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加性脉冲噪声驱动的线性分数阶调和振子的扩散

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摘　要：本文研究了加性脉冲噪声驱动的线性分数阶调和振子的扩散行为. 利用 Laplace 变换、双 Laplace 变换技巧及脉冲微分方程的基本性质，本文得到了振子位移的均值、方差、关联函数及均方位移. 这些量均可以通过三参数的广义 Mittag-Leffler 函数来表示. 然后，基于 Mittag-Leffler 函数的渐进性质，本文研究了振子的短时和长时扩散行为. 研究表明，加性脉冲噪声增强振子的短时超扩散，并抬升振子的长时欠扩散均方位移.

关键词：分数阶调和振子；均方位移；脉冲噪声；扩散
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Diffusion of linear fractional harmonic oscillator driven by additive impulsive noise

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Abstract: Diffusion of a linear fractional harmonic oscillator driven by both thermal noise and additive impulsive noise is investigated. By using the Laplace and double Laplace transform techniques and some basic properties of impulsive differential equation, the mean, variance, correlation function and mean square displacement (MSD) of the oscillator are expressed by generalized Mittag-Leffler functions with three parameters. Furthermore, asymptotic diffusion of the oscillator is investigated in terms of the asymptotic properties of generalized Mittag-Leffler function. It is shown that the additive impulsive noise enhances the ballistic diffusion of the oscillator for short time-lag and adds a constant to the MSD for long time-lag.

Keywords: Fractional harmonic oscillator; Mean square displacement; Impulsive noise; Diffusion
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1　Introduction

The technique of optical tweezers has found wide application in the measurement of forces on single molecules and materials in physical and biophysical studies[1-5], where the mean square displacement (MSD) of the Brownian particle is one of the most valuable quantities. In the small-displacement regime, the particle-trap interaction is approximately that of harmonic potential[6]. Therefore we can consider the Brownian particle as a harmonic oscillator. In many of the measure-

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ments, the oscillator shows an anomalous diffusive behavior, say, its MSD has the asymptotic form $t^{\alpha}, \alpha \neq 1$. Note that the diffusion is called subdiffusive when $\alpha < 1$ and superdiffusive when $\alpha > 1$.

Anomalous diffusion can be exactly determined if the dynamic of the oscillator can be formulated in terms of linear generalized Langevin equation (LGLE) or linear fractional Langevin equation (LFLE)[7-10], which take both memory effects of the viscoelastic media through a power-law correlation function into account[11]. For example, Vinales, Wang, and Desposito considered the anomalous diffusion of harmonic oscillator in Refs. [12, 13]; Wang and Masoliver considered the Ornstein-Uhlenbeck noise in Ref. [14], Desposito and Vinales considered some abstract noises defined in the Laplace transform domain in Ref. [15], Sandev and Tomovski considered the generalized Mittag-Leffler noise in Ref. [16].

For a harmonic oscillator, the internal (thermal) noise is generally considered to be a continuous stochastic processes. However, the external noise can be a discrete one, for instance, impulsive noise, chaotic noise, Langevin-like noise, etc. Recently, we considered the anomalous diffusion of the following linear fractional harmonic oscillator with additive impulsive noise[17]

$$\ddot{x}(t) + \gamma\, {}_0^C D_t^{\alpha} x(t) + \lambda x(t) = L(t),\ 0<\alpha\leqslant 1 \tag{1}$$

here γ is the friction coefficient,

$${}_0^C D_t^{\alpha} x(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-t')^{-\alpha}\dot{x}(t')\mathrm{d}t', & 0<\alpha<1, \\ \dot{x}(t), & \alpha = 1 \end{cases} \tag{2}$$

is the Caputo fractional derivative, $\Gamma(x)$ is the gamma function, $\delta(t)$ is the Dirac delta function, and $L(t) = K\sum_{j=0}^{\infty} y_j \delta(t - j\nu)$ is an additive impulsive noise with frequency $2\pi/\nu$, amplitude K and weighted by a stochastic or deterministic sequence $\{y_j\}_{j=0}^{\infty}, \langle y_j\rangle = 0, \langle y_i y_j\rangle = 2D\delta_{ij}$, here δ_{ij} is the Kronecker delta function. If $\alpha = 1$, the linear fractional harmonic oscillator (1) recovers the classical one. In this paper, diffusion of the oscillator is further considered in the presence of thermal noise $\xi(t)$, which is a zero-centered stationary Gaussian process with correlation function

$$\langle \xi(t)\xi(t')\rangle = C(|t'-t|) = k_B T\gamma \frac{|t-t'|^{-\alpha}}{\Gamma(1-\alpha)} \tag{3}$$

due to the fluctuation-dissipation relation[18], here T is the absolute temperature and k_B is the Boltzmann constant. Specifically, given deterministic initial values $x|_{t=0} = x_0, \dot{x}|_{t=0} = v_0$, we consider the following fractional harmonic oscillator

$$\ddot{x}(t) + \gamma\, {}_0^C D_t^{\alpha} x(t) + \lambda x(t) = L(t) + \xi(t),\ 0<\alpha\leqslant 1 \tag{4}$$

here the impulsive noise $L(t)$ is statistically independent of $\xi(t)$. In order to analyze the effect of $L(t)$ on diffusion of the oscillator, the MSD

$$\rho(\tau) = \lim_{t\to\infty}\langle\langle [x(t+\tau) - x(t)]^2\rangle\rangle \tag{5}$$

is exactly expressed by Mittag-Leffler and generalized Mittag-Leffler functions with two and three parameters by using the Laplace and double Laplace transform techniques[19] and some basic properties on impulsive differential equation[20, 21], here $|x(t+\tau) - x(t)|$ is the displacement between two time points t and $t+\tau$, here t denotes the absolute time, τ is the time-lag, and the pairs of double brackets are averages over the internal and external noises respectively. Exact expressions for the mean, variance and correlation function of the oscillator are deduced. Then, asymptotic diffusion of the oscillator for short and long time-lags is analyzed underlying asymptotic expression of the MSD in terms of asymptotic properties of Mittag-Leffler and generalized Mittag-Leffler functions.

This paper is organized as follows. In section 2, we deduce the exact expression of $x(t)$, correlation function of the oscillator are also obtained. In section 3, we deduce the MSD of the oscillator. Effect of the impulsive noise on the asymptotic diffusion of the oscillator for short and long time-lags are considered in section 4. The summary is provided in section 5.

2 Mean of the oscillator

In order to obtain the exact solution of equation (3) by using the Laplace transform technique, let us recall the notion of Laplace convolution of causal functions, i. e., the convolution integral of any two casual functions $f_1(t), f_2(t)$ (say, $f_j(t) \equiv 0$ for $t < 0, j = 1, 2$), which reads in our notations as[22]

$$f_1(t) * f_2(t) = \int_0^t f_1(t-t') f_2(t') \mathrm{d}t' = f_2(t) * f_1(t).$$

The Laplace transform of a casual function $f(t)$, locally absolutely integrable in $\mathbf{R}^+$, reads

$$\hat{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty \mathrm{e}^{-st} f(t) \mathrm{d}t.$$

By using the sign " $\div$ " to denote the juxtaposition of $f(t)$ with its Laplace transform $\hat{f}(s)$, a Laplace transform pair reads $f(t) \div \hat{f}(s)$. Then, by the convolution theorem of the Laplace transform, we have the pair

$$f_1(t) * f_2(t) \div \hat{f}_1(s) \hat{f}_2(s).$$

Now, making Laplace transform in (3) and taking the initial values into account, we have

$$\hat{x}(s) = x_0 \hat{G}(s) + v_0 \hat{H}(s) + \hat{H}(s)[\hat{\xi}(s) + \hat{L}(s)] \tag{6}$$

where $H(t), I(t) = \int_0^t H(t') \mathrm{d}t'$, and $G(t)$ are defined by their corresponding Laplace transforms:

$$\hat{H}(s) = \frac{1}{s^2 + \gamma s^\alpha + \lambda} \tag{7}$$

$$\hat{I}(s) = \frac{\hat{H}(s)}{s} \tag{8}$$

$$\hat{G}(s) = \frac{1}{s} - \lambda \hat{I}(s) \tag{9}$$

With the help of the following

$$\mathcal{L}\left[\frac{s^{r-1}}{s^\alpha + a s^\beta + b}\right] = t^{\alpha - r} \sum_{j=0}^\infty (-a)^j t^{(\alpha-\beta)j} E^{j+1}_{\alpha, \alpha+(\alpha-\beta)j-r+1}(-bt^\alpha) \tag{10}$$

proposed by Haubold [23], we obtain

$$\hat{H}(s) \div H(t) = t \sum_{j=0}^\infty (-\gamma)^j t^{(2-\alpha)j} E^{j+1}_{2, 2+(2-\alpha)j}(-\lambda t^\alpha) \tag{11}$$

$$\hat{I}(s) \div I(t) = t^2 \sum_{j=0}^\infty (-\gamma)^j t^{(2-\alpha)j} E^{j+1}_{2, 3+(2-\alpha)j}(-\lambda t^\alpha) \tag{12}$$

$$\hat{G}(s) \div G(t) = 1 - \lambda t^2 \sum_{j=0}^\infty (-\gamma)^j t^{(2-\alpha)j} E^{j+1}_{2, 3+(2-\alpha)j}(-\lambda t^\alpha) \tag{13}$$

here

$$E^r_{\alpha, \beta}(t) = \sum_{j=0}^\infty \frac{(r)_j}{\Gamma(\alpha j + \beta)} \frac{t^j}{j!} \tag{14}$$

is the Mittag-Leffler function with three parameters[24], $(r)_j$ is the Pochammer symbol defined by

$$(r)_j = \frac{\Gamma(r+j)}{\Gamma(r)} = r(r+1)\cdots(r+j-1).$$

By the initial-value and final-value theorems, we have

$$H(0) = \lim_{s\to\infty} s\hat{H}(s) = 0, H(\infty) = \lim_{s\to 0} s\hat{H}(s) = 0 \tag{15}$$

$$I(0) = \lim_{s\to\infty} s\hat{I}(s) = 0, I(\infty) = \lim_{s\to 0} s\hat{I}(s) = 1/\lambda \tag{16}$$

$$G(0) = \lim_{s\to\infty} s\hat{G}(s) = 1, G(\infty) = \lim_{s\to 0} s\hat{G}(s) = 0 \tag{17}$$

Set $n\nu \leqslant t < (n+1)\nu$, here n is a non-negative integer. Since

$$H(t) * L(t) = K \sum_{j=0}^n y_j H(t - j\nu) \tag{18}$$

making Laplace inversion transform in (6), we have

$$x(t) = \langle\langle x(t) \rangle\rangle + H(t) * \xi(t) + K \sum_{j=0}^n y_j H(t - j\nu) \tag{19}$$

where

$$\langle\langle x(t) \rangle\rangle = x_0 G(t) + v_0 H(t) = x_0 [1 - \lambda t^2 \sum_{j=0}^\infty (-\gamma)^j t^{(2-\alpha)j} E^{j+1}_{2, 3+(2-\alpha)j}(-\lambda t^\alpha)] + v_0 t \sum_{j=0}^\infty (-\gamma)^j t^{(2-\alpha)j} E^{j+1}_{2, 2+(2-\alpha)j}(-\lambda t^\alpha) \tag{20}$$

is the mean of $x(t)$. Note that $\langle\langle x(t) \rangle\rangle$ is independent of $L(t)$.

3 MSD of the oscillator

To obtain the exact expression of MSD, we need to calculate the correlation function $R_x(t, t')$ of the oscillator $x(t)$ by using the double Laplace transform technique.

From (6), we have

$$\langle\langle \hat{x}(s)\hat{x}(s')\rangle\rangle = x_0^2\hat{G}(s)\hat{G}(s') + v_0^2\hat{H}(s)\hat{H}(s') + x_0v_0[\hat{G}(s)\hat{H}(s') + \hat{H}(s)\hat{G}(s')] + \hat{H}(s)\hat{H}(s')\langle\hat{\xi}(s)\hat{\xi}(s')\rangle + \langle\hat{H}(s)\hat{L}(s)\hat{H}(s')\hat{L}(s')\rangle \tag{21}$$

Since

$$\langle\hat{\xi}(s)\hat{\xi}(s')\rangle = k_BT\gamma\frac{s^{\alpha-1}+(s')^{\alpha-1}}{s+s'} \tag{22}$$

and

$$\langle\hat{H}(s)\hat{L}(s)\hat{H}(s')\hat{L}(s')\rangle = 2DK^2\sum_{j=0}^{n}[\hat{H}(s)\exp(-j\nu s)][\hat{H}(s')\exp(-j\nu s')] \tag{23}$$

after some algebra we can find that

$$\hat{H}(s)\hat{H}(s')\langle\hat{\xi}(s)\hat{\xi}(s')\rangle = k_BT[\frac{\hat{I}(s)}{s'}+\frac{\hat{I}(s')}{s}-\frac{\hat{I}(s)+\hat{I}(s')}{s+s'} - \hat{H}(s)\hat{H}(s')-\lambda\hat{I}(s)\hat{I}(s')] \tag{24}$$

It follows that

$$R_x(t,t') = x_0^2G(t)G(t')+v_0^2H(t)H(t') + x_0v_0[G(t)H(t')+H(t)G(t')] + k_BT[I(t)+I(t')-I(|t-t'|) - H(t)H(t')-\lambda I(t)I(t')] + 2DK^2\sum_{j=0}^{n}H(t-j\nu)H(t'-j\nu) \tag{25}$$

Thus we have

$$\langle\langle x^2(t)\rangle\rangle = \langle\langle x(t)\rangle\rangle^2 + k_BT[2I(t) - H^2(t)-\lambda I^2(t)] + 2DK^2\sum_{j=0}^{n}H^2(t-j\nu) \tag{26}$$

from (16), which means that

$$\sigma_x^2(t) = k_BT[2I(t)-H^2(t)-\lambda I^2(t)] + 2DK^2\sum_{j=0}^{n}H^2(t-j\nu) \tag{27}$$

From (25), (26), we have

$$\rho(t,\tau) = \langle\langle[x(t+\tau)-x(t)]^2\rangle = [x_0\Delta G(t,\tau)+v_0\Delta H(t,\tau)]^2 + k_BT[2I(|\tau|)-\Delta H^2(t,\tau)-\lambda\Delta I^2(t,\tau)] + 2DK^2\sum_{j=0}^{n}\Delta H^2(t-j\nu,\tau) \tag{28}$$

where

$$\Delta G(t,\tau) = G(t+\tau)-G(t),$$
$$\Delta H(t,\tau) = H(t+\tau)-H(t),$$
$$\Delta I(t,\tau) = I(t+\tau)-I(t).$$

In order to get the exact expression of MSD, we set $t = n\nu, \tau > 0$. By letting $t\to\infty$ in (28) and taking (15)～(17) into account, we have

$$\rho(\tau) = \lim_{t\to\infty}\rho(t,\tau) = 2k_BTI(\tau) + 2DK^2\sum_{j=0}^{\infty}\Delta H^2(j\nu,\tau) \tag{29}$$

which indicates that

$$\rho(\tau) \propto K^2.$$

Note that (29) recovers the corresponding result of Vinales and Desposito[17,18] when $K = 0$.

4 Asymptotic diffusion of the oscillator

Now we consider asymptotic diffusion of the oscillator underlying the asymptotic expressions of $I(t)$ and $H(t)$.

When $s\to 0$, we have

$$\hat{I}(s) \approx \frac{s^{-1}}{\gamma s^\alpha+\lambda} = \frac{1}{\omega}[\frac{1}{s}-\frac{s^{\alpha-1}}{s^\alpha+\omega}] \tag{30}$$

from (7), here $\omega = \lambda/\gamma$. Applying the inverse Laplace transform technique to (30) and using the following formula for Mittag-Leffler function with two parameters[14]

$$\mathcal{L}[t^{\beta-1}E_{\alpha,\beta}(\pm\omega t^\alpha)] = \frac{s^{\alpha-\beta}}{s^\alpha\mp\omega} \tag{31}$$

we have

$$I(t) \approx \frac{1}{\omega}[1-E_\alpha(-\omega t^\alpha)] \tag{32}$$

here $E_\alpha(z) = E_{\alpha,1}(z)$ is the Mittag-Leffler function. Then, by the following asymptotic property of Mittag-Leffler function[18]

$$E_\alpha(z) \sim -\sum_{j=1}^{\infty}\frac{z^{-k}}{\Gamma(1-j\alpha)}, \quad \alpha\pi/2 < \arg z < 2\pi-\alpha\pi/2,\ |z|\to\infty \tag{33}$$

it follows that

$$I(t) \approx \frac{1}{\omega} - \frac{\Gamma(\alpha)\sin\pi\alpha}{\pi\omega^2}t^{-\alpha}, t\to\infty \tag{34}$$

Consequently, we have

$$H(t) = \frac{\mathrm{d}}{\mathrm{d}t}I(t) \approx \frac{\Gamma(\alpha+1)\sin\pi\alpha}{\pi\omega^2}t^{-(\alpha+1)}, t\to\infty \tag{35}$$

On the other hand, by the definitions of $I(t), H(t)$, we also have

$$I(t) \approx \frac{t^2}{2} - \frac{\gamma t^{4-\alpha}}{\Gamma(5-\alpha)}, t\to 0 \tag{36}$$

$$H(t)=\frac{\mathrm{d}}{\mathrm{d}t}I(t)\approx t-\frac{\gamma t^{3-\alpha}}{\Gamma(4-\alpha)},t\to 0 \quad (37)$$

To investigate asymptotic behaviors of the oscillator for short and long time-lags, we substitute (34)～(37) into (29) and discuss the following two limit cases.

Case 1. $\tau\to\infty$. Since $H(\infty)=0, H(0)=0$, we have

$$2DK^2\sum_{j=0}^{\infty}\Delta^2 H(j\nu,\tau)\approx 2DK^2\sum_{j=0}^{\infty}H^2(j\nu)=\eta_1 \quad (38)$$

It follows that

$$\rho(\tau)\approx 2k_BT\left[\frac{1}{\omega}-\frac{\Gamma(\alpha)\sin\pi\alpha}{\pi\omega^2}\tau^{-\alpha}\right]+\eta_1 \quad (39)$$

which is a power-law decay to

$$\rho(\infty)\approx\frac{2k_BT}{\omega}+\eta_1 \quad (40)$$

for long time-lag. Note that the first term of the right side of (40) describes a diffusion of power-law resulting from the internal noise $\xi(t)$ as well as the second term describes the effect of impulsive noise $L(t)$.

Case 2. $\tau\to 0$. We have

$$\Delta H(j\nu,\tau)=H(j\nu+\tau)-H(j\nu)\approx H'(j\nu)\tau, j=0,1,2,\ldots$$

from the mean value theorem. It follows that

$$2DK^2\sum_{j=0}^{\infty}\Delta H^2(j\nu,\tau)\approx \tau^2\left[2DK^2\sum_{j=0}^{\infty}H'^2(j\nu)\right]=\eta_2\tau^2 \quad (41)$$

$$\rho(\tau)\approx(k_BT+\eta_2)\tau^2-\frac{2k_BT\gamma}{\Gamma(5-\alpha)}\tau^{4-\alpha}\to (k_BT+\eta_2)\tau^2, \tau\to 0 \quad (42)$$

which indicates an enhanced ballistic diffusion resulting from the cooperation of internal and external impulsive noises for short time-lag.

5 Summary

In this paper, by deriving the asymptotic expression of the MSD for short and long time-lags, anomalous diffusion of the linear fractional harmonic oscillator driven by additive impulsive noise is investigated. It is shown that the oscillator undergoes a enhanced ballistic diffusion for short time-lag and a lifted subdiffusion for long time-lag, which means that the additive impulsive noise can be used to amplify the experimental results of optical tweezers experiments.

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