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带变号格林函数的三阶三点差分方程边值问题的多解性

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摘　要：本文利用 Leggett-Williams 不动点定理得到了离散非线性三阶三点边值问题

$$\begin{cases}\Delta^3 u(t-1)=f(t,u(t)),t\in[1,T-2]_Z,\\ \Delta u(0)=u(T)=\Delta^2(\eta)=0\end{cases}$$

正解的存在性，这里 $T>4$ 是一个整数，$f\in[1,T-2]_Z\times[0,\infty),[0,\infty)$ 是连续函数并且 η 满足：若 T 是奇数，则 $\eta\in[\frac{T-1}{2},T-2]_Z$；若 T 是偶数，则 $\eta\in[\frac{T-2}{2},T-2]_Z$.

关键词：离散的三阶三点边值问题；格林函数变号；正解；锥
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Multiple positive solutions of a discrete third-order three-point BVP with sign-changing Green's function

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Abstract: In this paper, by using the Leggett-Williams fixed point theorem, we obtain existence of positive solutions of the following discrete nonlinear third-order three-point boundary value problems:

$$\begin{cases}\Delta^3 u(t-1)=f(t,u(t)),t\in[1,T-2]_Z,\\ \Delta u(0)=u(T)=\Delta^2(\eta)=0\end{cases}$$

where $T>4$ is an integer, $f\in[1,T-2]_Z\times[0,\infty),[0,\infty)$ is continuous and

$$\begin{cases}\eta\in[\frac{T-1}{2},T-2]_Z\ T\equiv 1(\mathrm{mod}\ 2),\\ \eta\in[\frac{T-2}{2},T-2]_Z\ T\equiv 0(\mathrm{mod}\ 2).\end{cases}$$

Keywords: Discrete third-order three-point boundary value problem; Sign-changing Green′s function; Positive solution; Cone
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1 Introduction

Let a,b be two integers with $b > a$. Let us employ $[a,b]_Z$ to denote the integer set $\{a,a+1,\ldots,b\}$. For any real number c, $[c]$ is the integer part of c. In this paper, we consider the existence of positive solutions of the discrete third-order three-point BVP

$$\begin{cases}\Delta^3 u(t-1)=f(t,u(t)),\ t\in[1,T-2]_Z\\ \Delta u(0)=u(T)=\Delta^2 u(\eta)=0\end{cases} \tag{1}$$

where $T>4$ is an integer, $f\in([1,T-2]_Z\times[0,\infty),[0,\infty))$ is continuous and η satisfies the condition:

(H_0) $\eta\in[\frac{T-1}{2},T-2]_Z$, if T is an odd number, or $\eta\in[\frac{T-2}{2},T-2]_Z$, if T is an even number.

Difference equations appear in many mathematical models in diverse fields, such as economy, biology, physics and finance[1-17]. In recent years, the existence of positive solutions of third-order boundary value problems has been discussed by several authors. For example, In Refs. [2, 8～15], by using different methods such as the Krasnosel'skiis fixed point theorem in cone, the iterative technique, and the fixed point theory, the authors obtained the existence of positive solutions of the boundary value problems for third-order differential equations. For the discrete case, there are also several excellent results on the existence of positive solutions of the discrete third-order boundary value problems, see, for instance, Refs. [3, 6～12, 16] and the references therein. For instance, by using the Guo-Krasnosel' skiis fixed point theorem, Agarwal and Henderson[7] considered the existence of positive solutions of the discrete third order boundary value problems

$$\begin{cases}\Delta^3 u(t-1)=\lambda a(t)f(t,u(t)),\\ \quad t\in[2,T+2]_Z,\\ u(0)=u(1)=u(T+3)=0\end{cases} \tag{2}$$

Later, by using the same theorem, Kong *et al*[9] considered the existence of positive solutions of the more general boundary value problems for third-order functional difference equations.

However, in order to obtain the positive solutions, the Green's functions they used are positive in all of the above papers. Now the question is how can we get the existence of positive solutions for the discrete problems when the Green's function changes it's sign. In 2015, by using the Guo-Krasnoselskii's fixed point theorem, Wang and Gao[16] first discussed the existence of positive solutions of the following third-order difference equation boundary value problems

$$\begin{cases}\Delta^3 u(t-1)=a(t)f(t,u(t)),\ t\in[1,T-1]_Z,\\ u(0)=\Delta u(T)=\Delta^2 u(\eta)=0\end{cases} \tag{3}$$

in which the Green's function $G(t,s)$ for (3) is sign-changing. This leads many difficulties, such as the positivity of the summation operator, the construction of the cone, the concavity of the solutions, and other computation difficulties in discussing the existence of positive solutions of (3).

Motivated by the above-mentioned results, we consider the existence of multiple positive solutions of (1). It will be shown that the Green's function for (1) is also sign-changing. To overcome the difficulties which are led by the sign-changing Green's function, a new cone is defined and the positivity and concavity of the solutions of the corresponding linear problems are discussed, see Section 2. The main tool is the Leggett-Williams fixed point theorem[18].

Now, let us state some fundamental concepts and the Leggett-Williams fixed point theorem[17].

Let E be a real Banach space with cone P. A map $\sigma: P\to(-\infty,+\infty)$ is said to be a concave functional if

$$\sigma(\lambda x+(1-\lambda)y)\geqslant\lambda\sigma(x)+(1-\lambda)\sigma(y)$$

for all $x,y\in P$ and $\lambda\in[0,1]$.

Let a and b be two number with $0<a<b$ and σ be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_a=\{x\in P:\ \|x\|<a\},$$

$$P(\sigma,a,b)=\{x\in P: a\leqslant\sigma(x),\ \|x\|<a\}.$$

Our main tool is the following well-known Leggett-Williams fixed point theorem.

Theorem 1.1 Let $A:\overline{P_r}\to\overline{P_r}$ be completely continuous and σ be a nonnegative continuous concave functional on P such that $\sigma(x) \leqslant \| x \|$ for all $x \in \overline{P_c}$. Suppose that there exist $0 < d < a < b \leqslant c$ such that

(i) $\{x \in P(\sigma,a,b): \sigma(x) > a\} \neq \varnothing$ and $\sigma(Ax) > a$ for $x \in P(\sigma,a,b)$;

(ii) $\| Ax \| < d$ for $\| x \| < d$;

(iii) $\sigma(Ax) > a$ for $x \in P(\sigma,a,b)$ with $\| Ax \| > b$.

Then A has at least three fixed points x_1, x_2, x_3 in $\overline{P_c}$ satisfying

$$\| x_1 \| < d, a < \sigma(x_2), \| x_3 \| > d, \sigma(x_3) < a.$$

The rest of this paper is arranged as follows: In section 2, we will show the expression and some properties of the Green′s function of (1). Specially, we show that the Green′s function changes its sign. Moreover, we give some other preliminaries. In Section 3, we demonstrate our main result and prove it.

2 Preliminaries

At first, let us consider the following linear problem

$$\begin{cases} \Delta^3 u(t-1) = y(t) \quad t \in [1, T-2]_Z \\ \Delta u(0) = u(T) = \Delta^2 u(\eta) = 0 \end{cases} \tag{4}$$

Define the Green's function $G(t,s)$ as follows.

If $(t,s) \in [2,T]_Z \times [\eta+1, T-2]_Z$, then

$$G(t,s) = \begin{cases} \dfrac{-(T-s)(T-s-1)}{2}, 0 \leqslant t-2 < s \leqslant T-2, \\ \dfrac{(t-T)(t+T-1-2s)}{2}, \eta < s \leqslant t-2 \leqslant T-2 \end{cases} \tag{5}$$

If $(t,s) \in [2,T]_Z \times [1,\eta]_Z$, then

$$G(t,s) = \begin{cases} \dfrac{t-t^2-s^2-s+2sT}{2}, 0 \leqslant t-2 < s \leqslant \eta, \\ s(T-t), 1 \leqslant s \leqslant t-2 \leqslant T-2 \end{cases} \tag{6}$$

Meanwhile,

$$G(0,s) = G(1,s) = \begin{cases} \dfrac{-(T-s)(T-s-1)}{2}, \eta < s \leqslant T-2, \\ \dfrac{2sT-s-s^2}{2}, 1 \leqslant s \leqslant \eta \end{cases} \tag{7}$$

If $\eta = T-2$, then the Green's function $G(t,s)$ is defined only by (6) and (7).

Lemma 2.1 The problem (4) has a unique solution

$$u(t) = \sum_{s=1}^{T-2} G(t,s) y(s) \tag{8}$$

where $G(t,s)$ is defined by (5)～(7).

Proof Summing from $s=1$ to $s=t-1$ at both sides of the equation in (4), we get

$$\Delta^2 u(t-1) = \Delta^2 u(0) + \sum_{s=1}^{t-1} y(s).$$

Repeating the above process, we obtain

$$\Delta u(t-1) = (t-1)\Delta^2 u(0) + \sum_{s=1}^{t-2} (t-s-1) y(s).$$

Summing from $s=1$ to $s=t$ at both sides of above equation, we have

$$u(t) = u(0) + \frac{t(t-1)}{2}\Delta^2 u(0) + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s).$$

By using the boundary condition $\Delta u(0) = u(T) = \Delta^2 u(\eta) = 0$, we get that

$$\begin{cases} \Delta^2 u(0) + \displaystyle\sum_{s=1}^{\eta} y(s) = 0, \\ u(0) = \displaystyle\sum_{s=1}^{\eta} \frac{T(T-1)}{2} y(s) - \displaystyle\sum_{s=1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s). \end{cases}$$

Therefore,

$$u(t) = \sum_{s=1}^{\eta} \frac{T(T-1)-t(t-1)}{2} y(s) - \sum_{s=1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s) + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s) \tag{9}$$

This implies that (8) holds.

Now, we can give some properties of $G(t,s)$.

If $(t,s) \in [2,T]_Z \times [\eta+1, T-2]_Z$, then

$$\Delta G(t,s) = \begin{cases} 0, 1 \leqslant t-2 < s \leqslant T-2, \\ t-s, 1 \leqslant s \leqslant t-2 \leqslant T-2; \end{cases}$$

If $(t,s) \in [2,T]_Z \times [1,\eta]_Z$, then

$$\Delta G(t,s) = \begin{cases} -t, 1 \leqslant t-2 < s \leqslant T-2, \\ -s, 1 \leqslant s \leqslant t-2 \leqslant T-2. \end{cases}$$

Thus, if $1 \leqslant s \leqslant \eta$, then $\Delta G(t,s) \leqslant 0$ for $t \in [0, T$

$-1]_Z$, and $G(t,s)\geqslant 0$ for $t\in[0,T]_Z$. If $\eta<s\leqslant T-2$, then $\Delta G(t,s)\geqslant 0$ for $t\in[0,T-1]_Z$, and $G(t,s)<0$ for $t\in[0,T]_Z$. Therefore $G(t,s)$ changges its sign. If $s>\eta$, then

$$\max_{t\in[0,T]_z} G(t,s)=G(T,s)=0$$

and

$$\min_{t\in[0,T]_z} G(t,s)=G(0,s)=\frac{-(T-s)(T-s-1)}{2}\geqslant\frac{-(T-\eta)(T-\eta-1)}{2}.$$

If $s\leqslant\eta$, then

$$\min_{t\in[0,T]_z} G(t,s)=G(T,s)=0$$

and

$$\max_{t\in[0,T]_z} G(t,s)=G(0,s)=\frac{-s^2-s+2sT}{2}\leqslant\frac{-\eta^2-\eta+2\eta T}{2}.$$

***Remark* 1** If $\eta=T-2$, then we find that $G(t,s)\geqslant 0$ and $G(t,s)\neq 0$. This case has been discussed by several authors, see, for instance, Refs.[3～11]. So, in the rest of this paper we could suppose that $\eta<T-2$.

***Remark* 2** Before we consider the existence of positive solutions of (4), we may discuss the existence of positive solution of a more special problem

$$\begin{cases}\Delta^3 u(t-1)=1\ t\in[1,T-2]_Z,\\ \Delta u(0)=u(T)=\Delta^2 u(\eta)=0\end{cases}\tag{10}$$

We will see that (H_0) is a necessary and sufficient condition for the existence of positive solutions to (10). To some extend, this explains why we choose such η which satisfies (H_0).

From Lemma 2.1, we know that (10) has a solution $u(t)$ as follows,

$$u(t)=\frac{t^3-3(1+\eta)t^2+(3\eta+2)t-T^3+3(1+\eta)T^2-(3\eta+2)T}{6}.$$

For the sake of convenience, let

$$\varphi(t)=t^3-3(1+\eta)t^2+(3\eta+2)t-T^3+3(1+\eta)T^2-(3\eta+2)T.$$

Obviously, $u(t)\geqslant 0\Leftrightarrow\varphi(t)\geqslant 0$. By direct computation, we get

$$\Delta\varphi(t)=t(3t-3-6\eta),$$
$$\Delta\varphi(t)\geqslant 0$$

for $t>1+2\eta$ and $\Delta\varphi(t)\leqslant 0$ for $0<t<1+2\eta$. Furthermore, if $1+2\eta$ is an integer, then $\Delta\varphi(1+2\eta)=0$.

Now we prove that (H_0) is a necessary and sufficient condition of $\varphi(t)\geqslant 0, t\in[0,T]_Z$. In fact, if $\varphi(t)\geqslant 0$, then $\varphi(T)=0$, which implies that $\varphi(T-1)\geqslant 0$. If $\varphi(T-1)>0$, then $\Delta\varphi(T-1)<0$. This implies that $\eta>\frac{T-2}{2}$. If $\varphi(T-1)=0$, then $\varphi(T-2)>0$. Otherwise, $\varphi(t)\equiv 0, t\in[0,T]_Z$. This contradicts to $\varphi(t)\neq 0$. Therefore, $\Delta\varphi(T-2)<0$ and $\Delta\varphi(T-1)=0$. These two equations imply that T is an even number and $\eta=\frac{T-2}{2}$.

Conversely, if $\eta\geqslant\frac{T-2}{2}$, then $T-1\leqslant 1+2\eta$ and $\Delta\varphi(T-1)\leqslant 0$. This combines the condition $\varphi(T)=0$ implies that $\varphi(t)\geqslant 0, t\in[0,T]_Z$.

Let $E=\{u:[0,T]_Z\to\mathbf{R}\mid\Delta u(0)=u(T)=\Delta^2 u(\eta)=0\}$. Then E is a Banach space under the norm $\|u\|=\max_{t\in[0,T]_z}|u(t)|$. Now we define a cone $\dot{P}\subset E$ as follows:

$$\dot{P}=\{y\in E: y(t)\geqslant 0, t\in[0,T]_Z, \Delta y(t)\leqslant 0, t\in[0,T-1]_Z\}.$$

Lemma 2.2 Assume that (H_0) holds. If $y\in\dot{P}$, then the unique solution $u(t)$ of (4) belongs to $\dot{P}$, where $u(t)$ is defined as (8). Moreover, $u(t)$ is concave on $[0,\eta+2]_Z$.

Proof First, if $0\leqslant t-2\leqslant\eta$, then

$$u(t)=\sum_{s=1}^{t-2}s(T-t)y(s)+\sum_{s=t-1}^{\eta}\frac{t-t^2-s^2-s+2sT}{2}y(s)-\sum_{s=\eta+1}^{T-2}\frac{(T-s)(T-s-1)}{2}y(s),$$

$$\Delta u(t)=-\sum_{s=1}^{t-2}sy(s)+(1-t)y(t-1)-$$

$$\sum_{s=t}^{\eta} ty(s) \leqslant 0 \tag{11}$$

and

$$\Delta^2 u(t-1) = -\sum_{s=t}^{\eta} y(s) \leqslant 0 \tag{12}$$

Second, if $\eta < t-2 \leqslant T-2$, then

$$u(t) = \sum_{s=1}^{\eta} s(T-t)y(s) + \sum_{s=\eta+1}^{t-2} \frac{(t-T)(t+T-1-2s)}{2} y(s) - \sum_{s=t-1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s).$$

Since η satisfies (H_0), we obtain that

$$\Delta u(t) = -\sum_{s=1}^{\eta} sy(s) + \sum_{s=\eta+1}^{t-2} (t-s)y(s) + y(t-1) \leqslant y(\eta) \frac{t(t-1-2\eta)}{2} \leqslant 0 \tag{13}$$

and

$$\Delta^2 u(t-1) = \sum_{s=\eta+1}^{t-3} y(s) + y(t-1) + y(t-2) \geqslant 0.$$

By (11) and (12), we get that $\Delta u(t) \leqslant 0$ for all $t \in [0, T-1]_Z$. Combining this with the boundary condition $u(T) = 0$, we get $u(t) \geqslant 0$ for $t \in [0, T]_Z$, which implies $u \in \hat{P}$. Moreover, by (12) and the condition $\Delta^2 u(\eta) = 0$, we get that $\Delta^2 u(t-1) \leqslant 0$ for $t \in [1, \eta+1]_Z$. Therefore, $u(t)$ is concave on $[0, \eta+2]_Z$.

Lemma 2.3 Suppose that (H_0) holds. If $y \in \hat{P}$, then the unique solution $u(t)$, defined as (8), satisfies the following inequality:

$$\min_{t \in [T-2-\theta, \theta]_Z} u(t) \geqslant \theta^* \| u \| \tag{14}$$

where $\theta \in [\frac{T+1}{2}, \eta+1]_Z$, for odd T and $\theta \in [\frac{T}{2}, \eta+1]_Z$ for even T. Moreever, $\theta^* = \frac{\eta+2-\theta}{\eta+2}$.

Proof By Lemma 2.2, $u(t)$ is concave on $[0, \eta+2]_Z$. Then

$$\frac{u(t)-u(0)}{t} \geqslant \frac{u(\eta+2)-u(0)}{\eta+2}, t \in [0, \eta+1]_Z \tag{15}$$

From Lemma 2.2, we get that $u(t)$ is non-increasing for $t \in [0, T]_Z$, which implies that $u(0) = \| u \|$. Combining this with (15), we obtain

$$u(t) \geqslant \frac{\eta+2-t}{\eta+2} u(0) = \frac{\eta+2-t}{\eta+2} \| u \|.$$

This implies

$$\min_{t \in [T-2-\theta, \theta]_z} u(t) = u(\theta) \geqslant \frac{\eta+2-\theta}{\eta+2} \| u \| = \theta^* \| u \|.$$

Since $\frac{T+1}{2} > \frac{T-2}{2}$ for odd T and $\frac{T}{2} > \frac{T-2}{2}$ for even T, we get $\theta > T-2-\theta$ and the set $[T-2-\theta, \theta]_z$ is well-defined.

3 Main results

In the reminder of this paper, we assume that $f: [1, T-2]_Z \times [0, \infty) \to [0, \infty)$ is continuous and satisfies the following conditions:

(D_1) For each $x \in [0, +\infty)$, the mapping $t \to f(t, x)$ is decreasing;

(D_2) For each $t \in [1, T-2]_Z$, the mapping $x \to f(t, x)$ is increasing.

Let

$$P = \{u \in \hat{P} \mid \min_{t \in [T-2-\theta, \theta]_Z} u(t) \geqslant \theta^* u\}.$$

Then it is easy to check that P is a cone in E. Now, we define an operator A on P by

$$(Au)(t) = \sum_{s=1}^{T-2} G(t,s) f(s, u(s)), t \in [0, T]_Z.$$

Form Lemma 2.2 to Lemma 2.3, we know that $Au: P \to P$. Meanwhile, since E is finite dimensional, $Au: P \to P$ is completely conutinuous. Obviously, if u is an fixed point of A in P, then u is a nonnegative solution of (1).

For convenience, we denote

$$H_1 = \sum_{s=1}^{\eta} \frac{-\eta^2 - \eta + 2\eta T}{2},$$

$$H_2 = \min_{t \in [T-2-\theta, \theta]_Z} \sum_{s=T-2-\theta}^{\theta} G(t, s).$$

Theorem 3.1 Assume that there exist numbers d, a and c with $0 < d < a < \frac{a}{\theta^*} < c$ such that

$$f(t,u) < \frac{d}{H_1}, t \in [1, \eta]_z, u \in [0, d] \tag{16}$$

$$f(t,u) > \frac{a}{H_2}, t \in [T-2-\theta, \theta]_z, u \in [a, \frac{a}{\theta^*}] \tag{17}$$

$$f(t,u) < \frac{c}{H_1}, t \in [1, \eta]_z, u \in [0, c] \tag{18}$$

Then (1) has at least three positive solutions u, v and w satisfying

$$\| u \| < d, a < \min_{t\in[T-2-\theta,\theta]_z} v(t),$$
$$\| w \| > d, \min_{t\in[T-2-\theta,\theta]_z} w(t) < a.$$

Proof For $u \in P$, we define

$$\sigma(u) = \min_{t\in[T-2-\theta,\theta]_Z} u(t).$$

It is easy to check that σ is a nonnegative continuous concave functional on P with $\sigma(u) \leqslant \| u \|$ for $u \in P$.

We first asssert that if there exists a positive number r such that

$$f(t,u) < \frac{r}{H_1} \text{ for } t\in[1,\eta]_Z \text{ and } u \in [0,r],$$

then $A:\overline{P_r}\to P_r$. Indeed, if $u\in\overline{P_r}$, then

$$\| Au \| = \max_{t\in[0,T]_Z} \left| \sum_{s=1}^{T-2} G(t,s)f(s,u(s)) \right| \leqslant \max_{t\in[0,T]_Z} \left| \sum_{s=1}^{\eta} G(t,s)f(s,u(s)) \right| + \max_{t\in[0,T]_Z} \left| \sum_{s=\eta+1}^{T-2} G(t,s)f(s,u(s)) \right| \leqslant \frac{-\eta^2-\eta+2\eta T}{2}\sum_{s=1}^{\eta} f(s,u(s)) + \frac{(T-\eta)(T-\eta-1)}{2}\sum_{s=\eta+1}^{T-2} f(s,u(s)).$$

Since η satisfies (H_0), we get that $-\eta^2-\eta+2\eta T \geqslant (T-\eta)(T-\eta-1)$. Then

$$\| Au \| \leqslant \frac{-\eta^2-\eta+2\eta T}{2}\sum_{s=1}^{T-2} f(s,u(s)) \leqslant \sum_{s=1}^{T-2} \frac{-\eta^2-\eta+2\eta T}{2}\frac{r}{H_1} = r.$$

Therefore $Au\in P_r$. Hence, if (16) and (18) hold, then A maps $\overline{P_d}$ into P_d and $\overline{P_c}$ into P_c. Next, we assert that $\{u \in P(\sigma,a,\frac{a}{\theta^*}):\sigma(u) > a\} \neq \varnothing$ and $\sigma(Au) > a$ for all $u \in P(\sigma,a,\frac{a}{\theta^*})$. In fact, the constant function $\frac{a+\frac{a}{\theta^*}}{2}$ belongs to $\{u \in P(\sigma,a,\frac{a}{\theta^*}):\sigma(u) > a\}$.

On the one hand, for $u \in P(\sigma,a,\frac{a}{\theta^*})$, we have

$$a \leqslant \sigma(u) = \min_{t\in[T-2-\theta,\theta]_z} u(t) \leqslant u(t) \leqslant \| u \| \leqslant \frac{a}{\theta^*} \tag{19}$$

for all $t\in[T-2-\theta,\theta]_z$. Also, for any $u \in P$ and $t \in[T-2-\theta,\theta]_z$, we have

$$\sum_{s=1}^{T-\theta-1} G(t,s)f(s,u(s)) + \sum_{s=\theta+1}^{\eta} G(t,s)f(s,u(s)) + \sum_{s=\eta+1}^{T-2} G(t,s)f(s,u(s)) \geqslant \sum_{s=1}^{T-\theta-1} s(T-t)f(s,u(s)) - \sum_{s=\eta+1}^{T-2} \frac{(T-s)(T-s-1)}{2}f(s,u(s)) \geqslant f(\eta,u(\eta))\left[\sum_{s=1}^{T-\theta-1} s(T-t) - \sum_{s=\theta+1}^{T-2} \frac{(T-s)(T-s-1)}{2}\right] = f(\eta,u(\eta))\frac{(T-\theta)(T-\theta-1)(T-\theta+1)}{3} \geqslant 0,$$

which together with (17) and (19) implies

$$\sigma(Au) = \min_{t\in[T-2-\theta,\theta]_z} \sum_{s=1}^{T-2} G(t,s)f(s,u(s)) \geqslant \min_{t\in[T-2-\theta,\theta]_z} \sum_{s=T-2-\theta}^{\theta} G(t,s)f(s,u(s)) \geqslant \frac{a}{H_2} \min_{t\in[T-2-\theta,\theta]_z} \sum_{s=T-2-\theta}^{\theta} G(t,s) = a$$

for $u \in P(\sigma,a,\frac{a}{\theta^*})$.

Finally, we verify that if $u \in P(\sigma,a,c)$ and $Au > \frac{a}{\theta^*}$, then $\sigma(Au) > a$. To see this, we suppose that $u \in P(\sigma,a,c)$ and $\| Au \| > \frac{a}{\theta^*}$. Then it follows from $A(u) \in P$ that

$$\sigma(Au) = \min_{t\in[T-2-\theta,\theta]_z} (Au)(t) \geqslant \theta^* \| Au \| > a.$$

To sum up, all the hypotheses of the Leggett-Willianma fixed point theorem are satisfied. Therefore, A has at least three fixed points, that is, (1) has at least three positive solutions u, v and w satisfying

$$\| u \| < d, a < \min_{t\in[T-2-\theta,\theta]_z} v(t),$$
$$\| w \| > d, \min_{t\in[T-2-\theta,\theta]_z} w(t) < a.$$

Theorem 3.2 Let m be an arbitrary positive integer. Assume that there exist numbers d_i ($1\leqslant i\leqslant m$) and a_j ($1\leqslant j\leqslant m-1$) with $0<d_1<a_1<\frac{a_1}{\theta^*}<d_2<a_2<\frac{a_2}{\theta^*}<\ldots<d_{m-1}<a_{m-1}<\frac{a_{m-1}}{\theta^*}<d_m$ such that

$$f(t,u)<\frac{d_i}{H_1},t\in[1,\eta]_z,u\in[0,d_i],1\leqslant i\leqslant m \tag{20}$$

$$f(t,u)>\frac{a_j}{H_2},t\in[T-2-\theta,\theta]_z,$$

$$u\in[a_j,\frac{a_j}{\theta^*}],1\leqslant j\leqslant m-1 \tag{21}$$

Then (1) has at least $2m-1$ positive solutions in $\overline{P_{d_m}}$.

Proof We use induction on m.

First, for $m=1$, we know from (20) that A: $\overline{P_{d_1}}\to P_{d_1}$. Then it follows from Schauder fixed point theorem that (1) has at least one positive solution in $\overline{P_{d_1}}$.

Next, we assume that this conclusion holds for $m=k$. To show that this conclusion also holds for $m=k+1$, we suppose that there exist numbers d_i ($1\leqslant i\leqslant k+1$) and a_j ($1\leqslant j\leqslant k$) with $0<d_1<a_1<\frac{a_1}{\theta^*}<d_2<a_2<\frac{a_2}{\theta^*}<\ldots<d_k<a_k<\frac{a_k}{\theta^*}<d_{k+1}$ such that

$$f(t,u)<\frac{d_i}{H_1},t\in[1,\eta]_z,u\in[0,d_i],$$

$$1\leqslant i\leqslant k+1 \tag{22}$$

$$f(t,u)>\frac{a_j}{H_2},t\in[T-2-\theta,\theta]_z,$$

$$u\in[a_j,\frac{a_j}{\theta^*}],1\leqslant j\leqslant k \tag{23}$$

By assumption, (1) has at least $2k-1$ positive solutions u_i ($i=1,2,\ldots,2k-1$) in $\overline{P_{d_k}}$. At the same time, it follows from Theorem 3.1, (22) and (23) that (1) has at least three positive solutions u, v and w in $\overline{P_{d_k}}$ such that

$$\|u\|<d_k,a_k<\min_{t\in[T-2-\theta,\theta]_z}v(t),$$

$$\|w\|>d_k,\min_{t\in[T-2-\theta,\theta]_z}w(t)<a_k.$$

Obviously, v and w are different from u_i ($i=1,2,\ldots,2k-1$). Therefore, (1.1) has at least $2k+1$ positive solutions in $\overline{P_{d_{k+1}}}$, which shows that this conclusion also holds for $m=k+1$.

Example 3.3 We consider the BVP

$$\begin{cases}\Delta^3u(t-1)=f(t,u(t)),\ t\in[1,4]_Z,\\ \Delta u(0)=u(6)=\Delta^2u(4)=0\end{cases} \tag{24}$$

where

$$f(t,u(t))=\begin{cases}\frac{1}{400}(4-t)(u+1)^2,(t,u)\in[1,4]_z\times[0,1],\\ (4-t)[\frac{37}{800}(u-1)+\frac{1}{100}],(t,u)\in[1,4]_z\times[1,2],\\ \frac{1}{160}(4-t)(u+1)^2,(t,u)\in[1,4]_z\times[2,10],\\ \frac{121}{160}(4-t),(t,u)\in[1,4]_z\times[10,\infty).\end{cases}$$

Let $\theta=4$. Then $\theta^*=\frac{1}{5}$. A simple calculation shows that $H_1=32.5$ and $H_2=18$. If we choose $d=1$, $a=2$, $c=1068$, then all the conditions of Theorem 3.1 are satisfied. Therefore, it follows Theorem 3.1 that (24) has at least three positive solutions. So we omit them.

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