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F-完备抛物仿射超球的 Bernstein 性质

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摘 要：设 $x:M\to\mathbf{R}^{n+1}$ 是局部强凸超曲面，由定义在凸域 $D\subset\mathbf{R}^n$ 上的局部强凸函数 $x_{n+1}=f(x_1,x_2,\cdots,x_n)$ 给出. 本文在 M 上定义 F-度量 $\widetilde{G}=F(\rho)\sum_{i,j}\frac{\partial^2 f}{\partial x_i\partial x_j}\mathrm{d}x_i\mathrm{d}x_j$，研究 F-完备抛物仿射超球并得到了相应的 Bernstein 性质.

关键词：F-完备；F-相对度量；抛物仿射超球
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Bernstien properties of F-complete parabolic affine hyperspheres

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Abstract: Let $x:M\to\mathbf{R}^{n+1}$ be a locally strongly convex hypersurface given by the graph of a locally strongly convex function $x_{n+1}=f(x_1,x_2,\cdots,x_n)$ defined in a convex domain $D\subset\mathbf{R}^n$. Defining the F-metric $\widetilde{G}=F(\rho)\sum_{i,j}\frac{\partial^2 f}{\partial x_i\partial x_j}\mathrm{d}x_i\mathrm{d}x_j$ on M, we derive the PDEs of the F-complete parabolic affine hyperspheres and obtain some Bernstein properties.

Keywords: F-complete; F-relative metric; Parabolic affine hyperspheres
(2010 MSC 53C55)

1 Introduction

It is interesting to study Bernstein properties of affine hyperspheres. In Ref. [1], Xiong and Yang considered hyperbolic relative hyperspheres with Li-normalization and classify the subclass which is Euclidean complete. In Ref. [2], Xu studied α-relative parabolic affine hyperspheres and obtained that if

$$M=\{(x_1,x_2,\cdots,x_{n+1})\,|\,x_{n+1}=f(x_1,x_2,\cdots,x_n),(x_1,x_2,\cdots,x_n)\in D\subset\mathbf{R}^n\}$$

is a α-relative parabolic affine hypersphere which complete with respect to the Calabi metric and $\alpha\notin[\frac{n+2}{n+1},\frac{n+2}{2}]$, then M must be an elliptic paraboloid.

In this paper, we consider a relative normalization of M induced by $\widetilde{U}=F(\rho)U$ (see section 2), where $F(\rho)$ be a C^3- function defined on M such that $F(\rho)>0$ everywhere. We call $F(p)$ an F-relative normalization of M. With F-relative normalization, the corresponding metric of M is given by

$$\widetilde{G}=F(\rho)\sum_{i,j}\frac{\partial^2 f}{\partial x_i\partial x_j}\mathrm{d}x_i\mathrm{d}x_j$$

and is called F-relative metric. Here and later we use the following notations:

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$$F'=\frac{dF}{d\rho},F''=\frac{d^2F}{d\rho^2},s_i=\frac{\partial s}{\partial x_i},s_{ij}=\frac{\partial^2 s}{\partial x_i\partial x_j},$$

$$h'=\frac{dh}{d\rho},h=\frac{n+2}{2}\frac{F'\rho}{F}-\frac{F''\rho}{F'}-\frac{n}{2}-1.$$

Parabolic affine hypersphere with F-relative normalization is called F-relative parabolic affine hypersphere. We study F-relative parabolic affine hypersphere and obtain the following

Theorem 1.1 Let $(M,\widetilde{G})$ is an F-complete parabolic affine hypersphere with F-Ricci curvature bounded from below by a negative constant $-N$. If κ and χ are all constants and $\chi>0$, where

$$\chi=\frac{2n^2+15n+4}{n-1}+\frac{4n}{n-1}(\frac{F''\rho}{F'})^2-\frac{n^2-2n-4}{n-1}(\frac{F'\rho}{F})^2-\frac{n^2+5n+14}{n-1}\frac{F'\rho}{F}-2\frac{F'''}{F'}\rho^2-\frac{8}{n-1}\frac{F''}{F}\rho^2+\frac{n^2+7n+12}{n-1}\frac{F''\rho}{F'} \quad (1)$$

$$\kappa=\frac{4}{n-1}\frac{F''\rho}{F'}+\frac{n^2+n-10}{2(n-1)}\frac{F'\rho}{F}+\frac{5n+10}{2(n-1)} \quad (2)$$

then M must be an elliptic paraboloid.

For α-complete parabolic affine hypersphere (the case $F(\rho)=\rho^\alpha$ in Theorem 1.1), by calculation we have

Corollary 1.2 Let $x:M\to\mathbf{R}^{n+1}$ is a α-relative parabolic affine hypersphere and is complete with respect to the α-metric. If $\alpha^2<\frac{n^2+8n-4}{n^2-4n+2}$, then M must be an elliptic paraboloid.

2 Preliminaries

Assume that $x_{n+1}=f(x_1,x_2,\cdots,x_n)$ is a smooth strictly convex function defined in a convex domain $D\subset\mathbf{R}^n$. f defines a locally strictly convex hypersurface $x:M\to\mathbf{R}^{n+1}$, given by a graph representation

$$M=\{(x_1,x_2,\cdots,x_{n+1})\,|\,x_{n+1}=f(x_1,x_2,\cdots,x_n),(x_1,x_2,\cdots,x_n)\in D\subset\mathbf{R}^n\}.$$

For every point $x\in M$, let $Y=(Y^1,\cdots,Y^{n+1})$ be a transversal vector field along M such that $dY\in T_xM$, then Y is called a relative normalization of M. Corresponding to the transversal field Y, there exists a unique conormal vector field U. Particularly, when $Y=(0,0,\cdots,1)$, the conormal field U and the relative Riemannian metric G' on M are defined respectively by

$$U=(-\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n},1),$$

$$G'=\sum\frac{\partial^2 f}{\partial x_i\partial x_j}dx_i dx_j,$$

here G' is called Calabi metric. Denote

$$\rho=[\det(f_{ij})]^{-\frac{1}{n+2}}.$$

Li first considered a relative normalization of M induced by $U^\alpha=\rho^\alpha U$, where α is a non-zero real constant. It was then called an α-relative normalization of M in Ref. [2], later called Li-normalization in Refs. [3,4]. With Li-normalization, the corresponding metric of M is given by[2]

$$G=\rho^\alpha\sum\frac{\partial^2 f}{\partial x_i\partial x_j}dx_i dx_j \quad (3)$$

and is called Li-metric, or α-metric. The corresponding geometry is called Li-geometry or α-relative geometry.

For partial derivation of the vector valued function x, we use the notation from above, while covariant derivation with respect to the Levi-Civita connection of the relative metric is denoted by $x_{,ij}$ etc. Following Ref. [2], assume that $q:=F(\rho)>0$. We consider the new conormal vector field $\widetilde{U}=F(\rho)U$. Then there exists a unique transversal vector field $\widetilde{Y}$ that satisfies the equations

$$\langle\widetilde{U}_i,\widetilde{Y}\rangle=0,\ \langle\widetilde{U},\widetilde{Y}\rangle=1.$$

Let $x=(x_1,x_2,\cdots,f(x_1,x_2,\cdots,x_n))$ denote the position vector of the graph hypersurface, then the relative normal satisfies

$$\widetilde{Y}=\frac{1}{F(\rho)}Y+\sum\frac{F'\rho_j}{(F(\rho))^2}f^{kj}x_k.$$

We consider this relative normal $\widetilde{Y}$ on M and its associated relative metric

$$\widetilde{G}=F(\rho)\sum f_{ij}dx_i dx_j.$$

With this geometry the relative Weingarten form $\widetilde{B}$ is given by

$$\widetilde{B}_{ij}=[\frac{2(F')^2}{F^2}-\frac{F''}{F}]\rho_i\rho_j-\frac{F'}{F}\rho_{ij}+\sum\frac{F'f^{kl}}{F}\rho_l\frac{\partial f_{ij}}{\partial x_k} \quad (4)$$

The Fubini-Pick tensor $\widetilde{A}$ is given by

$$\widetilde{A}_{ijk} = -\frac{1}{2}[F' f_{ij} \frac{\partial \rho}{\partial x_k} + F' f_{ik} \frac{\partial \rho}{\partial x_j} + F' f_{jk} \frac{\partial \rho}{\partial x_i} + F f_{ijk}] \tag{5}$$

The components of the Ricci tensor read

$$\widetilde{R}_{ik} = \sum_{m,l} (\widetilde{A}_{iml} \widetilde{A}_{mlk} - \widetilde{A}_{imk} \widetilde{A}_{mll}) + \frac{n-2}{2} \widetilde{B}_{ik} + \frac{n}{2} \widetilde{L} \widetilde{G}_{ik} \tag{6}$$

Under the F-relative normalization,

$$\sum_l \widetilde{A}_{mll} = \sum_{i,j} \widetilde{G}_{ij} \widetilde{A}_{ijm} = \frac{n+2}{2}(\frac{1}{\rho} - \frac{F'}{F})\rho_m \tag{7}$$

In local terms the Laplacian Δ with respect to the F-metric $\widetilde{G}$ reads

$$\Delta = \frac{1}{\sqrt{\det(\widetilde{G}_{kl})}} \sum \frac{\partial}{\partial x_i}(\widetilde{G}^{ij} \sqrt{\det(\widetilde{G}_{kl})} \frac{\partial}{\partial x_j}).$$

We define

$$\Phi = \frac{1}{F} \sum f^{ij} \frac{\rho_i}{\rho} \frac{\rho_j}{\rho}.$$

In this paper, we will consider the pair $\{\widetilde{U}, \widetilde{Y}\}$ and call it the F-relative normalization of the graph hypersurface M. The eigenvalues $\lambda_1, \cdots, \lambda_n$ of the associated Weingarten operator or relative shape operator are called the F-relative principal curvatures, and

$$\widetilde{L} = \frac{1}{n} \sum \lambda_i$$

is called the F-relative mean curvature. M is called a_n F-relative parabolic affine hypersphere if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ everywhere on M. Let $\widetilde{B} = 0$, we obtain the following proposition for a F-relative parabolic affine hypersphere.

Proposition 2.1　Choose $F(\rho)U$ as a relative normalization of M, then a locally strongly convex F-relative parabolic affine hypersphere satisfies the following system of PDEs:

$$\rho_{ij} = \frac{2(F')^2 - FF''}{FF'} \rho_i \rho_j + \sum f^{kl} f_{ijk} \rho_l, \quad \forall 1 \leqslant i, j \leqslant n \tag{8}$$

In the following we will use the F-metric to do the calculations. That is to say, the norms and the Laplacian operator are defined with respect to the F-metric. From (8), we have

$$\Delta \rho = h \frac{\| \nabla \rho \|^2}{\rho} \tag{9}$$

3　Estimation of $\Delta\Phi$

Proposition 3.1　Let $f(x)$ be a C^∞-strictly convex function, and satisfy the PDEs (8). Then we have

$$\begin{aligned} \frac{\Delta\Phi}{\Phi} \geqslant & \frac{n}{n-1} \sum \frac{|\nabla \Phi|^2}{\Phi^2} + [\frac{n^2+n-10}{2(n-1)} \frac{F'\rho}{F} + \frac{4}{n-1} \frac{F''\rho}{F'} + \frac{5n+10}{2(n-1)}] \langle \frac{\nabla\Phi}{\Phi}, \nabla \log \rho \rangle + \\ & [\frac{2n^2+15n+4}{n-1} + \frac{4n}{n-1}(\frac{F''\rho}{F'})^2 - \frac{n^2-2n-4}{n-1}(\frac{F'\rho}{F})^2 - \frac{n^2+5n+14}{n-1}\frac{F'\rho}{F} - \\ & 2\frac{F'''}{F'}\rho^2 - \frac{8}{n-1}\frac{F''}{F}\rho^2 + \frac{n^2+7n+12}{n-1}\frac{F''\rho}{F'}]\Phi \end{aligned} \tag{10}$$

Proof　Let $p \in M$ be any fixed point. We choose a local orthonormal frame field of the F-metric. Then

$$\Phi = \frac{\sum (\rho_{,j})^2}{\rho^2},$$

$$\Phi_{,i} = 2\sum \frac{\rho_{,j}\rho_{,ji}}{\rho^2} - 2\rho_{,i} \frac{\sum(\rho_{,j})^2}{\rho^3},$$

$$\Delta\Phi = 2\sum \frac{\rho_{,j}\rho_{,jii}}{\rho^2} + 2\frac{\sum(\rho_{,ji})^2}{\rho^2} - 8\sum \frac{\rho_{,i}\rho_{,j}\rho_{,ji}}{\rho^3} + (6-2h)\Phi^2,$$

where we used (9). For the case $\Phi(p) = 0$, it is easy to get (at p)

$$\Delta\Phi \geqslant 2\frac{\sum(\rho_{,ji})^2}{\rho^2}.$$

Now we assume that $\Phi(p) \neq 0$. Choose a local orthonormal frame field of the the F-metric such that (at p) $\rho_{,1} = \|\nabla\rho\| > 0, \rho_{,i} = 0, \forall i > 1$. Then

$$\Delta\Phi = 2\sum \frac{\rho_{,j}\rho_{,jii}}{\rho^2} + 2\frac{\sum(\rho_{,ji})^2}{\rho^2} - 8\sum \frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + (6-2h)\Phi^2 \tag{11}$$

Applying Schwarz's inequality we get

$$2\sum(\rho_{,ji})^2 \geqslant 2(\rho_{,11})^2 + 4\sum_{i>1}(\rho_{,1i})^2 + 2\sum_{i>1}(\rho_{,ii})^2 \geqslant 2(\rho_{,11})^2 + 4\sum_{i>1}(\rho_{,1i})^2 +$$

$$\frac{2}{n-1}(\Delta\rho-\rho_{,11})^2=\frac{2n}{n-1}(\rho_{,11})^2+4\sum_{i>1}(\rho_{,1i})^2+\frac{2}{n-1}h^2\frac{(\rho_{,1})^4}{\rho^2}-\frac{4h}{n-1}\frac{\rho_{,11}}{\rho}(\rho_{,1})^2 \quad (12)$$

An application of the Ricci identity shows that

$$2\sum\frac{\rho_{,j}\rho_{,jii}}{\rho^2}=2\frac{\rho_{,1}\rho_{,1ii}}{\rho^2}=\frac{2}{\rho^2}(\Delta\rho)_{,1}\rho_{,1}+2R_{11}\frac{\rho_{,1}}{\rho^2}=4h\rho_{,11}\frac{(\rho_{,1})^2}{\rho^3}+2R_{11}\Phi+2(h'\rho-h)\Phi^2 \quad (13)$$

Note that

$$\sum\frac{(\Phi_{,i})^2}{\Phi}=4\frac{\sum(\rho_{,1i})^2}{\rho^2}-8\sum\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3}+4\Phi^2 \quad (14)$$

Substituting (12), (13) into (11) yields

$$\Delta\Phi\geqslant\frac{2n}{n-1}\frac{(\rho_{,11})^2}{\rho^2}+4\sum_{i>1}\frac{(\rho_{,1i})^2}{\rho^2}+(6-4h+2h'\rho+\frac{2h^2}{n-1})\Phi^2+\frac{4(n-2)}{n-1}\frac{(\rho_{,1})^2}{\rho^3}h\rho_{,11}+2R_{11}\Phi\geqslant\frac{2n}{n-1}\sum\frac{(\rho_{,1i})^2}{\rho^2}+(6-4h+2h'\rho+\frac{2h^2}{n-1})\Phi^2+\frac{4(n-2)}{n-1}\frac{(\rho_{,1})^2}{\rho^3}h\rho_{,11}+2R_{11}\Phi.$$

From the above inequality and (14), we have

$$\Delta\Phi\geqslant\frac{n}{2(n-1)}\frac{|\nabla\Phi|^2}{\Phi}+[\frac{4(n-2)}{n-1}h-\frac{4(n-2)}{n-1}]\frac{(\rho_{,1})^2}{\rho^3}\rho_{,11}+(6-4h+2h'\rho+\frac{2h^2}{n-1}-\frac{2n}{n-1})\Phi^2+2R_{11}\Phi \quad (15)$$

Choosing a local orthonormal frame field of the F-metric, $Ff_{ij}=\delta_{ij}=F^{-1}f^{ij}$ and using (5), we have

$$-2\widetilde{A}_{ijl}\rho_{,l}=\frac{F'}{F}(\delta_{ij}\rho_l^2+\delta_{il}\rho_j\rho_l+\delta_{jl}\rho_i\rho_l)+f^{kl}f_{ijk}\rho_l.$$

From (8) and the above equality, we get

$$\rho_{,ij}=\rho_{ij}+\widetilde{A}_{ij1}\rho_{,1}=-\frac{F''}{F'}\rho_i\rho_j-\frac{F'}{F}\delta_{ij}|\nabla\rho|^2-\widetilde{A}_{ij1}\rho_{,1}.$$

From the definition of Φ and the above equality, we have

$$\Phi_{,i}=-2\widetilde{A}_{11i}\frac{(\rho_{,1})^2}{\rho^2}-2(1+\frac{F''}{F'}\rho)\frac{\rho_i\rho_1^2}{\rho^3}-2\frac{F'}{F}\delta_{1i}\frac{(\rho_{,1})^3}{\rho^2}.$$

Thus

$$\sum\frac{\Phi_{,i}\rho_{,i}}{\rho}=-2\widetilde{A}_{111}\frac{(\rho_{,1})^3}{\rho^3}-2(1+\frac{F''}{F'}\rho+\frac{F'}{F}\rho)\Phi^2 \quad (16)$$

$$\sum\frac{(\Phi_{,i})^2}{\Phi}=4\frac{\sum(\rho_{,1i})^2}{\rho^2}-8\sum\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3}+4\Phi^2=8(1+\frac{F''}{F'}\rho+\frac{F'}{F}\rho)\widetilde{A}_{111}\frac{(\rho_{,1})^3}{\rho^3}+4(1+\frac{F''}{F'}\rho+\frac{F'}{F}\rho)^2\Phi^2+4\sum(\widetilde{A}_{i11})^2\Phi \quad (17)$$

By the same method as deriving (12), we have

$$\sum(\widetilde{A}_{ml1})^2\geqslant(\widetilde{A}_{111})^2+2\sum_{i>1}(\widetilde{A}_{i11})^2+\sum_{i>1}(\widetilde{A}_{ii1})^2\geqslant(\widetilde{A}_{111})^2+2\sum_{i>1}(\widetilde{A}_{i11})^2+\frac{1}{n-1}(\sum\widetilde{A}_{ii1}-\widetilde{A}_{111})^2\geqslant\frac{n}{n-1}\sum(\widetilde{A}_{i11})^2-\frac{2}{n-1}\widetilde{A}_{111}\sum\widetilde{A}_{ii1}+\frac{1}{n-1}(\sum\widetilde{A}_{ii1})^2 \quad (18)$$

Then, from (16)~(18) and (6)~(7), we have

$$2R_{11}\Phi=\frac{2}{\rho^2}\sum(\widetilde{A}_{1ml})^2(\rho_{,1})^2-2\widetilde{A}_{11m}\widetilde{A}_{mll}\frac{(\rho_{,1})^2}{\rho^2}=[\frac{(n+1)(n+2)}{2(n-1)}(1-\frac{F'\rho}{F})+\frac{2n}{n-1}(1+\frac{F''\rho}{F'}+\frac{F'\rho}{F})]\langle\nabla\Phi,\nabla\log\rho\rangle+[\frac{8+14n+3n^2}{2(n-1)}-\frac{n(n-2)}{2(n-1)}(\frac{F'\rho}{F})^2-\frac{4+n^2}{n-1}\frac{F'\rho}{F}+\frac{2n}{n-1}(\frac{F''\rho}{F'})^2+\frac{n^2+7n+2}{n-1}\frac{F''\rho}{F'}-\frac{n^2-n+2}{n-1}\frac{F''\rho^2}{F}]\Phi^2+\frac{n}{2(n-1)}\sum\frac{|\nabla\Phi|^2}{\Phi} \quad (19)$$

Inserting (19) into (15), we obtain (10). The proof is complete.

4 Proof of Theorem 1.1

Let $p_0\in M$, denote by $r=r(p_0,p)$ the geodesic distance function from p_0 to p with respect to the F-metric $\widetilde{G}$. For any $a>0$, let $B_a(p_0)=\{p\in M\mid r(p_0,p)\leqslant a\}$. Consider the function

$$L=(a^2-r^2)^2\Phi$$

defined on $B_a(p_0)$. Obviously, L attains its maximum at some interior point $\bar{p}$. We may assume that r^2 is a C^2 function in a neighborhood of $\bar{p}$, and $\Phi>0$ at $\bar{p}$. Then, at $\bar{p}$, we have

$$0=L_{,i}=(a^2-r^2)^2\Phi_{,i}-4r(a^2-r^2)r_{,i}\Phi \tag{20}$$

and

$$0\geqslant\Delta L=(a^2-r^2)^2\Delta\Phi-8r(a^2-r^2)<\nabla r,\nabla\Phi>_G-4\Phi r(a^2-r^2)\Delta r+8r^2\|\nabla r\|_{\widetilde{G}}^2\Phi-4\Phi(a^2-r^2)\|\nabla r\|_{\widetilde{G}}^2 \tag{21}$$

Insert (20) into (21) one gets

$$\frac{\Delta\Phi}{\Phi}\leqslant\frac{24r^2}{(a^2-r^2)^2}+\frac{4}{a^2-r^2}+\frac{4r\Delta r}{a^2-r^2} \tag{22}$$

here we used the fact that $\|\nabla r\|_{\widetilde{G}}^2=1$. Writing

$$\chi=\frac{2n^2+15n+4}{n-1}+\frac{4n}{n-1}\left(\frac{\rho F''}{F'}\right)^2-\frac{n^2-2n-4}{n-1}\left(\frac{\rho F'}{F}\right)^2-\frac{n^2+5n+14}{n-1}\frac{\rho F'}{F}-2\frac{\rho^2F'''}{F'}-\frac{8}{n-1}\frac{\rho^2F''}{F}+\frac{n^2+7n+12}{n-1}\frac{\rho F''}{F'},$$

$$\kappa=\frac{5n+10}{2(n-1)}+\frac{4}{n-1}\frac{\rho F''}{F'}+\frac{n^2+n-10}{2(n-1)}\frac{\rho F'}{F}.$$

Applying Schwarz's inequality we have

$$\frac{4\kappa r}{a^2-r^2}\langle\nabla r,\frac{\nabla\rho}{\rho}\rangle\geqslant-\theta\Phi-\frac{4\kappa^2r^2}{\theta(a^2-r^2)^2} \tag{23}$$

Combining (10) with (22) and using (23) we have

$$\Phi(\chi-\theta)\leqslant\left(24-\frac{16n}{n-1}+\frac{4\kappa^2}{\theta}\right)\frac{r^2}{(a^2-r^2)^2}+\frac{4}{a^2-r^2}+\frac{4r\Delta r}{a^2-r^2} \tag{24}$$

Recall that $(M,\widetilde{G})$ is a complete Riemannian manifold with Ricci curvature bounded by a negative constant $-N$, then the Laplacian comparison theorem implies that

$$r\Delta r\leqslant(n-1)(1+\sqrt{N}r) \tag{25}$$

Combining (24) with (25), we have

$$\Phi(\chi-\theta)\leqslant\left(24-\frac{16n}{n-1}+\frac{4\kappa^2}{\theta}\right)\frac{r^2}{(a^2-r^2)^2}+\frac{4n}{a^2-r^2}+\frac{4(n-1)r\sqrt{N}}{a^2-r^2}.$$

Since χ is a positive constant, we may choose a sufficiently small number $\theta>0$ such that $\chi-\theta>0$. Therefore (at $\bar{p}$) we have

$$\Phi\leqslant d_1\frac{r^2}{(a^2-r^2)^2}+d_2\frac{r}{a^2-r^2}+d_3\frac{1}{a^2-r^2} \tag{26}$$

(26) holds everywhere on $B_a(p_0)$. Let $a\to\infty$ we get $\Phi\equiv0$ on M. By the well-known theorem of Calabi[10], we conclude that M is an elliptic paraboloid. This completes the proof.

5 Proof of Corollary

Let $\bar{a}=r(p_0,\bar{p})$. We will separate the discussion into two cases:

Case 1, $p_0=\bar{p}$. we have $\bar{a}=0$.

Case 2, $p_0\neq\bar{p}$. then $\bar{a}>0$. Let

$B_{\bar{a}}(p_0)=\{p\in M\mid r(p_0,p)\leqslant\bar{a}\}$.

Proposition 3.1 and the maximum principle gield

$\max_{B_{\bar{a}}(p_0)}\Phi=\max_{\partial B_{\bar{a}}(p_0)}\Phi$.

Note that $a^2-r^2=a^2-\bar{a}^2$ on $\partial B_{\bar{a}}(p_0)$, it follows that

$\max_{B_{\bar{a}}(p_0)}\Phi=\Phi(\bar{p})$.

Consider $p\in B_{\bar{a}}(p_0)$. We choose an affine coordinate neighborhood $\{U,\varphi\}$ with $p\in U$ such that $R_{ij}(p)=0$, for $i\neq j$ and $G_{ij}(\varphi(p))=\delta_{ij}$, $1\leqslant i,j\leqslant n$ in U. From (6)~(7) we have

$$\widetilde{R}_{ii}=\sum(\widetilde{A}_{iml})^2-\sum\widetilde{A}_{iim}\widetilde{A}_{mll}\geqslant\sum(\widetilde{A}_{iim})^2-\sum\widetilde{A}_{iim}\widetilde{A}_{mll}=\sum(\widetilde{A}_{iim})^2-\sum\widetilde{A}_{iim}\frac{n+2}{2}\left(\frac{\rho F'}{F}-1\right)\frac{\rho_{,m}}{\rho}\geqslant-\frac{(n+2)^2}{16}\left(\frac{\rho F'}{F}-1\right)^2\Phi.$$

If $F=\rho^\alpha$, we have

$$\widetilde{R}_{ii}\geqslant-\frac{(n+2)^2}{16}(\alpha-1)^2\Phi(\bar{p}),$$

i. e. α-Ricci curvature is bounded from below by a negative constant $-N$. From (1) and (2), we have

$$\chi=\frac{n^2+8n-4}{n-1}-\frac{n^2-4n+2}{n-1}\alpha^2.$$

$$\kappa=\frac{5n+2}{2(n-1)}+\frac{n^2+n-2}{2(n-1)}\alpha.$$

This completes the proof of Corollary1.2.

References:

[1] Xiong M, Yang B Y. Hyperbolic relative hyperspheres with Li-normalization [J]. Results Math, 2011, 59: 545.

[2] Xu R W. Bernstein properties for some relative parabolic affine hyperspheres [J]. Results Math, 2008, 52: 409.

[3] Wu Y D, Zhao G S. Hypersurfaces with Li-normalization and prescribed Gauss-Kronecker curvature [J]. Results Math, 2011, 59: 563.

[4] Wang B F, Xu R W. The Euclidean complete hypersurfaces with negative constant relative affine mean curvature [J]. J Sichuan Univ: Nat Sci Ed (四川大学学报:自然科学版), 2008, 45: 1001.

[5] Ji X, Hu C F. Some variational problems for α - relative hypersurfaces [J]. J Sichuan Univ: Nat Sci Ed (四川大学学报:自然科学版), 2015, 52: 487.

[6] Xu R W, Li A M, Li X X. Euclidean complete α - relative extremal hypersurfaces [J]. J Sichuan Univ: Nat Sci Ed (四川大学学报:自然科学版), 2009, 46: 1217.

[7] Xu R W, Cao L F. Bernstein properties for α -complete hypersurfaces [J]. J Inequal Appl, 2014, 2014: 159.

[8] Pogorelov A V. The Minkowski multidimensional problem [M]. New York/London: John Wiley and Sons, 1978.

[9] Pogorelov A V. On the improper convex affine hyperspheres [J]. Geometriae Dedicata, 1972, 1: 33.

[10] Calabi E. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens [J]. Mich Math J, 1958, 5: 550.