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# 对数 Bergman 型空间到 Bloch 空间上的 Stevic-Sharma 算子

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**摘 要**：设 $D$ 是复平面中的开单位圆盘，$\varphi$ 是 $D$ 到自身的解析映射，$H(D)$ 是 $D$ 上的解析函数空间. 为了统一研究复合算子、乘积算子和微分算子三者的乘积，Stevic 和 Sharma 引进了如下的 Stevic-Sharma 算子：$T_{\psi_1,\psi_2,\varphi}f(z)=\psi_1(z)f(\varphi(z))+\psi_2(z)f'(\varphi(z))$，$f\in H(D)$，其中 $\psi_1,\psi_2\in H(D)$. 本文利用符号函数给出了对数 Bergman 型空间到 Bloch 空间上 Stevic-Sharma 算子的有界性、紧性刻画.

**关键词**：对数 Bergman 型空间；Bloch 空间；Stevic-Sharma 算子；有界性；紧性

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# Stevic-Sharma operators from logarithmic Bergman-type spaces to Bloch spaces

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**Abstract**: Let $D$ be the open unit disk in the complex plane $\mathbf{C}$, $\varphi$ be an analytic self-map of $D$ and $H(D)$ the space of all analytic functions on $D$. In order to unify the products of composition, multiplication and differentiation operators, Stevic and Sharma introduced the following Stevic-Sharma operator: $T_{\psi_1,\psi_2,\varphi}f(z)=\psi_1(z)f(\varphi(z))+\psi_2(z)f'(\varphi(z))$, $f\in H(D)$, where $\psi_1,\psi_2\in H(D)$. Motivated by some recent results of this operator, the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ from logarithmic Bergman-type space to Bloch space are characterized in this paper.

**Keywords**: Logarithmic Bergman-type space; Bloch space; Stevic-Sharma operator; Boundedness; Compactness

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## 1 Introduction

Let $D=\{z\in\mathbf{C}: |z|<1\}$ be the open unit disk in the complex plane $\mathbf{C}$ and $H(D)$ the class of all analytic functions on $D$. Let $\varphi$ be an analytic self-map of $D$ and $\psi\in H(D)$. The weighted composition operator $W_{\varphi,\psi}$ on $H(D)$ is defined by

$$W_{\varphi,\psi}f(z)=\psi(z)f(\varphi(z)),\ z\in D.$$

If $\psi\equiv 1$, it becomes the composition operator, usually denoted by $C_\varphi$. If $\varphi(z)=z$, it becomes the multiplication operator, usually denoted by $M_\psi$. Hence, since $W_{\varphi,\psi}=M_\psi C_\varphi$, it is a product-type operator. A natural problem is to provide function theoretic characterizations when $\varphi$ and $\psi$ induce a bounded or compact weighted composition operator (see, e.g., Refs. [1～5] and the references therein).

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A systematic study of other product-type operators started by Stevic and his collaborators since the publication of papers[6,7]. Before that there were a few papers in the topic, e. g. , Ref. [8]. The differentiation operator on $H(D)$ is defined by

$$Df(z) = f'(z), z \in D.$$

The product-type operators $DC_\varphi$ and $C_\varphi D$ attracted some attention first (see, e. g. , Refs. [9～12] and the references therein). The publication of Ref. [7] attracted some attention in product-type operators involving integral-type ones (see, e. g. , Refs. [13～17] and the references therein). Since that time there has been a great interest in various product-type operators on spaces of holomorphic functions. For example, the following six product-type operators from Bergman spaces to Bloch type spaces

$$M_\psi C_\varphi D, M_\psi DC_\varphi, C_\varphi M_\psi D, C_\varphi DM_\psi, DC_\varphi M_\psi, DM_\psi C_\varphi \quad (1)$$

were studied by Sharma in Ref. [18]. The next product-type operators $W_{\varphi,\psi}D$ and $DW_{\varphi,\psi}$, which were considered in Refs. [19] and [20], are included in (1) as the first and sixth operators respectively. For some other studies of In order to treat operators in (1) in a unified manner, Stevic and Sharma introduced the following Stevic-Sharma operator

$$T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1 f(\varphi(z)) + \psi_2(z) f'(\varphi(z)), f \in H(D) \quad (2)$$

For example, in Refs. [21] and [22] the operator was studied on the weighted Bergman space.

By using Stevic-Sharma operator all six possible products of composition, multiplication and differentiation operators can be obtained. More specifically we have

$$M_\psi C_\varphi D = T_{0,\psi,\varphi}, M_\psi DC_\varphi = T_{0,\psi\varphi',\varphi}, C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi},$$
$$C_\varphi DM_\psi = T_{\psi'\circ\varphi,\psi\circ\varphi,\varphi}, DM_\psi C_\varphi = T_{\psi',\psi\varphi',\varphi}, DC_\varphi M_\psi = T_{\varphi'\psi'\circ\varphi,\varphi'\psi\circ\varphi,\varphi}.$$

We characterize the boundedness and compactness of the Stevic-Sharma operator from logarithmic Bergman-type space to Bloch space in this paper. As the applications of our main results, readers can obtain some characterizations for the boundedness and compactness for all six product-type operators in (1).

Now we present the needed spaces and some facts. The Bloch space B consists of all $f \in H(D)$ such that

$$b(f) := \sup_{z\in D}(1 - |z|^2) | f'(z) | < \infty.$$

It is a Banach space with the norm $f_B = |f(0)| + b(f)$. Obviously, the quantity $b(f)$ is a seminorm on the space $B$ and a norm on the quotient space $B/P_0$, where $P_0$ is the set of all constant functions. For some results on Bloch spaces and some concrete operators on them, see, for example, Refs. [1, 3, 10] and the references therein.

Let $dA(z) = \frac{1}{\pi} dx dy$ be the normalized Lebesgue measure on D. For $-1 < \gamma < \infty, \delta \leqslant 0$ and $0 < p < \infty$, the logarithmic Bergman-type space $A^p_{\omega_{\gamma,\delta}}$ consist of all $f \in H(D)$ such that

$$\| f \|^p_{A^p_{\omega_{\gamma,\delta}}} = \int_D |f(z)|^p \omega_{\gamma,\delta}(z) < \infty,$$

where the weight function $\omega_{\gamma,\delta}(z)$ is defined by

$$\omega_{\gamma,\delta}(z) = \left(\log \frac{1}{|z|}\right)^\gamma \left[\log\left(1 - \frac{1}{\log|z|}\right)\right]^\delta.$$

For $p \geqslant 1$ it is a Banach space, while for $0 < p < 1$ it is a translation invariant metric space with the metric given by $d(f,g) = \| f - g \|^p_{A^p_{\omega_{\gamma,\delta}}}$. From a calculation and the fact $\int_0^1 \omega_{\gamma,\delta}(r) r dr < \infty$, it is easily seen that $H^\infty \subset A^p_{\omega_{\gamma,\delta}}$. In fact, this containment is proper since the function $k_{w,t}(z)$ in Section 2 is in $A^p_{\omega_{\gamma,\delta}}$ but not in $H^\infty$. In particular, every complex polynomial function belongs to $A^p_{\omega_{\gamma,\delta}}$. Some properties of this kind of space were studied by Jiang in Ref. [24].

Let $X$ and $Y$ be two topological vector spaces whose topologies are given by the translation invariant metrics $d_X$ and $d_Y$. A linear operator $L: X \to Y$ is bounded if there exists a positive constant $K$ such that

$$d_Y(Tf, 0) \leqslant K d_X(f, 0)$$

for all $f \in X$. The operator $L: X \to Y$ is compact if it maps bounded sets into relatively compact sets.

Throughout this paper, positive constant $C$ may differ from one occurrence to the other.

## 2 Auxiliary results

In order to characterize the compactness, we need the following result which was proved in a standard way. So, the proof is omitted.

**Lemma 2.1** Let $\varphi$ be an analytic self-map of D and $\psi_1,\psi_2 \in H(D)$. Then the bounded operator $T_{\psi_1,\psi_2,\varphi}:A^p_{\omega_\gamma,\delta}\to B$ is compact if and only if for every bounded sequence $\{f_j\}$ in $A^p_{\omega_\gamma,\delta}$ such that $f_j\to 0$ uniformly on every compact subset of D as $j\to\infty$, it follows that

$$\lim_{j\to\infty}\|T_{\psi_1,\psi_2,\varphi}f_j\|_B=0.$$

The following useful results were obtained in Ref.[24].

**Lemma 2.2** Let $-1<\gamma<\infty,\delta\leqslant 0,0<p<\infty$ and $0<r<2/3$. Then for each $k\in\mathbf{N}_0$, there exists a positive constant $C_k=C(\gamma,\delta,p,r,k)$ independent of $f\in A^p_{\omega_\gamma,\delta}$ and $z\in\{z\in D:|z|>r\}$ such that

$$|f^{(k)}(z)|\leqslant\frac{C_k}{(1-|z|^2)^{\frac{\gamma+2}{p}+k}}\left[\log\left(1-\frac{1}{\log|z|}\right)\right]^{-\frac{\delta}{p}}\|f\|_{A^p_{\omega_\gamma,\delta}}.$$

The above lemma does not provide any relation between $|f^{(k)}(0)|$ and $\|f\|_{A^p_{\omega_\gamma,\delta}}$. But from this lemma and the maximum module theorem, we obtain the following result.

**Lemma 2.3** Let $-1<\gamma<\infty,\delta\leqslant 0,0<p<\infty$ and $0<r<2/3$. Then for all $f\in A^p_{\omega_\gamma,\delta}$, we have

$$|f^{(k)}(0)|\leqslant\frac{C_k}{(1-r^2)^{\frac{\gamma+2}{p}+k}}\left[\log\left(1-\frac{1}{\log r}\right)\right]^{-\frac{\delta}{p}}\|f\|_{A^p_{\omega_\gamma,\delta}},$$

where $C_k$ is the constant in Lemma 2.2.

The following function is in $A^p_{\omega_\gamma,\delta}$, which will be used in the proofs on the main results.

**Lemma 2.4** Let $-1<\gamma<\infty,\delta\leqslant 0,0<p<\infty$ and $0<r<1$. Then for every $t>0$ and $w\in D$ with $|w|>r$, the following function belongs to $A^p_{\omega_\gamma,\delta}$

$$k_{w,t}(z)=\left[\log\left(1-\frac{1}{\log|w|}\right)\right]^{-\frac{\delta}{p}}\frac{(1-|w|^2)^{-\frac{\delta}{p}+t}}{(1-\overline{w}z)^{\frac{\gamma-\delta+2}{p}+t}},z\in D.$$

Moreover, there exists a constant $C$ independent of $k_{w,t}$ such that

$$\sup_{\{w\in D:|w|>r\}}\|k_{w,t}\|_{A^p_{\omega_\gamma,\delta}}\leqslant C.$$

**Lemma 2.5** Let $-1<\gamma<\infty,\delta\leqslant 0,0<p<\infty$ and $0<r<1$. Let $w\in D$ with $|w|>r$ and $m\in\mathbf{N}_0$. Then for each $k\in\{0,1,\cdots,m+2\}$, there exist constants $a_{1,k},a_{1,k},\ldots,a_{m+3,k}$ such that the function

$$f_{w,k}(z)=\sum_{i=1}^{m+3}a_{i,k}k_{w,i}(z)$$

satisfies

$$f^{(k)}_{w,k}(w)=\left[\log\left(1-\frac{1}{\log|w|}\right)\right]^{-\frac{\delta}{p}}\frac{w^k}{(1-|w|^2)^{\frac{\gamma+2}{p}+k}}\text{ and }f^{(i)}_{w,k}(w)=0\qquad(3)$$

for each $j\in\{0,1,2\cdots,m+2\}\backslash\{k\}$. Moreover, there exists a constant C independent of $k_{w,k}$ such that $\sup_{\{w\in D:|w|>r\}}\|f_{w,k}\|_{A^P_{\omega_\gamma,\delta}}\leqslant C$.

***Remark* 1** As a special case, we will use the case m=0 in Lemma 5 in the proof of main results.

## 3 Main results

First we characterize the boundedness of operator $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$.

**Theorem 3.1** Let $\varphi$ be an analytic self-map of D and $\varphi_1,\varphi_2\in H(D)$. Then the following statements are equivalent:

(i) The operator $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is bounded;

(ii) For each $k\in\{0,1,2\}$, it follows that $M_k=\sup_{z\in D}M_k(z)<\infty$, where

$$M_0(z)=\frac{(1-|z|^2)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{r+2}{p}}}\left[\log\left(1-\frac{1}{\log|\varphi(z)|}\right)\right]^{-\frac{\delta}{p}},$$

$$M_1(z)=\frac{(1-|z|^2)|\psi_1'(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{r+2}{p}+1}}\left[\log\left(1-\frac{1}{\log|\varphi(z)|}\right)\right]^{-\frac{\delta}{p}},$$

and

$$M_2(z)=\frac{(1-|z|^2)|\varphi'(z)\|\psi_2(z)|}{(1-|\varphi(z)|^2)^{\frac{r+2}{p}+2}}$$

$$\left[\log\left(1-\frac{1}{\log|\varphi(z)|}\right)\right]^{-\frac{\delta}{p}}.$$

**Proof** (i)⇒(ii). Suppose that $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is bounded. Set $h_0(z)\equiv 1\in A^p_{w_{r,\delta}}$. Then we get

$$L_0=\sup_{z\in D}(1-|z|^2)|(T_{\psi_1,\psi_2,\varphi}h_0)'(z)|=\sup_{z\in D}(1-|z|^2)|\psi_1'(z)|\leqslant C\|T_{\psi_1,\psi_2,\varphi}\| \quad (4)$$

Setting $h_1(z)=z\in A^p_{w_{r,\delta}}$, we have

$$b(T_{\psi_1,\psi_2,\varphi}h1)=\sup_{z\in D}(1-|z|^2)|\psi_1'(z)\varphi(z)+\psi_1(z)\varphi'(z)+\psi_2'(z)|\leqslant C\|T_{\psi_1,\psi_2,\varphi}\| \quad (5)$$

By using (5), the boundedness of $\varphi$ and the triangle inequality, we have

$$L_1=\sup_{z\in D}(1-|z|^2)|\psi_1(z)\varphi'(z)+\psi_2'(z)|\leqslant C\|T_{\psi_1,\psi_2,\varphi}\| \quad (6)$$

Also, setting $h_2(z)=z^2$, we have

$$b(T_{\psi_1,\psi_2,\varphi}h_2)=\sup_{z\in D}(1-|z|^2)|\psi_1'(z)\varphi^2(z)+2(\psi_1(z)\varphi'(z)+\psi_2'(z))\varphi(z)+2\psi_2(z)\varphi'(z)|\leqslant C\|T_{\psi_1,\psi_2,\varphi}\| \quad (7)$$

Once again, by using (4), (6), (7), the boundedness of $\varphi$ and the triangle inequality, we obtain

$$L_2=\sup_{z\in D}(1-|z|^2)\psi_2(z)\|\varphi'(z)|\leqslant C\|T_{\psi_1,\psi_2,\varphi}\| \quad (8)$$

Let $r\in(0,2/3)$ be fixed. For a fixed $w\in D$ with $|\varphi(w)|>r$ and $k\in\{0,1,2\}$, by Lemma 2.5, there exist constants $a_{1,k},a_{2,k},a_{3,k}$ such that the function

$$f_{\varphi(w),k}(z)=\sum_{i=1}^{3}a_{i,k}k_{\varphi(w)}(z),$$

satisfies

$$f^{(k)}_{\varphi(w),k}(\varphi(w))=\frac{\overline{\varphi(w)}^k}{(1-|\varphi(w)|^2)^{\frac{r+2}{p}+k}}\left[\log\left(1-\frac{1}{\log|\varphi(w)|}\right)\right]^{-\frac{\delta}{p}}$$

and

$$f^{(j)}_{\varphi(w),k}(\varphi(w))=0 \quad (9)$$

for each $j\in\{0,1,2\}\backslash\{k\}$, moreover

$$\sup_{\{w\in D:|\varphi(w)|>r\}}\|f_{\varphi(w),k}\|_{A^p_{w_{r,\delta}}}\leqslant C \quad (10)$$

Then from (7), (10) and the boundedness of $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$, we have

$$\begin{aligned}
b(T_{\psi_1,\psi_2,\tilde\omega}f)&=\sup_{z\in D}(1-|z|^2)|(T_{\psi_1,\psi_2,\tilde\omega}f)'(z)|=\\
&\sup_{z\in D}(1-|z|^2)|\psi_1'(z)f(\tilde\omega(z))+[\psi_1(z)\tilde\omega'(z)+\psi_2{}'(z)]f'(\tilde\omega(z))+\psi_2(z)f''(\tilde\omega(z))|\leqslant\\
&\Big(\sup_{|\tilde\omega(z)|\leqslant r}+\sup_{|\tilde\omega(z)|>r}\Big)(1-|z|^2)|\psi_1'(z)f(\tilde\omega(z))+[\psi_1(z)\tilde\omega'(z)+\psi_2'(z)]f'(\tilde\omega(z))+\psi_2(z)f''(\tilde\omega(z))|\leqslant\\
&\sup_{|\tilde\omega(z)|\leqslant r}|\psi_1'(z)f(\tilde\omega(z))+[\psi_1(z)\tilde\omega'(z)+\psi_2'(z)]f'(\tilde\omega(z))+\psi_2(z)f''(\tilde\omega(z))|+\\
&\sup_{|\tilde\omega(z)|>r}|\psi_1'(z)f(\tilde\omega(z))+[\psi_1(z)\tilde\omega'(z)+\psi_2'(z)]f'(\tilde\omega(z))+\psi_2(z)f''(\tilde\omega(z))|\leqslant\\
&\max_{|\tilde\omega(z)|=r}|\psi_1'(z)f(\tilde\omega(z))+[\psi_1(z)\tilde\omega'(z)+\psi_2'(z)]f'(\tilde\omega(z))+\psi_2(z)f''(\tilde\omega(z))|+\sum_{k=0}^{2}C_kM_k\|f\|_{A_{w^p_{r,\delta}}}\leqslant\\
&\Big(\sum_{k=0}^{2}\frac{IC_k}{(1-r^2)^{\frac{\gamma+2}{p}+k}}[\log(1-\frac{1}{\log r})]^{-\frac{\delta}{p}}+\sum_{k=0}^{2}C_kM_k)\Big)\|f\|_{A_{\omega^p_{r,\delta}}} \quad (11)
\end{aligned}$$

where

$$I=\max\{\max_{|\varphi(z)|=r}|\psi_1'(z)|,\max_{|\varphi(z)=r|}|\psi_1(z)\varphi'(z)+\psi_2'(z)|,\max_{|\varphi(z)|=r}|\psi_2(z)\varphi'(z)|\}<\infty.$$

On the other hand, from Lemma 2.3, we see that if $\varphi(0)=0$, then

$$|T_{\psi_1,\psi_2,\varphi}f(0)|=|\psi_1(0)f(0)+\psi_2(0)f'(0)|\leqslant C\|f\|_{A^p_{\omega_{r,\delta}}} \quad (12)$$

where

$$C=\left(\frac{C_0|\psi_1(0)|}{(1-r^2)^{\frac{r+2}{p}}}+\frac{C_1|\psi_2(0)|}{(1-r^2)^{\frac{r+2}{p}+1}}\right)\left[\log\left(1-\frac{1}{\log r}\right)\right]^{-\frac{\delta}{p}}<\infty.$$

If $\varphi(0)\neq 0$, then by Lemma 2.2 it is clear that

$$|(T_{\psi_1,\psi_2,\varphi}f)(0)|\leqslant C\|f\|_{A^P_{\omega_{\gamma,\delta}}} \quad (13)$$

Hence, from (11)~(13) it follows that the operator $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is bounded. The proof is finished.

Next we characterize the compactness of operator $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$.

**Theorem 3.2** Let $\varphi$ be an analytic self-map of $D$ and $\psi_1,\psi_2\in H(D)$. Then the following statements are equivalent:

(i) The operator $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is compact;

(ii) For each $k\in\{0,1,2\}$, it follows that $L_k<\infty$ and $\lim_{|\varphi(z)|\to 1}M_K(z)=0$, where $L_k$ and $M_k(z)$ are defined in Theorem 3.1.

**Proof** (i)$\Rightarrow$(ii). Suppose that (i) holds. Then it is clear that the operator $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is bounded. In the proof of Theorem 3.1, we have shown that $L_k<\infty$ for each $k\in\{0,1,2\}$. Consider a sequence $\{\varphi(z_i)\}$ in $D$ such that $|\varphi(z_i)|\to 1$ as $i\to\infty$. If such sequence does not exist, then the last two conditions (ii) obviously hold. We may suppose, without loss of generality, that $|\varphi(z_i)|>1/2$ for all $i\in\mathbf{N}$. For each fixed $k\in\{0,1,2\}$, we define $f_{i,k}(z)=f_{\varphi(z_i),k}$. Then by Lemma 2.5 it follows that the function $f_{i,k}$ satisfies

$$f^{(k)}_{i,k}(\varphi(z_i))=\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^{\frac{\gamma+2}{p}+k}}\left[\log\left(1-\frac{1}{\log|\varphi(z_i)|}\right)\right]^{-\frac{\delta}{p}}$$

and

$$f^{(j)}_{i,k}(\varphi(z_i))=0 \tag{14}$$

For each $j\in\{0,1,2\}\backslash\{k\}$, moreover $\sup\limits_{i\in\mathbf{N}}\|f_{i,k}\|_{A^p_{\omega_{\gamma,\delta}}}\leqslant C$. From Remark 2.1 in Ref.[24], it follows that $f_i\to 0$ uniformly on every compact subset of $D$ as $i\to\infty$. Then by Lemma 2.1

$$\lim_{i\to\infty}\|T_{\psi_1,\psi_2,\varphi}f_i\|_B=0.$$

From this, Lemmas 2.2 and 2.3, and since $L_\mathrm{k}$ is finite, we obtain

$$\lim_{i\to\infty}M_k(z_i)=0 \tag{15}$$

(ii)$\Rightarrow$(i). We first prove that $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is bounded. We observe that the conditions in (ii) imply that for every $\varepsilon>0$, there is an $\eta\in(0,1)$, such that for all $z\in K=\{z\in D:|\varphi(z)|>\eta\}$ and $k\in\{0,1,2\}$ it follows that $M_k(z)<\varepsilon$. From the fact $L_k<\infty$, for each $k\in\{0,1,2\}$ we obtain we have

$$M_k\leqslant\varepsilon+\frac{L_k}{(1-\eta^2)^{\frac{\gamma+2}{p}+k}}\left[\log\left(1-\frac{1}{\log\eta}\right)\right]^{-\frac{\delta}{p}}.$$

Hence from Theorem 3.1 it follows that operator $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is bounded.

Next we proof that the operator $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is compact. For this purpose, by Lemma 1 we just need to prove that, if $\{f_i\}$ is a sequence in $A^p_{\omega_{\gamma,\delta}}$ such that $\sup_{i\in\mathbf{N}}\|f_i\|_{A^p_{\omega_{\gamma,\delta}}}\leqslant M$ and $f_i\to 0$ uniformly on any compact subset of $D$ as $i\to\infty$, then

$$\lim_{i\to\infty}\|T_{\psi_1,\psi_2,\varphi}f_i\|_B=0.$$

For such chosen $\varepsilon$ and $\eta$, by Lemma 2.2 we have

$$\begin{aligned}&(1-|z|^2)|(T_{\psi_1,\psi_2,\varphi}f_i)'(z)|=\\&(1-|z|^2)|\psi_1'(z)f_i(\varphi(z))+(\psi_1(z)\varphi'(z)+\psi_2'(z))f_i'(\varphi(z))+\psi_2(z)\varphi'(z)f_i''(\varphi(z))|\leqslant\\&(1-|z|^2)(|\psi_1'(z)\|f_i(\varphi(z))|+|\psi_1(z)\varphi'(z)+\psi_2'(z)\|f_i'(\varphi(z))|+|\psi_2(z)\|\varphi'(z)\|f_i''(\varphi(z))|)\leqslant\\&L_0\sup_{z\in D}|f_i(z)|+(\sup_{z\in K}+\sup_{z\in D\backslash K})(1-|z|^2)|\psi_1(z)\varphi'(z)+\psi_2'(z)\|f_i'(\varphi(z))|+\\&(\sup_{z\in K}+\sup_{z\in D\backslash K})(1-|z|^2)|\varphi'(z)\|\psi_2(z)\|f_i''(\varphi(z))|\leqslant\\&2\varepsilon+L_0\sup_{z\in D}|f_i(z)|+L_1\sup_{|z|\leqslant\eta}|f_i'(z)|+L_2\sup_{|z|\leqslant\eta}|f_i''(z)|\end{aligned} \tag{16}$$

Since $f_i\to 0$ uniformly on compact subsets of $D$ as $i\to\infty$ implies that for each $k\in\mathbf{N}$, $f_i^{(k)}\to 0$ uniformly on compact subsets of $D$ as $i\to\infty$, from (16) we get

$$\lim_{i\to\infty}\sup_{z\in D}(1-|z|^2)|(T_{\psi_1,\psi_2,\varphi}f_i)'(z)|=0 \tag{17}$$

It is clear that

$$\lim_{i\to\infty}|T_{\psi_1,\psi_2,\varphi}f_i(0)|=0 \tag{18}$$

From (17) and (18) we obtain

$$\lim_{i\to\infty}\|T_{\psi_1,\psi_2,\varphi}f_i\|_B=0 \tag{19}$$

Hence from (19) and Lemma 2.1, we obtain that $T_{\psi_1,\psi_2,\varphi}:A^P_{\omega_{\gamma,\delta}}\to B$ is compact. The proof is finished.

**References**:


[1] Colonna F, Li S. Weighted composition operators from the minimal Mobius invariant space into the Bloch space [J]. Mediterr J Math, 2013, 10: 395.

[2] Esmaeili K, Lindstrom M. Weighted composition operators between Zygmund type spaces and their essential norms [J]. Integr Equat Oper Th, 2013, 75: 473.

[3] Madigan K, Matheson A. Compact composition operators on the Bloch space [J]. T Am Math Soc, 1995, 347: 2679.

[4] Zhou F. Weighted composition operators from Bers-type spaces into Bergman-type spaces in the unit ball [J]. J Sichuan Univ: Nat Sci Ed(四川大学学报自然科学版), 2012, 49: 294.

[5] Zhang K, Zhou Z H. Equivalent condition for invertible weighted composition operators in the unit

ball [J]. J Sichuan Univ: Nat Sci Ed(四川大学学报自然科学版), 2016, 53: 899.

[6] Li S, Stevic S. Composition followed by differentiation between Bloch type spaces [J]. J Comput Anal Appl, 2007, 9: 195.

[7] Li S, Stevic S. Products of composition and integral type operators from $H^\infty$ to the Bloch space [J]. Complex Var Elliptic, 2008, 53: 463.

[8] Hibschweiler R A, Portnoy N. Composition followed by differentiation between Bergman and Hardy spaces [J]. Rocky MT J Math, 2005, 35: 843.

[9] Li S, Stevic S. Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces [J]. Appl Math Comput, 2010, 217: 3144.

[10] Ohno S. Products of composition and differentiation on Bloch spaces [J]. B Korean Math Soc, 2009, 46: 1135.

[11] Stevic S. Norm and essential norm of composition followed by differentiation from $\alpha$-Bloch spaces to $H_\mu^\infty$ [J]. Appl Math Comput, 2009, 207: 225.

[12] Yang Y, Jiang Z J. Boundedness of products of differentiation and composition from weighted Bergman spaces to Zygmund spaces [J]. J Sichuan Univ:Nat Sci Ed(四川大学学报自然科学版), 2015, 52: 731.

[13] Krantz S, Stevic S. On the iterated logarithmic Bloch space on the unit ball [J]. Nonlinear Anal-Theor, 2009, 71: 1772.

[14] Stevic S. On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces [J]. Nonlinear Anal-Theor, 2009, 71: 6323.

[15] Stevic S. Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces [J]. Siberian Math J+, 2009, 50: 726.

[16] Stevic S. On an integral operator from the Zygmund space to the Bloch-type space on the unit ball [J]. Glasgow J Math, 2009, 51: 275.

[17] Stevic S, Ueki S I. Integral-type operators acting between weighted-type spaces on the unit ball [J]. Appl Math Comput, 2009, 215: 2464.

[18] Sharma A K. Products of composition multiplication and differentiation between Bergman and Bloch type spaces [J]. Turk J Math, 2011, 35: 275.

[19] Jiang Z J. On a class of operators from weighted Bergman spaces to some spaces of analytic functions [J]. Taiwan J Math, 2011, 15: 2095.

[20] Jiang Z J. On a product-type operator from weighted Bergman-Orlicz space to some weighted type spaces [J]. Appl Math Comput, 2015, 256: 37.

[21] Stevic S, Sharma A K, Bhat A. Products of multiplication composition and differentiation operators on weighted Bergman spaces [J]. Appl Math Comput, 2011, 217: 8115.

[22] Stevic S, Sharma A K, Bhat A. Essential norm of multiplication composition and differentiation operators on weighted Bergman spaces [J]. Appl Math Comput, 2011, 218 :2386.

[23] Li S, Stevic S. Volterra type operators on Zygmund space [J] J Inequal Appl, 2007: 32124-10.

[24] Jiang Z J. Product-type operators from logarithmic Bergman-type spaces to Zygmund-Orlicz spaces [J]. Mediterr J Math, 2016, 13: 4639.