doi：10.3969/j.issn.0490-6756.2020.05.007

关于第二类 Stirling 数的 p-adic 赋值的一些新结果

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摘 要：设 n 和 k 为任意正整数. 第二类 Stirling 数，记作 $S(n,k)$，表示将 n 个元素划分为恰好 k 个非空集合的个数. 设 p 为奇素数，令 $v_p(n)$ 表示 n 的 p-adic 赋值，即 $v_p(n)$ 是能整除 n 的最大的 p 的方幂. 一般来说，计算 $S(n,k)$ 的 p-adic 赋值是很困难的. 有许多作者研究了第二类 Stirling 数 $S(n,k)$ 的算术性质，包括 Davis，Lengyel 以及 Hong 等. 在本文中，我们研究第二类 Stirling 数的 p-adic 赋值的一些性质. 事实上，我们通过对 $S(n,k)$ 进行 p-adic 分析证明了 $v_p(S(p,2))\geqslant 1$，其中等号成立当且仅当 p 为一个 Wieferich 素数. 当 $n\geqslant 2$ 时，我们还证明了 $v_p(S(p^n,2p))\geqslant n$，以及 $v_p(S(p^n,4p))\geqslant n-2(p\geqslant 5)$，这改进了 Adelberg 不久前的结果.

关键词：第二类 Stirling 数；p-adic 赋值；同余式；Wieferich 素数
中图分类号：O156.1 **文献标识码**：A **文章编号**：0490-6756(2020)05-0865-06

Some new results on the p-adic valuations of Stirling numbers of the second kind

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Abstract: Let n and k be positive integers. The Stirling number of the second kind, denoted by $S(n,k)$, is defined as the number of ways to partition a set of n elements into exactly k nonempty subsets. Let p be an odd prime, $v_p(n)$ stand for the p-adic valuation of n, *i.e.*, $v_p(n)$ is the biggest nonnegative integer r with p^r dividing n. It is difficult to evaluate $v_p(S(n,k))$ in general. There are many authors including Davis, Lengyel, Hong *et al*. who investigated $v_p(S(n,k))$. In this paper, we consider the p-adic valuations of some special Stirling numbers of the second kind. In fact, by providing a p-adic analysis of $S(n,k)$, we show that $v_p(S(p,2))\geqslant 1$ with the equality holding if and only if p is a Wieferich prime. For $n\geqslant 2$, we also prove that $v_p(S(p^n,2p))\geqslant n$, and $v_p(S(p^n,4p))\geqslant n-2$ with $p\geqslant 5$, which improve the result given by Adelberg recently.

Keywords: Stirling number of the second kind; p-adic valuation; Congruence; Wieferich prime
(2010 MSC 11B73, 11A07)

1 Introduction

Let n and k be nonnegative integers. The Stirling number of the first kind, denoted by $s(n,k)$, counts the number of permutations of n elements with k disjoint cycles. One can also

收稿日期：2020-04-21
基金项目：国家自然科学基金(11771304)
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characterize $s(n,k)$ by

$$(x)_n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k,$$

where $(x)_n$ is the falling factorial which is defined by

$$(x)_n := x(x-1)(x-2)\cdots(x-n+1)$$

for $n \geqslant 1$ and $(x)_0 := 1$. The Stirling number of the second kind is defined as the number of ways to partition a set of n elements into exactly k non-empty subsets, and we have

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \qquad (1)$$

where $\binom{k}{i}$ represents the binomial coefficient, which is defined by

$$\binom{k}{i} = \frac{(k)_i}{i!} = \frac{k!}{i!\ (k-i)!},\ k \geqslant i.$$

Note that the exact value of Stirling number of the second kind $S(n,k)$ equals the sum of all products of $n-k$ not necessarily distinct integers from $\{1, 2, \cdots, k\}$, *i. e.*, the following explicit formula holds:

$$S(n,k) = \sum_{\substack{c_1+c_2+\cdots+c_k=n-k \\ (c_1,c_2,\cdots,c_k)\in \mathbf{N}^k}} 1^{c_1} 2^{c_2} \cdots k^{c_k},$$

where **N** stands for the set of all the nonnegative integers. One can also characterize the Stirling number of the second kind by

$$x^n = \sum_{k=0}^{\infty} S(n,k)\ (x)_k,$$

and there holds the recurrence relation

$$S(0,0)=1,\ S(n,0)=0$$

and

$$S(n,k) = kS(n-1,k) + S(n-1,k-1)\ ,n \geqslant k \geqslant 1.$$

Furthermore, we have the following two generating functions of Stirling numbers of the second kind:

$$(e^x - 1)^k = k! \sum_{j=k}^{\infty} S(j,k) \frac{x^j}{j!},$$

$$\prod_{i=1}^{k} \frac{1}{1-ix} = \sum_{j=0}^{\infty} S(j+k,k) x^j.$$

The Stirling numbers of the first and second kind can be considered to be inverse of one another:

$$\sum_{i\geqslant 0} (-1)^{n-i} s(n,i) \cdot S(i,k) = \sum_{i\geqslant 0} (-1)^{i-k} S(n,i) \cdot s(i,k) = \delta_{nk},$$

where δ_{nk} is the Kronecker delta function, which is defined by $\delta_{nk} := 1$ if $n=k$ and $\delta_{nk} := 0$ if $n \neq k$. See Refs. [1-14] for more results on this topic.

Divisibility properties of Stirling numbers have been studied from a number of different perspectives. Amdeberhan, Manna and Moll[1] studied the 2-adic valuations of Stirling numbers of the second kind, they also conjectured that $v_2(S(4n,5)) \neq v_2(S(4n+3,5))$ if and only if $n \in \{32j+7 : j \in \mathbf{N}\}$. Hong *et al*. proved this conjecture in Ref. [7]. Lengyel[9] conjectured, proved by Wannemacker[12], a special case of the 2-adic valuation of $S(n,k)$: $v_2(S(2^n,k)) = s_2(k) - 1$, independently of n, where $s_2(k)$ means the base 2 digital sum of k. By using Wannemacker's result, Hong *et al*.[7] proved that

$$v_2(S(2^n+1,k+1)) = s_2(k) - 1$$

holds for all k with $1 \leqslant k \leqslant 2^n$, which confirmed another conjecture of Amdeberhan, Manna and Moll[1]. We also note that the 2-adic valuation of the Stirling number of the second kind was studied by Zhao, Hong and Zhao in Refs. [13-14].

Given a prime p and a nonzero integer m, there exist unique integers a and r, with $p \nmid a$ and $r \geqslant 0$, such that $m = ap^r$. The number r is called the p-adic valuation of m, denoted by $r = v_p(m)$. Define $v_p(0) := \infty$. If $x = \frac{m_1}{m_2}$, where m_1 and m_2 are integers and $m_2 \neq 0$, then we define $v_p(x) := v_p(m_1) - v_p(m_2)$ (see, for example, Ref. [15]). It is easy to see that

$$v_p(m_1 m_2) = v_p(m_1) + v_p(m_2).$$

Furthermore, for any rational number x and y we have

$$v_p(x+y) \geqslant \min\{v_p(x), v_p(y)\},$$

and if $v_p(x) \neq v_p(y)$ then one has

$$v_p(x+y) = \min\{v_p(x), v_p(y)\}.$$

The above property is also known as the isosceles triangle principle[16].

For every odd prime p, we have $2^{p-1} \equiv 1 \pmod p$. An odd prime p such that $2^{p-1} \not\equiv 1$(mod

p^2) is called a Wieferich prime, see Ref. [17]. For example, 3, 5 and 7 are Wieferich primes. We have the following result, which gives a new method of determining Wieferich prime.

Theorem 1.1 For any odd prime p, we have

$$v_p(S(p,2))\geqslant 1,$$

where the equality holds if and only if p is a Wieferich prime.

Let p be an odd prime. For any given real number y, let $\lceil y\rceil$ be the smallest integer no less than y. Recently, by using the study of the higher order Bernoulli numbers $B_n^{(l)}$, Adelberg[18] proved that

$$v_p(S(n,k))\geqslant\left\lceil\frac{s_p(k)-s_p(n)}{p-1}\right\rceil \tag{2}$$

where $s_p(k)$ and $s_p(n)$ stand for the base p digital sum of k and n, respectively. Now for the case that n is a power of p, we arrive at the following two results, which improve Adelberg's result in this case, and are the main results of this paper.

Theorem 1.2 Let n be an integer with $n\geqslant 2$. For any odd prime p, we have

$$v_p(S(p^n,2p))\geqslant n.$$

Theorem 1.3 Let n be an integer with $n\geqslant 2$. For any prime $p\geqslant 5$, we have

$$v_p(S(p^n,4p))\geqslant n-2.$$

Evidently, for $n\geqslant 4$, the lower bounds of $v_p(S(p^n,2p))$ and $v_p(S(p^n,4p))$ in Theorems 1.2 and 1.3 are better than Adelberg's result (2).

We organize this paper as follows. Firstly, in Section 2 we show some preliminary lemmas which are needed in the proofs of Theorems 1.1 to 1.3. Then in Section 3, we give the proofs of Theorems 1.1 to 1.3.

2 Preliminaries

In this section, we present several auxiliary lemmas that are needed.

Let n and k be positive integers. By convention, we set $S(0,0)=1$ and $S(n,0)=S(0,k)=0$. It is also clear to see that $S(n,k)=0$ if $n<k$ and $S(n,n)=1$. Equation (1) also gives us that

$$S(n,1)=1$$

and

$$S(n,2)=2^{n-1}-1,\ n\geqslant 2 \tag{3}$$

Let m be a positive integer. The Euler phi function $\varphi(m)$ counts the number of integers in the set $\{1,\ldots,m\}$ that are relatively prime to m. For example, we have $\varphi(1)=1$ and $\varphi(6)=2$. If p is a prime number, then $(a,p)=1$ holds for any a with $a\in\{1,\ldots,p-1\}$, and so $\varphi(p)=p-1$. Let $r\geqslant 2$ be an integer. If p^r is a prime power, then $\varphi(p^r)=p^r-p^{r-1}$.

Lemma 2.1 (Euler) Let m be a positive integer and let a be an integer relatively prime to m. Then

$$a^{\varphi(m)}\equiv 1\pmod m.$$

The following result plays a crucial role in the proofs of Theorem 1.2 and Theorem 1.3.

Lemma 2.2 Let p be an odd prime. Let n, k, i be positive integers such that $1\leqslant k\leqslant p-1$, $1\leqslant i\leqslant kp-1$ and $(i,p)=1$. We have

$$v_p((kp-i)^{p^n}+i^{p^n})=n+1.$$

Proof Let i be an integer with $1\leqslant i\leqslant kp-1$ and $(i,p)=1$. Since p is odd, we can deduce that

$$\begin{aligned}(kp-i)^{p^n}+i^{p^n}&=\sum_{j=0}^{p^n}\binom{p^n}{j}(kp)^j(-i)^{p^n-j}+i^{p^n}=\\&\sum_{j=1}^{p^n}\binom{p^n}{j}(kp)^j(-i)^{p^n-j}+(-i)^{p^n}+i^{p^n}=\\&\sum_{j=1}^{p^n}\binom{p^n}{j}(kp)^j(-i)^{p^n-j}=\\&kp^{n+1}\cdot i^{p^n-1}+\sum_{j=2}^{p^n}\binom{p^n}{j}(kp)^j(-i)^{p^n-j}\end{aligned} \tag{4}$$

For any integer j with $1\leqslant j\leqslant p^n$, it follows from $(i,p)=1$ that

$$v_p(i^{p^n-j})=(p^n-j)v_p(i)=0 \tag{5}$$

Since $1\leqslant k\leqslant p-1$, one then derives that

$$v_p(kp^{n+1}\cdot i^{p^n-1})=v_p(k)+v_p(p^{n+1})+v_p(i^{p^n-1})=n+1 \tag{6}$$

In what follows, let $2\leqslant j\leqslant p^n$. Note that

$$\binom{p^n}{j}=\frac{p^n(p^n-1)\cdots(p^n-(j-1))}{j!}$$

and $v_p(j')<n$ holds for any integer j' with $1\leqslant j'\leqslant j-1\leqslant p^n-1$, which infers that $v_p(p^n-j')=v_p(j')$. Thus we obtain that

$$v_p\left(\binom{p^n}{j}\right)=v_p(p^n)-v_p(j)=n-v_p(j) \tag{7}$$

Then it follows from (5) together with $1\leqslant k\leqslant p-1$ and (7) that

$$v_p(\binom{p^n}{j}(kp)^j(-i)^{p^n-j})=v_p(\binom{p^n}{j})+j+v_p(i^{p^n-j})=n+j-v_p(j) \quad (8)$$

It is easy to check that $j-v_p(j)\geqslant 2$. In fact, for the case that $v_p(j)=0$, we have $j-v_p(j)\geqslant 2$ since $j\geqslant 2$, and if $v_p(j)\geqslant 1$, then by $p\geqslant 3$ one deduces that $j\geqslant p^{v_p(j)}\geqslant v_p(j)+2$. Hence by (8) and (6) we derive that

$$v_p(\sum_{j=2}^{p^n}\binom{p^n}{j}(kp)^j(-i)^{p^n-j})\geqslant \min_{2\leqslant j\leqslant p^n}\{v_p(\binom{p^n}{j}(kp)^j(-i)^{p^n-j})\}= n+j-v_p(j)\geqslant n+2>n+1= v_p(kp^{n+1}\cdot i^{p^n-1}) \quad (9)$$

Using the isosceles triangle principle together with (4), (6) and (9), one then arrives at

$$v_p((kp-i)^{p^n}+i^{p^n})=v_p(kp^{n+1}\cdot i^{p^n-1}+\sum_{j=2}^{p^n}\binom{p^n}{j}(kp)^j(-i)^{p^n-j})= v_p(kp^{n+1}\cdot i^{p^n-1})=n+1.$$

This finishes the proof of Lemma 2. 2.

3 The proof of the main results

In this section, we give the proofs of Theorems 1. 1 to 1. 3. We begin with the proof of Theorem 1. 1.

Proof of Theorem 1. 1 For any given odd prime p, we have

$$S(p,2)=2^{p-1}-1.$$

By Lemma 2. 1, one knows that

$$S(p,2)=2^{p-1}-1\equiv 0 \pmod p.$$

It infers that $v_p(S(p,2))\geqslant 1$ with the equality holding if and only if

$$2^{p-1}-1\not\equiv 0 \pmod{p^2},$$

i. e.,

$$2^{p-1}\not\equiv 1 \pmod{p^2},$$

which is equivalent to p being a Wieferich prime. So Theorem 1. 1 is proved.

Then we present the proof of Theorem 1. 2.

Proof of Theorem 1. 2 Let p be an odd prime. Replacing n by p^n and k by $2p$ in (1), one gets that

$$S(p^n,2p)=\frac{1}{(2p)!}\sum_{i=0}^{2p}(-1)^i\binom{2p}{i}(2p-i)^{p^n}=\frac{1}{(2p)!}\sum_{i=0}^{2p-1}(-1)^i\binom{2p}{i}(2p-i)^{p^n}=\frac{1}{(2p)!}((2p)^{p^n}+\sum_{i=1}^{p-1}(-1)^i\binom{2p}{i}(2p-i)^{p^n}-\binom{2p}{p}p^{p^n}+\sum_{i=p+1}^{2p-1}(-1)^i\binom{2p}{i}(2p-i)^{p^n}) \quad (10)$$

Since $\binom{2p}{2p-i}=\binom{2p}{i}$ holds for $1\leqslant i\leqslant p-1$, it is easy to obtain that

$$\sum_{i=p+1}^{2p-1}(-1)^i\binom{2p}{i}(2p-i)^{p^n}=\sum_{i=1}^{p-1}(-1)^{2p-i}\binom{2p}{2p-i}i^{p^n}=\sum_{i=1}^{p-1}(-1)^i\binom{2p}{i}i^{p^n} \quad (11)$$

Then it follows from (10) and (11) that

$$S(p^n,2p)=\frac{1}{(2p)!}((2p)^{p^n}-\binom{2p}{p}p^{p^n}+\sum_{i=1}^{p-1}(-1)^i\binom{2p}{i}((2p-i)^{p^n}+i^{p^n})) \quad (12)$$

Let i be an integer with $1\leqslant i\leqslant p-1$. By setting $k=2$ in Lemma 2. 2 we deduce that

$$v_p((2p-i)^{p^n}+i^{p^n})=n+1 \quad (13)$$

Also note that $v_p(\binom{2p}{i})=1$ and $v_p((2p)!)=2$. Hence (12) and (13) tell us that

$$v_p(S(p^n,2p))=v_p((2p)^{p^n}-\binom{2p}{p}p^{p^n}+\sum_{i=1}^{p-1}(-1)^i\binom{2p}{i}((2p-i)^{p^n}+i^{p^n}))-v_p((2p)!)\geqslant \min\{v_p((2p)^{p^n}),v_p(\binom{2p}{p}p^{p^n}),v_p(\sum_{i=1}^{p-1}(-1)^i\binom{2p}{i}((2p-i)^{p^n}+i^{p^n}))\}-2\geqslant \min\{p^n,p^n,\min_{1\leqslant i\leqslant p-1}\{v_p(\binom{2p}{i}((2p-i)^{p^n}+i^{p^n}))\}\}-2= \min\{p^n,v_p(\binom{2p}{i})+v_p((2p-i)^{p^n}+i^{p^n})\}-2= \min\{p^n,n+2\}-2=n.$$

This completes the proof of Theorem 1.2.

Finally, we give the proof of Theorem 1.3.

Proof of Theorem 1.3 Let $p\geqslant 5$ be an odd prime. Replacing n by p^n and k by $4p$ in equation (1), one obtains that

$$S(p^n,4p)=\frac{1}{(4p)!}\sum_{i=0}^{4p}(-1)^i\binom{4p}{i}(4p-i)^{p^n}=\frac{1}{(4p)!}\sum_{i=0}^{4p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}=\frac{1}{(4p)!}((4p)^{p^n}+\sum_{i=1}^{p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}-\binom{4p}{p}(3p)^{p^n}+\sum_{i=p+1}^{2p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}+\binom{4p}{2p}(2p)^{p^n}+\sum_{i=2p+1}^{3p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}-\binom{4p}{3p}p^{p^n}+\sum_{i=3p+1}^{4p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n})=\frac{1}{(4p)!}(\Delta_1+\Delta_2+\Delta_3) \tag{14}$$

where

$$\Delta_1:=(4p)^{p^n}-\binom{4p}{p}(3p)^{p^n}+\binom{4p}{2p}(2p)^{p^n}-\binom{4p}{3p}p^{p^n},$$

$$\Delta_2:=\sum_{i=1}^{p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}+\sum_{i=3p+1}^{4p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}$$

and

$$\Delta_3:=\sum_{i=p+1}^{2p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}+\sum_{i=2p+1}^{3p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}.$$

Note that $\binom{4p}{4p-i}=\binom{4p}{i}$ holds for any integer i with $1\leqslant i\leqslant p-1$ and $p+1\leqslant i\leqslant 2p-1$. We deduce that

$$\sum_{i=3p+1}^{4p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}=\sum_{i=1}^{p-1}(-1)^{4p-i}\binom{4p}{4p-i}i^{p^n}=\sum_{i=1}^{p-1}(-1)^i\binom{4p}{i}i^{p^n} \tag{15}$$

and

$$\sum_{i=2p+1}^{3p-1}(-1)^i\binom{4p}{i}(4p-i)^{p^n}=\sum_{i=p+1}^{2p-1}(-1)^{4p-i}\binom{4p}{4p-i}i^{p^n}=\sum_{i=p+1}^{2p-1}(-1)^i\binom{4p}{i}i^{p^n} \tag{16}$$

Then it follows from (15) and (16) that

$$\Delta_2=\sum_{i=1}^{p-1}(-1)^i\binom{4p}{i}((4p-i)^{p^n}+i^{p^n}) \tag{17}$$

and

$$\Delta_3=\sum_{i=p+1}^{2p-1}(-1)^i\binom{4p}{i}((4p-i)^{p^n}+i^{p^n}) \tag{18}$$

For any integer i such that $1\leqslant i\leqslant p-1$ or $p+1\leqslant i\leqslant 2p-1$, by using Lemma 2.2 one derives that

$$v_p((4p-i)^{p^n}+i^{p^n})=n+1 \tag{19}$$

Now from (19) together with (17) and (18) we obtain that

$$v_p(\Delta_2)=v_p(\sum_{i=1}^{p-1}(-1)^i\binom{4p}{i}((4p-i)^{p^n}+i^{p^n}))\geqslant\min_{1\leqslant i\leqslant p-1}\{v_p((-1)^i\binom{4p}{i}((4p-i)^{p^n}+i^{p^n}))\}=\min_{1\leqslant i\leqslant p-1}\{v_p(\binom{4p}{i})+v_p((4p-i)^{p^n}+i^{p^n})\}=n+2 \tag{20}$$

and

$$v_p(\Delta_3)=v_p(\sum_{i=p+1}^{2p-1}(-1)^i\binom{4p}{i}((4p-i)^{p^n}+i^{p^n}))\geqslant\min_{p+1\leqslant i\leqslant 2p-1}\{v_p((-1)^i\binom{4p}{i}((4p-i)^{p^n}+i^{p^n}))\}=\min_{p+1\leqslant i\leqslant 2p-1}\{v_p(\binom{4p}{i})+v_p((4p-i)^{p^n}+i^{p^n})\}=n+2 \tag{21}$$

since $v_p(\binom{4p}{i})=1$ holds for any integer i with $1\leqslant i\leqslant p-1$ and $p+1\leqslant i\leqslant 2p-1$. Also note that

$$v_p(\Delta_1)=v_p((4p)^{p^n}-\binom{4p}{p}(3p)^{p^n}+\binom{4p}{2p}(2p)^{p^n}-\binom{4p}{3p}p^{p^n})\geqslant\min\{v_p((4p)^{p^n}),v_p(\binom{4p}{p}(3p)^{p^n}),$$

$$v_p(\binom{4p}{2p}(2p)^{p^n}),v_p(\binom{4p}{3p}p^{p^n})\}=p^n \quad (22)$$

It then follows from (14) together with $v_p((4p)!)=4$ and (20) to (22) that

$$v_p(S(p^n,4p))= v_p(\Delta_1+\Delta_2+\Delta_3)-v_p((4p)!)\geqslant \min\{v_p(\Delta_1),v_p(\Delta_2),v_p(\Delta_3)\}-4\geqslant \min\{p^n,n+2,n+2\}-4=n-2.$$

This complete the proof of Theorem 1.3.

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引用本文格式:

中　文:赵伟,邱敏. 关于第二类 Stirling 数的 p-adic 赋值的一些新结果[J]. 四川大学学报:自然科学版,2020,57:865.

英　文:Zhao W, Qiu M. Some new results on the p-adic valuations of Stirling numbers of the second kind [J]. J Sichuan Univ: Nat Sci Ed, 2020, 57: 865.