e\left(\frac{ma^2}{p}\right)=\sum_{a=0}^{p-1}(1+\chi_2(a))e\left(\frac{ma}{p}\right)=\sum_{a=0}^{p-1}\chi_2(a)e\left(\frac{ma}{p}\right)=\tau(\chi_2)\chi_2(m).$$

Let a and b be two integer and $\gcd(2a,N)=1$, we have

$$\left|\sum_{n=1}^{N} e\left(\frac{an^2+bn}{N}\right)\right| = \sqrt{N}.$$

Lemma 2.1 Let p be a prime with $3\mid(p-1)$, integer r with $r^3\equiv 1 \pmod p$ and $r\not\equiv 1 \pmod p$, then

$$\left|\sum_{b=1}^{p-1} e\left(\frac{b^2(r^2-1)+b(r-1)}{p}\right)\right|^2 = p+1-e\left(\frac{4^{-1}(r^2-r)}{p}\right)\sum_{a=0}^{p-1} e\left(\frac{a^2(r-r^2)}{p}\right) - e\left(\frac{4^{-1}(r-r^2)}{p}\right)\sum_{a=0}^{p-1} e\left(\frac{a^2(r^2-r)}{p}\right).$$

Proof According to the properties of the reduced residue system mod p, if b pass through a reduced residue system mod p, then $a=br^2$ and $a=a-2^{-1}$ also pass through a reduced residue system mod p. We obtain

$$\left|\sum_{b=1}^{p-1} e\left(\frac{b^2(r^2-1)+b(r-1)}{p}\right)\right|^2 = \left|\sum_{a=1}^{p-1} e\left(\frac{(a-2^{-1})^2(r-r^2)-4^{-1}(r-r^2)}{p}\right)\right|^2 = p+1-e\left(\frac{4^{-1}(r^2-r)}{p}\right)\sum_{a=0}^{p-1} e\left(\frac{a^2(r-r^2)}{p}\right) - e\left(\frac{4^{-1}(r-r^2)}{p}\right)\sum_{a=0}^{p-1} e\left(\frac{a^2(r^2-r)}{p}\right).$$

This proves the lemma.

Lemma 2.2 Let p be a prime with $3\mid(p-1)$, integer r with $r^3\equiv 1 \pmod p$ and $r\not\equiv 1 \pmod p$, then

$$\sum_{\substack{a=1\\ a\neq 1,r,r^2}}^{p-1}\sum_{c=1}^{p-1} e\left(\frac{c^2(r^2-1)(a^2-1)+c(r-1)(a-1)}{p}\right) = 4-\tau(\chi_2)\left(\frac{1-r}{p}\right)\left[e\left(\frac{4^{-1}(r^2-r)}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{-4^{-1}(r^2-r)}{p}\right)\right] - \tau(\chi_2)\left(\frac{3}{p}\right)\left[e\left(\frac{-3\cdot 4^{-1}}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{3\cdot 4^{-1}}{p}\right)\right].$$

Proof According to the properties of the reduced residue system mod p, if c pass through a reduced residue system mod p, then cr also pass through a reduced residue system mod p and $r^2+r+1\equiv 0 \pmod p$. Thus we obtain

$$\sum_{\substack{a=1\\ a\neq 1,r,r^2}}^{p-1}\sum_{c=1}^{p-1} e\left(\frac{c^2(r^2-1)(a^2-1)+c(r-1)(a-1)}{p}\right) = \sum_{a=1}^{p-1}\sum_{c=1}^{p-1} e\left(\frac{c^2(r^2-1)(a^2-1)+c(r-1)(a-1)}{p}\right) - (p-1) - \sum_{c=1}^{p-1} e\left(\frac{-3c^2r^2-3cr}{p}\right) - \sum_{c=1}^{p-1} e\left(\frac{3c^2+3c}{p}\right) = -1+\sum_{a=2}^{p-2}\sum_{c=1}^{p-1} e\left(\frac{c^2(r^2-1)(a^2-1)+c(r-1)(a-1)}{p}\right) - \sum_{c=1}^{p-1} e\left(\frac{-3c^2-3c}{p}\right) - \sum_{c=1}^{p-1} e\left(\frac{3c^2+3c}{p}\right) \quad (2)$$

From the properties of Gauss sum we know that

$$\sum_{a=0}^{p-1} e\left(\frac{ma^2}{p}\right) = \tau(\chi_2)\chi_2(m).$$

From the properties of the reduced residue system mod p, we have

$$\sum_{a=2}^{p-2}\sum_{c=1}^{p-1} e\left(\frac{c^2(r^2-1)(a^2-1)+c(r-1)(a-1)}{p}\right) = \sum_{a=2}^{p-2}\sum_{c=0}^{p-1} e\left[\frac{(r^2-r)\dfrac{(a-1)}{4(a+1)}}{p}\right] e\left(\frac{(r^2-r)(a^2-1)c^2}{p}\right) - (p-3) = \sum_{a=2}^{p-2} e\left[\frac{(r^2-r)\dfrac{(a-1)}{4(a+1)}}{p}\right]\chi_2((r-r^2)\cdot(a^2-1))\tau(\chi_2) - (p-3) = \tau(\chi_2)\left(\frac{1-r}{p}\right)\cdot \sum_{a=2}^{p-2} e\left[\frac{(r^2-r)\dfrac{(a-1)}{4(a+1)}}{p}\right]\chi_2\left(\frac{a-1}{4(a+1)}\right) - (p-3).$$

Let $b=a+1=3,4,\cdots,p-1$. We have $\dfrac{(a-1)}{4(a+1)}=\dfrac{b-2}{4b}=4^{-1}-(2b)^{-1}$. Let $a=(2b)^{-1}$. If b pass through a reduced residue system mod p, then $a=(2b)^{-1}$ also pass through a reduced residue system mod p, so

$$\tau(\chi_2)\left(\frac{1-r}{p}\right)\sum_{b=3}^{p-1} e\left(\frac{(r^2-r)(4^{-1}-2^{-1}b^{-1})}{p}\right)\cdot \chi_2(4^{-1}-2^{-1}b^{-1}) - (p-3) = \tau(\chi_2)\cdot \left(\frac{1-r}{p}\right)\sum_{a=0}^{p-1} e\left(\frac{(r^2-r)(4^{-1}-a)}{p}\right)\cdot \chi_2(4^{-1}-a) - \tau(\chi_2)\left(\frac{1-r}{p}\right)e\left(\frac{(r^2-r)4^{-1}}{p}\right)\cdot \chi_2(4^{-1}) - \tau(\chi_2)\left(\frac{1-r}{p}\right)e\left(\frac{-(r^2-r)4^{-1}}{p}\right)\cdot \chi_2(-4^{-1}) - \tau(\chi_2)\left(\frac{1-r}{p}\right)e(0)\chi_2(0) - (p-3) =$$

$$3-\tau(\chi_2)\left(\frac{1-r}{p}\right)e\left(\frac{(r^2-r)4^{-1}}{p}\right)-$$
$$\tau(\chi_2)\left(\frac{r-1}{p}\right)e\left(\frac{-(r^2-r)4^{-1}}{p}\right),$$

the thirth part and fourth parts of (2). Then, from the properties of Gauss sum, we get

$$\sum_{c=1}^{p-1}e\left(\frac{-3c^2-3c}{p}\right)+\sum_{c=1}^{p-1}e\left(\frac{3c^2+3c}{p}\right)=$$
$$\tau(\chi_2)\left(\frac{3}{p}\right)\left[e\left(\frac{-3\cdot 4^{-1}}{p}\right)+\right.$$
$$\left.\left(\frac{-1}{p}\right)e\left(\frac{-3\cdot 4^{-1}}{p}\right)\right]-2.$$

This proves the lemma.

Lemma 2.3 Let p be an odd prime with $4\mid(p-1)$, integer r with $r^2\equiv-1\pmod p$, then

$$\left|\sum_{b=1}^{p-1}e\left(\frac{-2b^2+b(r-1)}{p}\right)\right|^2=$$
$$p+1-\sqrt{p}\left(\frac{2}{p}\right)\left[e\left(\frac{4^{-1}r}{p}\right)+e\left(\frac{-4^{-1}r}{p}\right)\right].$$

Proof According to the properties of the reduced residue system mod p, if b pass through a reduced residue system mod p, then $a=b^{-1}$ also pass through a reduced residue system mod p. From the properties of Gauss sum we know that

$$\sum_{a=0}^{p-1}e\left(\frac{ma^2}{p}\right)=\sum_{a=0}^{p-1}\chi_2(a)e\left(\frac{ma}{p}\right).$$

Then, from the orthogonality of the characters mod p, we obtain that

$$\left|\sum_{b=1}^{p-1}e\left(\frac{-2b^2+b(r-1)}{p}\right)\right|^2=$$
$$\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{-2c^2(b^2-1)+(r-1)c(b-1)}{p}\right)=$$
$$1+\sum_{b=2}^{p-2}e\left(\frac{\frac{(r-1)^2}{4}\cdot\frac{b-1}{2(b+1)}}{p}\right)\tau(\chi_2)\chi_2(2-2b^2)=$$
$$p+1-\sqrt{p}\left(\frac{2}{p}\right)\left[e\left(\frac{4^{-1}r}{p}\right)+e\left(\frac{-4^{-1}r}{p}\right)\right].$$

Lemma 2.4 Let p be an odd prime with $4\mid(p-1)$, integer r with $r^2\equiv-1\pmod p$, then we have

$$\sum_{\substack{a=2\\a\neq r,r^3}}^{p-2}\sum_{c=1}^{p-1}e\left(\frac{-2c^2(a^2-1)+c(r-1)(a-1)}{p}\right)=$$
$$5-\sqrt{p}\left[e\left(\frac{4^{-1}r}{p}\right)\chi_2(2)+e\left(\frac{4^{-1}}{p}\right)+\right.$$
$$\left.e\left(\frac{-4^{-1}r}{p}\right)\chi_2(2)+e\left(\frac{-4^{-1}}{p}\right)\right].$$

Proof According to the properties of the reduced residue system mod p, if a pass through a reduced residue system mod p, then $b=4^{-1}a^{-1}$ also pass through a reduced residue system mod p. From the properties of Guass sums and reduced system, we obtain

$$\sum_{\substack{a=2\\a\neq r,r^3}}^{p-2}\sum_{c=1}^{p-1}e\left(\frac{-2c^2(a^2-1)+c(r-1)(a-1)}{p}\right)=$$
$$\sum_{\substack{a=2\\a\neq r,r^3}}^{p-2}e\left(\frac{(r-1)^2\cdot\frac{a-1}{8(a+1)}}{p}\right)\cdot$$
$$\tau(\chi_2)\chi_2(-2(a^2-1))-(p-5)=$$
$$5-\sqrt{p}\left[e\left(\frac{4^{-1}r}{p}\right)\chi_2(2)+e\left(\frac{4^{-1}}{p}\right)+\right.$$
$$\left.e\left(\frac{-4^{-1}r}{p}\right)\chi_2(2)+e\left(\frac{-4^{-1}}{p}\right)\right].$$

The proof is end.

3 The proof of the main results

Proof of Theorem 1.1 By the orthogonality of characters mod p, we obtain

$$\sum_{m=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4=$$
$$(p-1)\sum_{m=1}^{p}\sum_{a=1}^{p-1}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^n(a^n-1)+b^2(a^2-1)+b(a-1)}{p}\right)\right|^2=$$
$$p(p-1)^3+(p-1)\sum_{m=1}^{p}\sum_{a=2}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{mc^n(b^n-1)(a^n-1)+c^2(b^2-1)(a^2-1)+c(b-1)(a-1)}{p}\right)=$$
$$p(p-1)^2(2p-3).$$

This finishes the proof of Theorem 1.1.

Proof of Theorem 1.2 By the orthogonality of characters mod p, we obtain

$$\sum_{m=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4=$$
$$(p-1)\sum_{m=1}^{p}\sum_{a=1}^{p-1}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^n(a^n-1)+b^2(a^2-1)+b(a-1)}{p}\right)\right|^2=p(p-1)^3+p(p-1)+$$
$$(p-1)\sum_{m=1}^{p}\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{mc^n(b^n-1)(a^n-1)+c^2(b^2-1)(a^2-1)+c(b-1)(a-1)}{p}\right)=$$
$$p(p-1)(2p^2-7p+8).$$

This finishes the proof of Theorem 1.2.

Proof of Theorem 1.3 By the orthogonality of characters mod p, we obtain

$$\sum_{m=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4=p(p-1)^3+p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{b^2(r^2-1)+b(r-1)}{p}\right)\right|^2+$$
$$p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{b^2(r-1)+b(r^2-1)}{p}\right)\right|^2+$$
$$(p-1)\sum_{\substack{a=1\\a\neq 1,r,r^2}}^{p-1}\sum_{m=1}^{p}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^n(a^n-1)+b^2(a^2-1)+b(a-1)}{p}\right)\right|^2 \tag{3}$$

According to Lemma 2.2, we can calculate the second part and third part of (3) as follows.

$$p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{b^2(r^2-1)+b(r-1)}{p}\right)\right|^2=p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{b^2(r-1)+b(r^2-1)}{p}\right)\right|^2=$$
$$p(p-1)(p+1)-p(p-1)\tau(\chi_2)\left(\frac{1-r}{p}\right)\left[e\left(\frac{4^{-1}(r^2-r)}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{4^{-1}(r-r^2)}{p}\right)\right].$$

According to Lemma 2.3, we focus on the fourth part of (3), which is equal to

$$(p-1)\sum_{\substack{a=1\\a\neq 1,r,r^2}}^{p-1}\sum_{m=1}^{p}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^n(a^n-1)+b^2(a^2-1)+b(a-1)}{p}\right)\right|^2=$$
$$(p-1)\sum_{\substack{a=1\\a\neq 1,r,r^2}}^{p-1}\sum_{m=1}^{p}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{mc^n(b^n-1)(a^n-1)+c^2(b^2-1)(a^2-1)+c(b-1)(a-1)}{p}\right)=$$
$$p(p-1)(p^2-5p+12)-2p(p-1)\tau(\chi_2)\left(\frac{1-r}{p}\right)\left[e\left(\frac{4^{-1}(r^2-r)}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{-4^{-1}(r^2-r)}{p}\right)\right]-$$
$$2p(p-1)\tau(\chi_2)\left(\frac{3}{p}\right)\left[e\left(\frac{-3\cdot 4^{-1}}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{3\cdot 4^{-1}}{p}\right)\right].$$

Thus

$$\sum_{m=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4=$$
$$p(p-1)(2p^2-5p+15)-4p(p-1)\tau(\chi_2)\left(\frac{1-r}{p}\right)\left[e\left(\frac{4^{-1}(r^2-r)}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{4^{-1}(r-r^2)}{p}\right)\right]-$$
$$2p(p-1)\tau(\chi_2)\left(\frac{3}{p}\right)\left[e\left(\frac{-3\cdot 4^{-1}}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{3\cdot 4^{-1}}{p}\right)\right].$$

This finishes the proof of Theorem 1.3.

Proof of Theorem 1.4 Let $1, r, r^3, -1$ be different root of $x^4\equiv 1 \pmod p$. By the orthogonality of characters mod p, we obtain

$$\sum_{m=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4=$$

$$(p-1)\sum_{a=1}^{p-1}\sum_{m=1}^{p}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^{n}(a^{n}-1)+b^{2}(a^{2}-1)+b(a-1)}{p}\right)\right|^{2}=$$
$$p(p-1)(p^{2}-2p+2)+p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{-2b^{2}+b(r-1)}{p}\right)\right|^{2}+p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{-2b^{2}+b(r^{3}-1)}{p}\right)\right|^{2}+$$
$$(p-1)\sum_{m=1}^{p}\sum_{\substack{a=2\\ a\neq r,r^{3}}}^{p-2}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^{n}(a^{n}-1)+b^{2}(a^{2}-1)+b(a-1)}{p}\right)\right|^{2}\qquad(4)$$

According to Lemma 2.3, we can calculate the sum of the second and the third part of equation (4) as follows.

$$p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{-2b^{2}+b(r-1)}{p}\right)\right|^{2}+p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{-2b^{2}+b(r^{3}-1)}{p}\right)\right|^{2}=2p(p-1)(p+1)-2p(p-1)\sqrt{p}\left(\frac{2}{p}\right)\left[e\left(\frac{4^{-1}r}{p}\right)+e\left(\frac{-4^{-1}r}{p}\right)\right].$$

According to Lemma 2.4, we can obtain the fourth part of equation (4) as follows.

$$(p-1)\sum_{m=1}^{p}\sum_{\substack{a=2\\ a\neq r,r^{3}}}^{p-2}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^{n}(a^{n}-1)+b^{2}(a^{2}-1)+b(a-1)}{p}\right)\right|^{2}=$$
$$(p-1)\sum_{m=1}^{p}\sum_{\substack{a=2\\ a\neq r,r^{3}}}^{p-2}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{mc^{n}(b^{n}-1)(a^{n}-1)+c^{2}(b^{2}-1)(a^{2}-1)+c(b-1)(a-1)}{p}\right)=$$
$$p(p-1)(p^{2}-7p+20)-2p(p-1)\sqrt{p}\left[e\left(\frac{4^{-1}r}{p}\right)\chi_{2}(2)+e\left(\frac{4^{-1}}{p}\right)+e\left(\frac{-4^{-1}}{p}\right)+e\left(\frac{-4^{-1}r}{p}\right)\chi_{2}(2)\right].$$

Hence

$$\sum_{m=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^{n}+a^{2}+a}{p}\right)\right|^{4}=p(p-1)(p^{2}-7p+24)-2p(p-1)\sqrt{p}\cdot\left[e\left(\frac{4^{-1}}{p}\right)+e\left(\frac{-4^{-1}}{p}\right)+2\left(\frac{2}{p}\right)e\left(\frac{4^{-1}r}{p}\right)+2\left(\frac{2}{p}\right)e\left(\frac{-4^{-1}r}{p}\right)\right].$$

This finishes the proof of Theorem 1.4.

Proof of Theorem 1.5 Let d be the smallest positive integer such that $r^{d}\equiv 1(\bmod p)$. By the orthogonality of characters mod p, we obtain

$$\sum_{m=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^{n}+a^{2}+a}{p}\right)\right|^{4}=$$
$$(p-1)\sum_{a=1}^{p-1}\sum_{m=1}^{p}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^{n}(a^{n}-1)+b^{2}(a^{2}-1)+b(a-1)}{p}\right)\right|^{2}=p(p-1)^{3}$$
$$+p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{b^{2}(r^{2}-1)+b(r-1)}{p}\right)\right|^{2}+p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{b^{2}(r^{4}-1)+b(r^{2}-1)}{p}\right)\right|^{2}+\cdots$$
$$+p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{b^{2}(r^{2d-2}-1)+b(r^{d-1}-1)}{p}\right)\right|^{2}+$$
$$p(p-1)\sum_{m=1}^{p}\sum_{\substack{a=1\\ a\neq 1,r,r^{2},\cdots,r^{d-1}}}^{p-1}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^{n}(a^{n}-1)+b^{2}(a^{2}-1)+b(a-1)}{p}\right)\right|^{2}\qquad(5)$$

Then we focus on the second part of (5). By the method similar to Lemma 2.3, we obtain

$$p(p-1)\left|\sum_{b=1}^{p-1}e\left(\frac{b^{2}(r^{2}-1)+b(r-1)}{p}\right)\right|^{2}=$$

$$p(p-1)\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{c^2(b^2-1)(r^2-1)+c(b-1)(r-1)}{p}\right)=$$

$$p(p-1)+p(p-1)\tau(\chi_2)\sum_{b=2}^{p-2}e\left(\frac{-\frac{(b-1)(r-1)}{4(b+1)(r+1)}}{p}\right)\chi_2((b^2-1)(r^2-1))=p^3+O(p^{\frac{5}{2}}),$$

and

$$-2(p^{\frac{5}{2}}-p^{\frac{3}{2}})-p\leqslant O(p^{\frac{5}{2}})\leqslant 2(p^{\frac{5}{2}}-p^{\frac{3}{2}})-p.$$

The rest of the proof is similar to the proofs of Theorem 1.1～Theorem 1.4, except the proof of the last part of (5). By orthogonality of characters mod p, we know

$$(p-1)\sum_{m=1}^{p}\sum_{\substack{a=1\\ a\neq 1,r,r^2,\cdots,r^{d-1}}}^{p-1}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^n(a^n-1)+b^2(a^2-1)+b(a-1)}{p}\right)\right|^2=$$

$$(p-1)\sum_{m=1}^{p}\sum_{\substack{a=1\\ a\neq 1,r,r^2,\cdots,r^{d-1}}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{mc^n(b^n-1)(a^n-1)+c^2(b^2-1)(a^2-1)+c(b-1)(a-1)}{p}\right)=$$

$$p(p-1)^2(p-d-1)+p(p-1)\sum_{\substack{a=1\\ a\neq 1,r,r^2,\cdots,r^{d-1}}}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{c^2(r^2-1)(a^2-1)+c(r-1)(a-1)}{p}\right)+\cdots+$$

$$p(p-1)\sum_{\substack{a=1\\ a\neq 1,r,r^2,\cdots,r^{d-1}}}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{c^2(r^{2p_1-2}-1)(a^2-1)+c(r^{p_1-1}-1)(a-1)}{p}\right)\tag{6}$$

In what follows we calculate the second part of (6) the rest parts can be done in the same way. From the properties of Gauss sum and residues system, we know the second part is

$$p(p-1)\sum_{\substack{a=1\\ a\neq 1,r,r^2,\cdots,r^{d-1}}}^{p-1}\sum_{c=1}^{p-1}e\left(\frac{c^2(r^2-1)(a^2-1)+c(r-1)(a-1)}{p}\right)=$$

$$p(p-1)\tau(\chi_2)\sum_{\substack{a=1\\ a\neq 1,r,r^2,\cdots,r^{d-1}}}^{p-1}e\left(\frac{-\frac{(r-1)(a-1)}{4(r+1)(a+1)}}{p}\right)\cdot$$

$$\chi_2((r^2-1)(a^2-1))-p(p-1)(p-d-1)=O(p^{\frac{5}{2}}),$$

where

$$(d+1)p(p-1)-d(p^{\frac{5}{2}}-p^{\frac{3}{2}})\leqslant O(p^{\frac{5}{2}})\leqslant(d+1)p(p-1)+d(p^{\frac{5}{2}}-p^{\frac{3}{2}}).$$

Hence, the last part of (4) is

$$(p-1)\sum_{m=1}^{p}\sum_{\substack{a=1\\ a\neq 1,r,r^2,\cdots,r^{d-1}}}^{p-1}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^n(a^n-1)+b^2(a^2-1)+b(a-1)}{p}\right)\right|^2=$$

$$p(p-1)^2(p-d-1)+O(p^{\frac{5}{2}})=p^4-(d+3)p^3+O(p^{\frac{5}{2}}),$$

where

$$(2d+3)p^2-(d+1)p+(d^2-1)p(p-1)-d(d-1)(p^{\frac{5}{2}}-p^{\frac{3}{2}})\leqslant O(p^{\frac{5}{2}})\leqslant$$

$$(2d+3)p^2-(d+1)p+(d^2-1)p(p-1)+d(d-1)(p^{\frac{5}{2}}-p^{\frac{3}{2}}).$$

Thus

$$\sum_{m=1}^{p}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4=$$

$$2p^4-7p^3+O(p^{\frac{5}{2}}),$$

where

$$-(d+2)(d-1)(p^{\frac{5}{2}}-p^{\frac{3}{2}})+(d^2+2d+2)p^2-(d^2+2d-1)p\leqslant O(p^{\frac{5}{2}})\leqslant(d+2)(d-1)(p^{\frac{5}{2}}-p^{\frac{3}{2}})+(d^2+2d+2)p^2-(d^2+2d-1)p.$$

This finishes the proof.

Corollary 3.1 Let p is a prime with $3\nmid(p-1)$, then

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a^2+a}{p}\right)\right|^4=p(p-1)^2(2p-3).$$

Corollary 3.2 Let p is a prime with $3\mid(p-1)$, integer r with $r^3\equiv1(\bmod p)$ and $r\not\equiv1(\bmod p)$, then

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a^2+a}{p}\right)\right|^4=p(p-1)(2p^2-5p+15)-4p(p-1)\tau(\chi_2)\cdot\left(\frac{1-r}{p}\right)\left[e\left(\frac{4^{-1}(r^2-r)}{p}\right)+\left(\frac{-1}{p}\right)\cdot e\left(\frac{4^{-1}(r-r^2)}{p}\right)\right]-2p(p-1)\tau(\chi_2)\left(\frac{3}{p}\right)\cdot\left[e\left(\frac{-3\cdot4^{-1}}{p}\right)+\left(\frac{-1}{p}\right)e\left(\frac{3\cdot4^{-1}}{p}\right)\right].$$

Corollary 3.3 For any prime p, we have the following asymptotic formula

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a^2+a}{p}\right)\right|^4=2p^4-7p^3+O(p^{\frac{5}{2}}).$$

Corollary 3.4 Let p be an odd prime with $4\nmid(p-1)$, then

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+a^2+a}{p}\right)\right|^4=p(p-1)(2p^2-7p+8).$$

Corollary 3.5 Let p is a prime with $4\mid(p-1)$, integer r with $r^2\equiv-1(\bmod p)$, then

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+a^2+a}{p}\right)\right|^4=p(p-1)(2p^2-7p+24)-2p(p-1)\cdot\sqrt{p}\left[e\left(\frac{4^{-1}}{p}\right)+e\left(\frac{-4^{-1}}{p}\right)+2\left(\frac{2}{p}\right)e\left(\frac{4^{-1}r}{p}\right)+2\left(\frac{2}{p}\right)e\left(\frac{-4^{-1}r}{p}\right)\right].$$

Corollary 3.6 For any prime p, such that the following asymptotic formula

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+a^2+a}{p}\right)\right|^4=2p^4-9p^3+O(p^{\frac{5}{2}}).$$

Problem In Theorem 1.5, we discussed the $2\mid d$. when $2\nmid d$, the situation will be more complicated. By now we still have not found an effective approach to calculate it up to now, so it is still an open problem.

On the other hard, we can only obtain the asymptotic formula for Theorem 1.5.

Conjecture We leave the following conjecture : let p be an odd prime and $gcd(n,p-1)=d$ is a fixed positive integer. If $2\mid d$, then we have the asymptotic formula

$$\sum_{m=1}^{p}\sum_{\chi\bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^n+a^2+a}{p}\right)\right|^4=2p^4-9p^3+O(p^{\frac{5}{2}}).$$

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