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Rayleigh 型时滞平均曲率方程周期解的存在性

孔凡超[1]，鲁世平[2]

(1. 安徽师范大学数学计算机科学学院，芜湖 241003；2. 南京信息工程大学数理学院，南京 210044)

摘 要：本文研究了如下 Rayleigh 型时滞平均曲率方程

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f(t,u'(t)) + g(u(t-\tau(t))) = p(t)$$

周期解的存在性问题. 运用 Mawhin 重合度扩展定理，本文给出了证明方程至少存在一个 T-周期解的充分性条件. 最后本文给出例子验证了文章的主要结论.

关键词：周期解；重合度拓展定理；Rayleigh 型平均曲率方程；时滞

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Periodic solutions for prescribed mean curvature Rayleigh equations with a deviating argument

KONG Fan-Chao[1], *LU Shi-Ping*[2]

(1. Department of Mathematics, Anhui Normal University, Wuhu 241003, China;
2. College of Mathematics and Statistics, NUIST, Nanjing 210044, China)

Abstract: In this paper, we give certain sufficient conditions for the existence of periodic solutions to the following prescribed mean curvature Rayleigh equations with a deviating argument

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f(t,u'(t)) + g(u(t-\tau(t))) = p(t).$$

By using Mawhin's continuation theorem, we prove that the given equation has at least one T- periodic solution. At last, we give an example to illustrate the application of our main results.

Key words: Periodic solutions; Continuation theorem; Prescribed mean curvature Rayleigh equation; Deviating arguments

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1 Introduction

Considering the following prescribed mean curvature Rayleigh equations with a deviating argument

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f(t,u'(t)) + g(u(t-\tau(t))) = p(t) \tag{1}$$

where$g,\tau,p \in C(\mathbf{R},\mathbf{R})$ are T- periodic, $f \in C(\mathbf{R}\times\mathbf{R},\mathbf{R})$ is a T- periodic function in the first argument and $T > 0$ is a given constant. In recent years, there are so many results about the existence of periodic solutions for the Rayleigh equations(see[1-9]). For example, in Ref. [3], Lu and Ge studied the periodic solutions of the following Rayleigh equation with a deviating argument:

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作者简介：孔凡超，安徽六安人，硕士，主要研究方向为泛函微分方程. E-mail：fanchaokong88@sohu.com

$$x''(t)+f(x'(t))+g(x(t-\tau(t)))=p(t),$$

and in Ref. [8], Lu and Gui discussed the existence of periodic solutions to p-Laplacian Rayleigh differential equation with a delay of the form:

$$(\varphi_p(x'(t)))'+f(x'(t))+ g(x(t-\tau(t)))=e(t).$$

Nowadays, the prescribed mean curvature $\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)'$ of a function $u(t)$ frequently appears in different geometry and physics[10-12], so it is interesting and worthwhile to consider the existence of periodic solutions of prescribed mean curvature equations. In Ref. [13], Feng discussed the periodic solution for the prescribed mean curvature Lie′nard equation of the form

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)'+f(u(t))u'(t)+ g(t,u(t-\tau(t)))=e(t),$$

and Liang and Lu[14] studied the homoclinic solution for the prescribed mean curvature Duffing-type equation of the form

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)'+cu'(t)+f(u(t))=p(t).$$

However, to best of our knowledge, the studies of prescribed mean curvature Rayleigh equation is relatively infrequent, and the method of finding a priori bounds for Rayleigh equations is different from Liénard equations and Duffing-type equations, so it′s worthy to study the Eq. (1).

The rest of this paper organized as follows. In Section 2, we shall state some necessary definitions and lemmas. In Section 3, we shall prove the main result.

2 Preliminary

In order to use Mawhin′s continuation theorem, we first recall it.

Let X and Y be two Banach space, a linear operator$L:D(L)\subset X\to Y$ is said to be a Fredholm operator of index zero provided that

(a) ImL is a closed subset of Y;

(b) dimKer$L$ = codimIm$L<\infty$.

Let$N:\Omega\subset X\to Y$ be a continuous operator, $N$is said to be L-compact and continuous in $\overline{\Omega}$ provided that

(c) $K_p(I-Q)N(\overline{\Omega})$ is a relative compact set of X,

(d) $QN(\overline{\Omega})$ is a bounded set of Y, where we denote $X_1=\mathrm{Ker}L, Y_2=\mathrm{Im}L$, then we have the decomposition $X=X_1\oplus X_2, Y=Y_1\oplus Y_2$, let $P: X\to X_1$, $Q:Y\to Y_1$ are continuous linear projectors (meaning $P^2=P$ and $Q^2=Q$), and

$$K_p=L|_{KerP\cap D(L)}.$$

Lemma 2.1 Let X and Y be two Banach spaces and$\Omega$is an open and bounded set of X, and let $L:D(L)\subset X\to Y$ be a Fredholm operator of index zero and let$N:\Omega\subset X\to Y$ be-compact on $\overline{\Omega}$. In addition, if the following conditions hold:

(h_1) $Lx\neq\lambda Nx, \forall(x,\lambda)\in\partial\Omega\times(0,1)$;

(h_2) $QNx\neq 0, \forall x\in KerL\cap\partial\Omega$;

(h_3) $\deg(JQN,\Omega\cap \mathrm{Ker}L,0)\neq 0$,

where $J:\mathrm{Im}Q\to \mathrm{Ker}L$is just any homeomorphism, then $Lx=Nx$ has at least one solution in $D(L)\cap\overline{\Omega}$.

Lemma 2.2[15] Suppose that $x(t)\in C^1([0,T])$, and $x(0)=x(T)$. Then

$$\int_0^T|x(t)|^2dt\leqslant\frac{T^2}{\pi^2}\int_0^T|x'(t)|^2dt.$$

Lemma 2.3[16] Let$s\in C(\mathbf{R},\mathbf{R})$ with $s(t+T)\equiv s(t)$ and $s(t)\in[0,T], \forall t\in\mathbf{R}$. Suppose $p\in(1,+\infty)$, $\alpha=\max_{t\in[0,T]}|s(t)|$ and $u\in C^1(\mathbf{R},\mathbf{R})$ with $u(t+T)\equiv u(t)$. Then

$$\int_0^T|u(t)-u(t-s(t))|^p dt\leqslant \alpha^p\int_0^T|u'(t)|^p dt.$$

In order to use Lemma 2.1, Let′s consider the problem

$$\begin{cases}u'(t)=\varphi(v(t))=\dfrac{v(t)}{\sqrt{1-v^2(t)}},\\ v'(t)=-f(t,u'(t))-g(u(t-\tau(t)))+p(t)\end{cases}\qquad(2)$$

Obviously, if $(u(t),v(t))^T$ is a solution of (2), then $u(t)$ is a solution of (1).

Let

$$X=Y=\{z:z(t)=(u(t),v(t))^T\in C^1(\mathbf{R},\mathbf{R}^2), z(t)=z(t+T)\},$$

where the normal $\|z\|=\max\{\|u\|_0,\|v\|_0\}$, and

$$\| u \|_0 = \max_{t\in[0,T]} | u(t) |,$$
$$\| v \|_0 = \max_{t\in[0,T]} | v(t) |.$$

It is obviously that X and Y are Banach space. Now we define the operator

$$L: D(L) \subset X \to Y, Lz = z' = (u'(t), v'(t))^T,$$

where

$$D(L) = \{z \mid z = (u(t), v(t))^T \in C^1(\mathbf{R}, \mathbf{R}^2), z(t) = z(t+T)\}.$$

Let

$$X_0 = \{z \mid z = (u(t), v(t))^T \in C^1(\mathbf{R}, \mathbf{R} \times (-1,1)), z(t) = z(t+T)\}.$$

Define a nonlinear operator $N: \overline{\Omega} \in (X \cap X_0) \subset X \to Y$ as follows:

$$Nz = (\frac{v(t)}{\sqrt{1-v^2(t)}}, -f(t, \frac{v(t)}{\sqrt{1-v^2(t)}}) - g(u(t-\tau(t))) + p(t))^T,$$

where $\overline{\Omega} \subset X_0 \subset X$ and Ω is an open and bounded set. Then problem (2) can be written as $Lz = Nz$ in $\overline{\Omega}$. We know

$$\mathrm{Ker}L = \{z \mid z \in X, z' = (u'(t), v'(t))^T = (0,0)^T\},$$

then $\forall t \in \mathbf{R}$, we have $u'(t) = 0, v'(t) = 0$. Obviously $u \in \mathbf{R}, v \in \mathbf{R}$, thus KerL= $\mathbf{R}^2$, and it is also easy to prove that $\mathrm{Im}L = \{y \in Y, \int_0^T y(s)\mathrm{d}s = 0\}$. Therefore, L is a Fredholm operator of index zero.

Let

$$P: X \to \mathrm{Ker}L, Pz = \frac{1}{T}\int_0^T z(s)\mathrm{d}s,$$

$$Q: Y \to \mathrm{Im}Q, Qy = \frac{1}{T}\int_0^T y(s)\mathrm{d}s.$$

$$K_p = L|^{-1}_{KerL \cap D(L)}.$$

Then it is easy to see that

$$(K_p y)(t) = \int_0^T G_k(t,s) y(s)\mathrm{d}s,$$

where

$$G_k(t) = \begin{cases} \dfrac{s-T}{T}, 0 \leqslant t \leqslant s, \\ \dfrac{s}{T}, s \leqslant t \leqslant T. \end{cases}$$

For all $\overline{\Omega}$ such that $\overline{\Omega} \subset (X_0 \cap X) \subset X$, we have $K_p(I-Q)N(\overline{\Omega})$ is a relative compact set of X, $QN(\overline{\Omega})$ is a bounded set of Y, so the operator N is L- compact in $\overline{\Omega}$.

For the sake of convenience, we list the following assumptions which will be used by us in studying the existence of periodic solutions to the (1) in Section 3.

[H_1]There exists a constant d such that $x(g(x) - p(t)) < 0$, for $|x| > d$ and $t \in \mathbf{R}$.

[H_2] $\forall t \in \mathbf{R}, xf(t,x) \leqslant 0$ (or $xf(t,x) \geqslant 0$). And there exists $r_2 > r_1 > 0$ and $h > 0$ suchthat

$$r_1 |x| - h \leqslant | f(t,x) | \leqslant r_2 |x| + h, \quad \forall (t,x) \in \mathbf{R}^2.$$

[H_3] $g(0) = 0$ and there exists a constant $l$such that

$$| g(x_1) - g(x_2) | \leqslant l | x_1 - x_2 |, \forall x_1, x_2 \in \mathbf{R}.$$

Throughout this paper, we define

$$A: = (\int_0^T | p(t) |^2 \mathrm{d}t)^{\frac{1}{2}} + \sup_{t\in[0,T]} | p(t) | < +\infty.$$

3　Main results

Theorem 3.1　Assume that conditions [H_1] ~[H_3] hold, $r_1 > \alpha l$ and

$$\frac{(\pi r_1 + \pi r_2 + lT)(A\sqrt{T} + Th) + \pi d l T(r_1 - \alpha l)}{\pi(r_1 - \alpha l)} < 1,$$

then (1) has at least one T- periodic solution.

Proof　Let $\Omega_1 = \{z \in \overline{\Omega}, Lz = \lambda Nz, \lambda \in (0,1)\}$. If $z \in \Omega_1$, we have

$$\begin{cases} u'(t) = \lambda\varphi(v(t)) = \lambda \dfrac{v(t)}{\sqrt{1-v^2(t)}}, \\ v'(t) = -\lambda f(t, \lambda\varphi(v(t))) - \lambda g(u(t-\tau(t))) + \lambda p(t) \end{cases} \tag{3}$$

Integrating the first equation of (3) from 0 to T, we have

$$\int_0^T \frac{v(t)}{\sqrt{1-v^2(t)}} \mathrm{d}t = 0 \tag{4}$$

Thus, there exist $t_1, t_2 \in [0,T]$ such that

$$v(t_1) \geqslant 0, v(t_2) \leqslant 0.$$

Let t_3 and t_4 be, respectively, the maximum and minimum points of $v(t)$. Then

$$v(t_3) \geqslant 0, v'(t_3) = 0 \tag{5}$$

and

$$v(t_4) \leqslant 0, v'(t_4) = 0 \tag{6}$$

It follows from the second equation of (3) that

$$0 = v'(t_3) = -\lambda f(t, \lambda \frac{v(t_3)}{\sqrt{1-v^2(t_3)}},) - \lambda g(u(t_3 - \tau(t_3))) + \lambda p(t_3).$$

Combining (5) and [H_2], we can obtain

$$g(u(t_3 - \tau(t_3))) - p(t_3) \geqslant 0 \quad (7)$$

In a similar way, we also have

$$g(u(t_4 - \tau(t_4))) - p(t_4) \leqslant 0 \quad (8)$$

In view of [H_1], (7) and (8), we can get

$$u(t_3 - \tau(t_3)) < d$$

and

$$u(t_4 - \tau(t_4)) > -d.$$

Since $u(t-\tau(t))$ is a continuous function on R, it follows that there must exist a constant $\xi \in \mathbf{R}$ such that

$$|u(\xi - \tau(\xi))| < d.$$

Note that there exist an integer m and $\bar{\xi} \in [0, T]$ such that $\xi - \tau(\xi) = mT + \bar{\xi}$. Then we have

$$|u(t)| = |u(\bar{\xi}) + \int_{\bar{\xi}}^{t} u'(s)\mathrm{d}s| \leqslant d + \int_{\bar{\xi}}^{t} |u'(s)| \mathrm{d}s, t \in [\bar{\xi}, \bar{\xi}+T],$$

and

$$|u(t)| = |u(t-T)| = |u(\bar{\xi}) - \int_{t-T}^{\bar{\xi}} u'(s)\mathrm{d}s| \leqslant d + \int_{t-T}^{\bar{\xi}} |u'(s)| \mathrm{d}s.$$

It follows from the above two inequalities that

$$\|u\|_0 = \max_{t\in[0,T]} |u(t)| = \max_{t\in[\bar{\xi},\bar{\xi}+T]} |u(t)| \leqslant \max_{t\in[\bar{\xi},\bar{\xi}+T]} [d + \frac{1}{2}(\int_{\bar{\xi}}^{t} |u'(s)| \mathrm{d}s + \int_{t-T}^{\bar{\xi}} |u'(s)| \mathrm{d}s)] \leqslant d + \frac{1}{2}\sqrt{T}\|u'\|_2 \quad (9)$$

Furthermore, by Lemma 2.2, we have

$$\|u\|_2 = (\int_0^T |u(t)|^2 \mathrm{d}t)^{\frac{1}{2}} = (\int_0^T |u(t+\bar{\xi})|^2 \mathrm{d}t)^{\frac{1}{2}} = (\int_0^T |u(t+\bar{\xi}) - u(\bar{\xi}) + u(\bar{\xi})|^2 dt)^{\frac{1}{2}} \leqslant (\int_0^T |u(t+\bar{\xi}) - u(\bar{\xi}) + d|^2 \mathrm{d}t)^{\frac{1}{2}} < [\int_0^T (|u(t+\bar{\xi}) - u(\bar{\xi})| + d)^2 \mathrm{d}t]^{\frac{1}{2}} < (\int_0^T (|u(t+\bar{\xi}) - u(\bar{\xi})|)^2 \mathrm{d}t)^{\frac{1}{2}} + (\int_0^T d^2 \mathrm{d}t)^{\frac{1}{2}} \leqslant \frac{\pi}{T}\|u'\|_2 + d\sqrt{T} \quad (10)$$

Multiplying the second equation of (3) by $u'(t)$ and integrating on the interval $[0, T]$, we have

$$0 = \int_0^T v'(t)u'(t)\mathrm{d}t = -\lambda\int_0^T f(t, \frac{u'(t)}{\lambda})\, u'(t)\mathrm{d}t - \lambda\int_0^T g(u(t-\tau(t)))u'(t)\mathrm{d}t + \lambda\int_0^T g(u(t-\tau(t)))u'(t)\mathrm{d}t + \lambda\int_0^T p(t)u'(t)\mathrm{d}t = -\lambda\int_0^T f(t, \frac{u'(t)}{\lambda})\, u'(t)\mathrm{d}t - \lambda\int_0^T [g(u(t-\tau(t))) - g(u(t))]u'(t)\mathrm{d}t - \lambda\int_0^T g(u(t))u'(t)\mathrm{d}t + \lambda\int_0^T pu'(t)\mathrm{d}t.$$

It follows from [H_2],[H_3]and Lemma 2.3 that

$$r_1\int_0^T |u'(t)|^2 \mathrm{d}t \leqslant \int_0^T |g(u(t-\tau(t))) - g(u(t))| \cdot |u'(t)| \mathrm{d}t + \int_0^T |p(t)| \cdot |u'(t)| \mathrm{d}t + \int_0^T |h| \cdot |u'(t)| \mathrm{d}t \leqslant l\int_0^T |u(t-\tau(t)) - u(t)| \cdot |u'(t)| \mathrm{d}t + \int_0^T |p(t)| \cdot |u'(t)| \mathrm{d}t + h\int_0^T |u'(t)| \mathrm{d}t \leqslant l(\int_0^T |u(t-\tau(t)) - u(t)|^2 \mathrm{d}t)^{\frac{1}{2}}(\int_0^T |u'(t)|^2 \mathrm{d}t)^{\frac{1}{2}} + (\int_0^T |p(t)|^2 \mathrm{d}t)^{\frac{1}{2}}(\int_0^T |u'(t)|^2 \mathrm{d}t)^{\frac{1}{2}} + h\sqrt{T}(\int_0^T |u'(t)|^2 \mathrm{d}t)^{\frac{1}{2}} \leqslant \alpha l\int_0^T |u'(t)|^2 \mathrm{d}t + (A + h\sqrt{T})(\int_0^T |u'(t)|^2 \mathrm{d}t)^{\frac{1}{2}}.$$

Since $r_1 > \alpha l$, thus

$$\|u'\|_2 \leqslant \frac{A + h\sqrt{T}}{r_1 - \alpha l} \quad (11)$$

which together with (9) and (10) leads to

$$\|u\|_0 \leqslant d + \frac{1}{2}\frac{A\sqrt{T} + Th}{r_1 - \alpha l} := \rho_1 \quad (12)$$

and

$$\|u\|_2 \leqslant \frac{AT + T\sqrt{T}h}{\pi(r_1 - \alpha l)} + d\sqrt{T} \quad (13)$$

Multiplying the second equation of (3) by $v'(t)$ and integrating on the interval $[0, T]$, in view of [H_2],[H_3]and Lemma 2.3,then we have

$$\int_0^T |v'(t)|^2 \mathrm{d}t = -\lambda\int_0^T f(t,\frac{u'(t)}{\lambda})\, v'(t)\mathrm{d}t - \lambda\int_0^T g(u(t-\tau(t)))\, v'(t)\mathrm{d}t + \lambda\int_0^T p(t)\, v'(t)\mathrm{d}t \leqslant$$
$$\lambda\int_0^T |f(t,\frac{u'(t)}{\lambda})|\cdot|v'(t)|\mathrm{d}t + \int_0^T |g(u(t-\tau(t))) - g(u(t))|\, v'(t)|\mathrm{d}t + \int_0^T |g(u(t))|\cdot|v'(t)|\mathrm{d}t + \int_0^T |p(t)|\cdot|v'(t)|\mathrm{d}t \leqslant$$
$$r_2\int_0^T |u'(t)|\cdot|v'(t)|\mathrm{d}t + h\int_0^T |v'(t)|\mathrm{d}t + l\int_0^T |u(t-\tau(t)) - u(t)|\cdot|v'(t)|\mathrm{d}t + l\int_0^T |u(t)|\cdot|v'(t)|\mathrm{d}t + \int_0^T |p(t)|\cdot|v'(t)|\mathrm{d}t \leqslant$$
$$r_2\|u'\|_2\|v'\|_2 + h\sqrt{T}\|v'\|_2 + \alpha l\|u'\|_2\|v'\|_2 + l\|u\|_2\|v'\|_2 + A\|v'\|_2,$$

which yields

$$\|v'\|_2 \leqslant (r_{22} + \alpha l)\|u'\|_2 + l\|u\|_2 + h\sqrt{T} + A.$$

From (11) and (13), we obtain

$$\|v'\|_2 \leqslant \frac{(\pi r_1 + \pi r_2 + lT)(A + h\sqrt{T}) + \pi dl\sqrt{T}(r_1 - \alpha l)}{\pi(r_1 - \alpha l)} \tag{14}$$

It follows from (4) that there exists $\zeta \in [0,T]$ such that $v(\zeta) = 0$. It implies that

$$|v(t)| = |v(\zeta) + \int_\zeta^t v'(s)\mathrm{d}s| \leqslant \int_0^T |v'(s)|\mathrm{d}s.$$

Thus

$$\|v\|_0 \leqslant \sqrt{T}\|v'\|_2.$$

Combining with (14), we can have

$$\|v\|_0 \leqslant \frac{(\pi r_1 + \pi r_2 + lT)(A\sqrt{T} + hT) + \pi dlT(r_1 - \alpha l)}{\pi(r_1 - \alpha l)} := \rho_2.$$

It follows from

$$\frac{(\pi r_1 + \pi r_2 + lT)(A\sqrt{T} + hT) + \pi dlT(r_1 - \alpha l)}{\pi(r_1 - \alpha l)} < 1$$

that

$$\|v\|_0 := \rho_2 < 1 \tag{15}$$

Let $\Omega_2 = \{z \in \mathrm{Ker}L : Nz \in \mathrm{Im}L\}$. If $z \in \Omega_2$, then $z \in$ KerL and $QNz = 0$. Obviously,

$$|u(t)| \leqslant \rho_1,\ v(t) = 0 < \rho_2.$$

Set

$$\Omega = \{z = (u,v)^{\mathrm{T}} \in X : \|u\|_0 \leqslant \rho_1 + 1,\ \|v\|_0 < \frac{1+\rho_2}{2} < 1\}.$$

Then the condition (h_1) and (h_2) of Lemma 2.1 are satisfied. In order to verify the condition (h_3) of Lemma 2.1, we define $J : \mathrm{Im}Q \to \mathrm{Ker}L$is a linear isomorphism $J(u,v) = (v,u)^T$, and define

$$H(z,\mu) = \mu z + (1-\mu)JQNz,\quad \forall (z,\mu) \in \Omega \times [0,1].$$

It follows from [H_1] that $z^{\mathrm{T}}H(z,\mu) \neq 0$, $\forall (z,\mu) \in \partial(\Omega \cap \mathrm{Ker}L) \times [0,1]$, then

$$\deg(JQN,\Omega \cap \mathrm{Ker}L,0) = \deg(H(z,0),\Omega \cap \mathrm{Ker}L,0) = \deg(H(z,1),\Omega \cap \mathrm{Ker}L,0) \neq 0,$$

which implies that the condition (h_3) of Lemma 2.1 is also satisfied. Therefore, Eq. (1) has at least one T- periodic solution.

As applications, we list the following equation:

Example 3.2 Consider the following equation:

$$\left(\frac{x'(t)}{\sqrt{1+(x'(t))^2}}\right) + f(t,x'(t)) + g(x(t-\tau(t))) = p(t) \tag{16}$$

The Eq. (16) has has at least one $\frac{\pi}{4}$-periodic solution.

Proof Corresponding to Theorem 3.1 and (1), we have

$$f(t,x'(t)) = (2 + \frac{1}{2}\sin 8t)x'(t),$$

$$g(x(t)) = -\frac{1}{64}x(t),$$

$$p(t) = \frac{1}{256}\sin 8t \text{ and } \tau(t) = \sin 8t.$$

Then we have $T = \frac{\pi}{4}, r_1 = 1, r_2 = 3, l = \frac{1}{64}, \alpha = \max\limits_{t\in[0,\frac{\pi}{4}]} |\sin 8t| = 1.$

It is easy to see that $r_1 > \alpha l$. Furthermore,

$$A = \left(\int_0^{\frac{\pi}{4}} \left|\frac{1}{256}\sin 8t\right|^2 dt\right)^{\frac{1}{2}} + \sup_{t\in[0,\frac{\pi}{4}]} \left|\frac{1}{256}\sin 8t\right| < \frac{1}{128}.$$

If we choose $d = \frac{1}{4}$ and $h = \frac{1}{256}$, then

$$\frac{(\pi r_1 + \pi r_2 + lT)(A\sqrt{T} + hT) + \pi dlT(r_1 - \alpha l)}{\pi(r_1 - \alpha l)} < \frac{(\pi + 3\pi + \frac{1}{64}\cdot\frac{\pi}{4})\cdot\frac{1}{128}}{\pi(1 - \frac{1}{64})} + \frac{1}{4}\cdot\frac{1}{64}\cdot\frac{\pi}{4} < 1.$$

Therefore the conditions [H_1]~[H_3] of Theorem 3.1 are satisfied. It follows that (16) has at least one $\frac{\pi}{4}$-periodic solution.

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