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一类半线性分数阶微分方程解的存在性

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摘 要：本文考虑如下含有两项分数阶导数的半线性分数阶微分方程解的存在性问题：

$$\begin{cases} {}^cD_t^\alpha u(t) + \lambda {}^cD_t^\beta u(t) = f(t,u(t)), 0<t\leqslant h, \\ u(0) = x_0, u'(0) = y_0, \end{cases}$$

其中 $1<\alpha\leqslant 2,\alpha>\beta>0,{}^cD_t^\alpha$ 为 Caputo 分数阶导数. 利用 Schauder 不动点定理,作者证明了在适当条件下解存在.所得结果改进了已有结论.

关键词：分数阶微分方程；Caputo 导数；Schauder 不动点定理
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Existence of solutions to a class of semilinear fractional differential equations

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Abstract: In this paper, based on the Schauder fixed point theorem, we consider existence of the solutions for a class of two-term fractional initial boundary value problem:

$$\begin{cases} {}^cD_t^\alpha u(t) + \lambda {}^cD_t^\beta u(t) = f(t,u(t)), 0<t\leqslant h, \\ u(0) = x_0, u'(0) = y_0, \end{cases}$$

where $\alpha\in(1,2], \alpha>\beta>0$, and the fractional derivative is in the sense of Caputo. Our results improve some recent results.

Keywords: Fractional differential equations; Caputo fractional derivatives; Schauder fixed point theorem
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1 Introduction

During the past decades, fractional differential equations[1-4] have attracted much attention due to their application in various sciences, such as physics, mechanics, chemistry, engineering, and many other branches of science.

Recently, there are many papers devoted to the existence of solutions to fractional differential equations, see for example, Refs. [5-15]. Some techniques of nonlinear analysis, such as fixed-point theorems, Leray-Schauder theory, and topological degree theory are applied to research the existence of the solutions. In most of the papers, a Lipschitz-type condition is assumed as the basic hypothesis. For example, Diethlm and Ford[2] proved the existence of the fractional differential equation

$$\begin{cases} D_t^\alpha u(t) = f(t,u(t)), \\ u^{(k)}(0) = u_0^{(k)}, k=0,1,\ldots,m-1, \end{cases}$$

where $0<\alpha<1$, m is the integer defined by $m-1$

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$<\alpha\leqslant m$, the differential operator D_t^α are taken in the Riemann - Liouville sense and the initial conditions are specified according to Caputo's suggestion, provided the Lipschitz condition holds:

$$|f(t,x)-f(t,y)|\leqslant L|x-y|.$$

Next in 2005, Yu[16] considered the existence of the same equation mentioned above but in a less weak condition of $f(t,u(t))$:

$$|f(t,x)-f(t,y)|\leqslant\lambda(t)h(|x-y|),$$

where h is continuous and λ satisfies an integral condition. By the lower and upper solution method and a fixed point theorem, Ref. [17] discussed the existence for a class of fractional initial value problem:

$$\begin{cases}D_t^\alpha u(t)=f(t,u(t)),t\in(0,h),\\ t^{2-\alpha}u(t)|_{t=0}=b_1,D_0^{\alpha\mp1}u(t)|_{t=0}=b_2,\end{cases}$$

where $f\in C([0,h]\times\mathbf{R})$, D_t^α is the Riemann-Liouville fractional derivative, $1<\alpha<2$. In this paper they used the assumption as follows: $f:[0,h]\times\mathbf{R}\rightarrow\mathbf{R}$ and there exist constants $A,B,C\geqslant0$ and $0<r_1\leqslant1<r_2<1/(2-\alpha)$ such that for $t\in[0,h]$

$$|f(t,u)-f(t,v)|\leqslant A|u-v|^{r_1}+B|u-v|^{r_2},u,v\in\mathbf{R}$$

and $|f(t,0)|\leqslant C$ for $t\in[0,h]$.

On the other hand, it appears that for some processes the order of the time-fractional derivative from the corresponding model equation does not remain constant. A possible approach to handle these phenomena is to employ the multi-term time-fractional differential equation, see for example Ref. [18]. For instance, Babakhani and Daftardar-Gejji[19] considered the initial value problem of nonlinear fractional differential equation:

$$\begin{cases}L(D)u=f(t,u),0<t<1,\\ u(0)=0,\end{cases}$$

where $L(D)={}^cD^{s_n}-a_{n-1}^c D^{s_{n-1}}-\cdots-a_1^c D^{s_1}$, $0<s_1<s_2<\cdots<s_n<1$, and $a_j>0$, $j=1,\cdots,n-1$. ${}^cD^{s_n}$ denotes the Caputo fractional derivative of order s_n. The mathematical analysis of such equations has been carried out extensively by many authors[20,21].

Motivated by the works mentioned above, we consider in this paper the existence of solutions for a class of two-term fractional initial boundary value problem

$$\begin{cases}{}^cD_t^\alpha u(t)+\lambda{}^cD_t^\beta u(t)=f(t,u(t)),0<t\leqslant h,\\ u(0)=x_0,u'(0)=y_0\end{cases}\tag{1}$$

where ${}^cD_t^\alpha$, ${}^cD_t^\beta$ are Caputo fractional derivatives, $1<\alpha<2$ and $\alpha>\beta>0$. Lemma 3.2 clarifies how the regularity of a function can be improved after integrating the function for fractional times, which may be of independent interest. In our main result our assumption is more weaker than those in Refs. [2,17]. It is remarkable that our methods also apply to multi-term fractional differential equations.

2 Preliminaries

We begin with the definitions of fractional integrals and fractional derivatives[1]. Let $\alpha>0$, $m=\lceil\alpha\rceil$, say, the smallest integer greater than or equal to α, and $I=(0,T)$ for some $T>0$.

Definition 2.1 (Riemann-Liouville fractional integral) The Riemann-Liouville fractional integral of order $\alpha>0$ is defined as follows:

$$I_t^\alpha f(t):=(g_\alpha * f)(t)=\int_0^t g_\alpha(t-s)f(s)\mathrm{d}s,$$

for $f\in L^1(I)$, $t>0$, where

$$g_\alpha(t)=\begin{cases}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)},t>0,\\ 0,t\leqslant0\end{cases}$$

and $\Gamma(\alpha)$ is the Gamma function.

Definition 2.2 (Riemann-Liouville fractional derivative) The Riemann Liouville fractional derivative of order α for all f satisfying

$$f\in L^1(I),g_{m-\alpha}*f\in W^{m,1}(I)$$

by

$$D_t^\alpha f(t):=D_t^m(g_{m-\alpha})*f(t)=D_t^m I_t^{m-\alpha}f(t),$$

where $D_t^m:=\dfrac{\mathrm{d}^m}{\mathrm{d}t^m}$, $m\in\mathbf{N}$.

Definition 2.3 (Caputo fractional derivative) The Caputo fractional derivative of order $\alpha>0$ is defined by

$${}^cD_t^\alpha f(t):=I_t^{m-\alpha}D_t^m f(t).$$

The basic properties of fractional integrals and derivatives are collected in the following lemma.

Lemma 2.4 If $f\in C^{m-1}(I)$, $\alpha>0$, $\beta>0$, then we have the following conclusions:

(i) $I_t^{\alpha} I_t^{\beta} f(t) = I_t^{\alpha+\beta} f(t)$;

(ii) $I_t^{\alpha} I_t^{\beta} f(t) = I_t^{\beta} I_t^{\alpha} f(t)$;

(iii) ${}^cD_t^{\alpha} I_t^{\alpha} f(t) = f(t)$;

(iv) $I_t^{\alpha\,c} D_t^{\alpha} f(t) = f(t) - \sum_{i=1}^{m-1} f^{(k)}(0) g_{k+1}(t)$.

Next we recall the definition of compact operator and Arzela-Ascoli theorem.

Definition 2.5 A compact operator is a linear operator T from a Banach space X to another Banach space Y, such that the image under T of any bounded subset of x is a relatively compact subset of Y. That is, if $\{x_n\}$ is a bounded sequence in X, then there is a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges.

Lemma 2.6 (Arzela-Ascoli theorem) Consider a sequence of real-valued continuous functions $\{f_n\}$ defined on a closed and bounded interval $[a,b]$ of the real line. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly. The converse is also true, in the sense that if every subsequence of $\{f_n\}$ itself has a uniformly convergent subsequence, then $\{f_n\}$ is uniformly bounded and equicontinuous.

Finally, the following fixed point theorem is needed in the sequel.

Lemma 2.7 (Schauder fixed point theorem) If U is a nonempty convex subset of a Banach space and T is a compact operator of U into itself such that $T(U)$ is contained in a compact subset of, then T has a fixed point.

3 Main results

We will use the following assumption:

[H] $f: [0,h] \times \mathbf{R} \to \mathbf{R}$ and there exist constants $A,B,C \geqslant 0$ and $0 < r_1 \leqslant 1 < r_2$ such that for $t \in [0,h]$,

$$|f(t,u) - f(t,v)| \leqslant A|u-v|^{r_1} + B|u-v|^{r_2} \tag{2}$$

for $u,v \in \mathbf{R}$ and

$$|f(t,0)| \leqslant C$$

for $t \in [0,h]$.

***Remark*1** Assume that $f(t,u) = f(t)$ and f is a Hölder continuous function, then (2) holds.

Lemma 3.1 Let ${}^cD_t^{\alpha}$, ${}^cD_t^{\beta}$ and f be as in (1). Then $u(t)$ is a solution of (1) if and only if $u(t)$ is the solution to the integral equation

$$u(t) = x_0\left[1 + \frac{ct^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + y_0\left[t + \frac{ct^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] - c\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} u(s)\mathrm{d}s + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\mathrm{d}s \tag{3}$$

when $1 < \beta < 2$, or

$$u(t) = x_0\left[1 + \frac{ct^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + y_0 t - c\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} u(s)\mathrm{d}s + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\mathrm{d}s \tag{4}$$

when $0 < \beta < 1$.

Proof We suppose that $1 < \beta < 2$. By integration α-times on both sides of (1), we can obtain

$$I_t^{\alpha\,c} D_t^{\alpha} u(t) + cI_t^{\alpha-\beta} I_t^{\beta\,c} D_t^{\beta} u(t) = I_t^{\alpha} f(t,u(t)),$$

it then follows from Lemma 2.4 (iv) that

$$u(t) - u(0) - u'(0)t + cI_t^{\alpha-\beta}(u(t) - u(0) - u'(0)t) = I_t^{\alpha} f(t,u(t)).$$

By inserting the initial data $u(0) = x_0, u'(0) = y_0$ we get

$$u(t) - x_0 - y_0 t + cI_t^{\alpha-\beta}(u(t) - x_0 - y_0 t) = I_t^{\alpha} f(t,u(t)),$$

that is

$$u(t) = x_0 + y_0 t - cI_t^{\alpha-\beta}(u(t) - x_0 - y_0 t) + I_t^{\alpha} f(t,u(t)) = x_0[1 + cg_{\alpha-\beta+1}(t)] + y_0[1 + cg_{\alpha-\beta+2}(t)] - c\int_0^t g_{\alpha-\beta}(t-s)u(s)\mathrm{d}s + \int_0^t g_{\alpha}(t-s) f(s,u(s))\mathrm{d}s.$$

This is (3). Using the same method one can prove (4) for the case $0 < \beta < 1$.

In the following lemma, we discuss how the regularity of a function can be improved when integration the function for γ-times.

Lemma 3.2 Let $f \in L^{\infty}[0,T]$, $\gamma > 0$ and define the function $g(t) = I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s)\mathrm{d}s$.

(i) if $0 < \gamma \leqslant 1$ and $f(t) \equiv \Gamma(\gamma+1)$, the function $g(t)=t^\gamma$ is γ-Hölder continuous with

$$|g(t)-g(s)|\leqslant|t-s|^\gamma,\quad t,s\in[0,T];$$

(ii) if $0<\gamma\leqslant 1$, then g is γ-Hölder continuous on $[0,T]$. More precisely,

$$|g(t)-g(s)|\leqslant\frac{2\|f\|_\infty}{\Gamma(\gamma+1)}|t-s|^\gamma,\quad t,s\in[0,T];$$

(iii) if $\gamma>1$, then g is Lipschitz continuous on $[0,T]$

$$|g(t)-g(s)|\leqslant g_\gamma(T)\|f\|_\infty|t-s|,\quad t,s\in[0,T];$$

(iv) if $\gamma>1$ and $\gamma\notin\mathbf{N}$, $n=\lceil\gamma\rceil$ then $g\in C^n[0,T]$ and $g^{(n)}$ is $(\gamma-n)$-Hölder continuous:

$$|g^{(n)}(t)-g^{(n)}(s)|\leqslant\frac{2\|f\|_\infty}{\Gamma(\gamma-n+1)}|t-s|^{\gamma-n},\quad t,s\in[0,T].$$

Proof (i) This follows directly from the inequality $(a+b)^\gamma\leqslant a^\gamma+b^\gamma$ for $0<\gamma\leqslant 1$ and $a,b\geqslant 0$.

(ii) Let $f\in L^\infty[0,T]$ and let $t_1,t_2\in[0,T]$ such that $t_1\geqslant t_2$, then

$$|g(t_1)-g(t_2)|=\left|\frac{1}{\Gamma(\gamma)}\int_0^{t_1}(t_1-s)^{\gamma-1}f(s)\mathrm{d}s-\frac{1}{\Gamma(\gamma)}\int_0^{t_2}(t_2-s)^{\gamma-1}f(s)\mathrm{d}s\right|\leqslant\left|\frac{1}{\Gamma(\gamma)}\int_0^{t_2}[(t_1-s)^{\gamma-1}-(t_2-s)^{\gamma-1}]f(s)\mathrm{d}s\right|+\left|\frac{1}{\Gamma(\gamma)}\int_{t_2}^{t_1}(t_1-s)^{\gamma-1}f(s)ds\right|\leqslant\frac{\|f\|_\infty}{\Gamma(\gamma)}\left\{\int_0^{t_2}[(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}]\mathrm{d}s+\int_{t_2}^{t_1}(t_1-s)^{\gamma-1}\mathrm{d}s\right\}=-\frac{\|f\|_\infty}{\Gamma(\gamma+1)}[(t_2-s)^\gamma|_0^{t_2}-(t_1-s)^\gamma|_0^{t_2}+(t_1-s)^\gamma|_{t_2}^{t_1}]=\frac{\|f\|_\infty}{\Gamma(\gamma+1)}[2(t_1-t_2)^\gamma-(t_1^\gamma-t_2^\gamma)]\leqslant\frac{2\|f\|_\infty}{\Gamma(\gamma+1)}(t_1-t_2)^\gamma.$$

(iii) For $\gamma>1$, we have

$$|g(t_1)-g(t_2)|\leqslant\frac{\|f\|_\infty}{\Gamma(\gamma)}\left\{\int_0^{t_2}[(t_1-s)^{\gamma-1}-(t_2-s)^{\gamma-1}]\mathrm{d}s+\int_{t_2}^{t_1}(t_1-s)^{\gamma-1}\mathrm{d}s\right\}=-\frac{\|f\|_\infty}{\Gamma(\gamma+1)}[(t_1-s)^\gamma|_0^{t_2}-(t_2-s)^\gamma|_0^{t_2}+(t_1-s)^\gamma|_{t_2}^{t_1}]=\frac{\|f\|_\infty}{\Gamma(\gamma+1)}(t_1^\gamma-t_2^\gamma)\leqslant\frac{T^{\gamma-1}\|f\|_\infty}{\Gamma(\gamma)}(t_1-t_2)=g_\gamma(T)\|f\|_\infty(t_1-t_2).$$

(iv) Let $t\geqslant 0$ and $h>0$. Since $\gamma>1$, we have

$$\frac{g(t+h)-g(t)}{h}=\frac{1}{h\Gamma(\gamma)}\left[\int_0^{t+h}(t+h-s)^{\gamma-1}f(s)\mathrm{d}s-\int_0^t(t-s)^{\gamma-1}f(s)\mathrm{d}s\right]=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{(t+h-s)^{\gamma-1}-(t-s)^{\gamma-1}}{h}f(s)\mathrm{d}s-\frac{1}{h\Gamma(\gamma)}\int_t^{t+h}(t+h-s)^{\gamma-1}f(s)\mathrm{d}s\to\frac{1}{\Gamma(\gamma-1)}\int_0^t(t-s)^{\gamma-2}f(s)\mathrm{d}s$$

by dominated convergence theorem. By induction we have $g\in C^n[0,T]$, for $n=\lfloor\gamma\rfloor$ and

$$g^{(k)}(t)=\frac{1}{\Gamma(\gamma-k)}\int_0^t(t-s)^{\gamma-k-1}f(s)\mathrm{d}s,$$

$k=1,\cdots,n$. And in particular,

$$g^{(n)}(t)=\frac{1}{\Gamma(\gamma-n)}\int_0^t(t-s)^{\gamma-n-1}f(s)\mathrm{d}s$$

since $0<\gamma-n<1$. By (ii) we have

$$|g^{(n)}(t)-g^{(n)}(s)|\leqslant\frac{2\|f\|_\infty}{\Gamma(\gamma-n+1)}|t-s|^{\gamma-n}.$$

***Remark* 2** The results of the above lemma are easily extended to vector-valued function $f\in L^\infty([0,T],X)$, where X is a Banach space.

Lemma 3.3 Suppose [H] holds. Define the operator T_1,T_2 by

$$(T_1u)(t)=x_0\left[1+\frac{ct^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]+y_0\left[t+\frac{ct^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right]-c\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}u(s)\mathrm{d}s+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s\qquad(5)$$

and

$$(T_2u)(t)=x_0\left[1+\frac{ct^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]+y_0t-c\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}u(s)\mathrm{d}s+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s.$$

Then the operators T_1 and T_2 are compact operators on $C[0,h]$

Proof We only show that the operator T_1 is compact from $C[0,h]$ to $C[0,h]$. The proof for T_2 is analogous. If $u\in C[0,h]$, then it follows from the definition of $(T_1u)(t)$ and Lemma 3.2 that $(T_1u)(t)$ is also continuous. Now given a sequence $\{u_n\}\subset C[0,h]$ such that $u_n\to u$ in $C[0,h]$, then by the definition of T_1 one has

$$|T_1u_n(t)-T_1u(t)|\leqslant|c\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(u_n(s)-u(s))\mathrm{d}s|+|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(s,u_n(s))-f(s,u(s))\mathrm{d}s|\leqslant\frac{|c|t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|u_n-u\|+\frac{t^\alpha}{\Gamma(\alpha+1)}(A\|u_n-u\|^{r_1}+B\|u_n-u\|^{r_2})\leqslant|c|g_{\alpha-\beta+1}(h)\|u_n-u\|+g_{\alpha+1}(h)(A\|u_n-u\|^{r_1}+B\|u_n-u\|^{r_2}),$$

it follows that T_1 is continuous on $C[0,h]$. Suppose that $U=\{u\in C[0,h]:|u(t)|\leqslant R\}$ is a bounded set, then $T_1(U)$ is uniformly bounded since T_1 is continuous.

Finally we prove the equicontinuity of $T_1(U)$. For every $u\in U$, and $t_1,t_2\in[0,h]$, by Lemma 3.2,

$$|T_1u(t_1)-T_1u(t_2)|=|cx_0(g_{\alpha-\beta+1}(t_1)-g_{\alpha-\beta+1}(t_2))+y_0(t_1-t_2)+cy_0(g_{\alpha-\beta+2}(t_1)-g_{\alpha-\beta+2}(t_1))-c(\int_0^{t_1}\frac{(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}u(s)\mathrm{d}s-\int_0^{t_2}\frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}u(s)\mathrm{d}s)+(\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s)|\leqslant\frac{|cx_0|}{\Gamma(\alpha-\beta+1)}\cdot|t_1-t_2|^{\alpha-\beta}+|y_0|\cdot|t_1-t_2|+|cy_0|g_{\alpha-\beta+2}(h)\cdot|t_1-t_2|+\frac{2|c|R}{\Gamma(\alpha-\beta+1)}\cdot|t_1-t_2|^{\alpha-\beta}+(AR^{r_1}+BR^{r_2})g_\alpha(h)\cdot|t_1-t_2|,$$

which gives the equicontinuity of $T_1(U)$. It thus follows from the Arzela-Ascoli theorem (Lemma 2.6) that T_1 is compact on $C[0,h]$.

Now we are able to give the main result of this paper.

Theorem 3.4 Suppose [H] holds, if there exists some constant R such that

$$|x_0|(1+|c|g_{\alpha-\beta+1}(h))+|y_0|(h+|c|g_{\alpha-\beta+2}(h))+|c|g_{\alpha-\beta+1}(h)R+g_{\alpha+1}(R)(AR^{r_1}+BR^{r_2}+C)\leqslant R$$

when $\beta>1$ or

$$|x_0|(1+|c|g_{\alpha-\beta+1}(h))+|y_0|h+|c|g_{\alpha-\beta+1}(h)R+g_{\alpha+1}(R)(AR^{r_1}+BR^{r_2}+C)\leqslant R$$

when $0<\beta<1$, then the fractional initial boundary value problem (1) has at least a solution in $C[0,h]$.

Proof Suppose $U=\{u\in C[0,h]:|u(t)|\leqslant R\}$, then U is a bounded convex set. For $u\in U$, by assumption [H],

$$|f(t,u)|\leqslant AR^{r_1}+BR^{r_2}+C.$$

Therefore, by (5) and Lemma 3.2,

$$|(T_1u)|(t)=|x_0(1+\frac{ct^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)})+y_0(t+\frac{ct^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)})-c\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}u(s)\mathrm{d}s+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\mathrm{d}s|\leqslant|x_0|(1+\frac{|c|h^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)})+|y_0|(h+\frac{|c|h^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)})+|c|R\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\mathrm{d}s+(AR^{r_1}+BR^{r_2}+C)\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\leqslant|x_0|(1+|c|g_{\alpha-\beta+1}(h))+|y_0|(h+|c|g_{\alpha-\beta+2}(h))+|c|Rg_{\alpha-\beta+1}(h)+g_{\alpha+1}(h)(AR^{r_1}+BR^{r_2}+C)\leqslant R.$$

That is to say, $T_1(U)$ is also contained in U. Use the same method, we can obtain the same conclusion for $T_2u(t)$.

Since the operators T_1 and T_2 are compact operators by Theorem 3.3, according to the Schauder fixed point theorem (Lemma 2.7), we know that the fractional initial boundary value problem (1) has at least a solution in $C[0,h]$.

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