doi：10.3969/j.issn.0490-6756.2018.06.001

非经典扩散方程的时间依赖强全局吸引子的存在性

王　静，马巧珍，姚晓斌
(西北师范大学数学与统计学院，兰州 730070)

摘　要：本文在具有光滑边界 $\partial\Omega$ 的有界域 $\Omega\subset\mathbf{R}^3$ 上研究非经典扩散方程 $u_t-\varepsilon(t)\Delta u_t-\Delta u+\lambda u=f(u)+g(x)$ 并在强拓扑空间中讨论了该问题解的长时行为.所用方法基于 Meng 和 Liu 引入并证明的时间依赖全局吸引子存在性的充分条件.

关键词：非经典扩散方程；时间依赖全局吸引子，存在性
中图分类号：O175.29　　**文献标识码**：A　　**文章编号**：0490-6756(2018)06-1133-04

Existence of time-dependent strong global attractors for nonclassical diffusion equation

WANG Jing, *MA Qiao-Zhen*, *YAO Xiao-Bin*
(College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China)

Abstract: We consider the nonclassical diffusion equation $u_t-\varepsilon(t)\Delta u_t-\Delta u+\lambda u=f(u)+g(x)$ on bounded domain $\Omega\subset\mathbf{R}^3$ with smooth boundary $\partial\Omega$, discuss the long-time behavior of solutions for this problem in strong topological space. The used method is a sufficient condition for the existence of time-dependent global attractor introduced and proved by Meng and Liu.

Keywords: Nonclassical diffusion equation; Time-dependent global attractor; Existence
(2010 MSC 35B25, 37L30, 45K05)

1　Introduction

Let $\Omega\subset\mathbf{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following nonclassical diffusion equation

$$\begin{cases} u_t-\varepsilon(t)\Delta u_t-\Delta u+\lambda u=f(u)+g(x), x\in\Omega,\\ u|_{\partial\Omega}=0, t\geqslant\tau, \\ u(x,\tau)=u_\tau(x), x\in\Omega \end{cases} \tag{1}$$

where $\lambda>0,\tau\in\mathbf{R}$ and $g\in\boldsymbol{L}^2(\Omega)$. Let $\varepsilon(t)$ be a decreasing bounded function satisfying

$$\lim_{t\to+\infty}\varepsilon(t)=0 \tag{2}$$

The nonlinear function $f\in C^1(\mathbf{R})$ satisfies the following growth conditions

$$\limsup_{|s|\to\infty}\frac{f(s)}{s}<\lambda_1, \forall s\in\mathbf{R} \tag{3}$$

$$|f'(s)|\leqslant C(1+|s|^4), \forall s\in\mathbf{R} \tag{4}$$

and

$$|f(s)|\leqslant C(1+|s|^r), \forall s\in\mathbf{R} \tag{5}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H^2(\Omega)\cap H_0^1(\Omega)$, C is a positive constant.

The nonclassical diffusion equation arises as a mathematical model to describe the physical phenomenaes, such as non-Newtonian flows, solid mechanics, and heat conduction[1-3].

Since (1) contains the term $-\varepsilon(t)\Delta u_t$, it is different from the usual reaction diffusion equa-

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收稿日期：2017-07-11
基金项目：国家自然科学基金(11561064,11361053)；西北师范大学青年教师科研能力提升计划项目(NWNU-LKQN-14-6)
作者简介：王静(1990－)，女，博士研究生，主要研究方向为无穷维动力系统. E-mail：wj414 029715@163.com
通讯作者：马巧珍. E-mail：maqzh@nwnu.edu.cn

tion ($\varepsilon(t)=0$). For instance, the reaction diffusion equations have some smoothing property: although the initial data only belongs to a weaker topological space, the solution will belong to a stronger topological space with the higher regularity. In addition, we can not understand its dynamics in the standard semigroup framework because the coefficient $\varepsilon(t)$ of $-\Delta u_t$ depends on the time t, which makes the problem complex.

When $\varepsilon(t)$ is a positive constant, the long-time behavior of solutions for the nonclassical diffusion equations has been extensively studied by several authors[4-10]. when $\varepsilon(t)=1$ and $\lambda=0$, Xiao[4] investigated the global attractor of the problem (1) in $H_0^1(\Omega)$ for $g\in L^2(\Omega)$. Recently, Xie *et al.* [10] studied the existence of global attractors on unbounded domain when the nonlinear function satisfied the arbitrary polynomial growth.

When $\varepsilon(t)$ depends on time t, Ding and Liu[11] obtained the existence of time-dependent global attractors for (1) in phase space $H_0^1(\Omega)$, in which the nonlinearity satisfies (3) and the following condition

$$|f'(s)|\leqslant C(1+|s|^2), \forall s\in \mathbf{R} \tag{6}$$

The asymptotic structure of time-dependent global attractors and the further regularity of u_t have been obtained in Ref. [12]. Simultaneously, Ma *et al.* [13] investigated the existence and regularity as well as asymptotic structure of time-dependent attractor with lower forcing term.

In this paper, we will take advantage of the method different from Refs. [14～16]to prove the existece of time-dependent global attractors in a strong Hilbert space, which has been introduced by Meng and Liu[17]. We extend and improve the results of Ref. [13].

For convenience, we choose C as the positive constant which may be different from line to line or in the same line throughout our paper.

2 Preliminaries

We write $H=L^2(\Omega)$ and $V=H_0^1(\Omega)$, the scalar products and norms on H and V are denoted by $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $\|\cdot\|$, respectively. Without loss of generality, we introduce the family of Hilbert spaces $I_s=D(A^{s/2})$, $\forall s\in\mathbf{R}$, with the standard inner products and norms, respectively,

$$(\cdot,\cdot)_{D(A^{s/2})}=(A^{s/2}\cdot,A^{s/2}\cdot),$$
$$\|\cdot\|_{D(A^{s/2})}=\|A^{s/2}\cdot\|.$$

Then, for $t\in\mathbf{R}$ and $-1\leqslant s\leqslant 1$, let $I_t^s=I_{s+1}$, endowed with the time-dependent norm

$$\|u\|_{I_t^s}^2=\|u\|_s^2+\varepsilon(t)\|u\|_{s+1}^2.$$

Especially, $H=I_0=L^2(\Omega)$, $V=I_1=H_0^1(\Omega)$, $D(A)=I_2=H_0^1(\Omega)\cap H^2(\Omega)$.

Definition 2.1 (Condition(C_t))[17] The process $U(t,\tau)$ is said to satisfy Condition (C_t) in time-dependent space if for any bounded set B_τ of X_τ and for any $\zeta>0$ there exists $\tau_\zeta<t$ and a finite dimensional subspace X_t' of X_t, such that $\{\|PU(t,\tau)B_\tau\|\}$ is bounded and

$$\|(I-P)U(t,\tau)x\|_{X_t}<\zeta, \tau\leqslant\tau_\zeta, x\in B_\tau,$$

where $P:X_t\to X_t'$ is a bounded projector.

Theorem 2.2[17] Let $U(t,\tau)$ be a process in a family of Banach spaces $\{X_t\}_{t\in\mathbf{R}}$. Then there is a time-dependent global attractor $A=\{A_t\}_{t\in\mathbf{R}}$ for $U(\cdot,\cdot)$ in X_t if the following conditions hold:

(i) $U(t,\tau)$ has a pullback absorbing family $B=\{B_t\}_{t\in\mathbf{R}}$;

(ii) $U(t,\tau)$ is Condition (C_t).

In order to obtain the main results, we need the following properties of compactness about the nonlinear operator f. It can be easily proved according to (2)～(5), so we omit it.

Lemma 2.3 Suppose that $f\in C^1(\mathbf{R},\mathbf{R})$ with (5) and let $f:D(A)\to V$ be defined by

$$((f(u),v))=\int_\Omega f'(u)\nabla u\nabla v\mathrm{d}x, \forall u\in D(A), \forall v\in V,$$

Then f is continuous compact.

Lemma 2.4[18] Let Φ, r_1, r_2 be nonnegative, locally summable functions on $[\tau,+\infty)$, $\tau\in\mathbf{R}$, which satisfy the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)+a\Phi(t)\leqslant r_1(t)+r_2(t)\Phi^{1-b}(t), t\in[\tau,+\infty)$$

for $a>0$ and $0<b<1$. Assume also that

$$m_j=\sup_{t\geqslant\tau}\int_t^{t+1}r_j(y)\mathrm{d}y<\infty, j=1,2,$$

then

$$\Phi(t)\leqslant\frac{1}{b}\Phi(\tau)e^{-a(t-\tau)}+\frac{1}{b}m_1C(a)$$
$$+[m_2C(ab)]1/b,t\in[\tau,+\infty),$$

where $C(\nu)=\frac{e^{\nu}}{1-e^{-\nu}}$.

3 Main results

Lemma 3.1[12,13] Assume that (2)～(5) hold. Then for any initial data $u_\tau\in I_\tau$ and any $T>\tau$, there exists a unique solution u of (1) such that $u\in C([\tau,T];I_t)$. Furthermore, if $u_\tau\in I_\tau^1$, then u satisfies $u\in C([\tau,T];I_t^1)$, and the solution depends on the initial data continuously in I_t^1.

It follows from Lemma 3.1 that a family of maps $U(t,\tau)$: $I_\tau\to I_t$ acting as $U(t,\tau)u_\tau=u(t)$ define a strongly continuous process on a family of spaces I_t^1.

Lemma 3.2[13] Assume that (2)～(4) hold. Let $U(t,\tau)u_\tau$ be the solution of (1) with initial value $u_\tau\in I_\tau$. Then there exists a positive constant K, such that

$$\lambda\|u(t)\|^2+(1+\varepsilon(t))\|u(t)\|_1^2\leqslant K,$$
$$\forall\tau\leqslant t-t_0 \tag{7}$$

Lemma 3.3 Assume that (2), (3) and (5) hold. Then there exists a time-dependent absorbing set $B=\{B_t(\rho_1)\}_{t\in\mathbf{R}}$ in I_t^1 for the process $U(t,\tau)$ corresponding to the problem (1).

Proof Multiplying the equation (1) by $2Au$ and integrating on Ω, we find that

$$\frac{d}{dt}[\|u\|_1^2+\varepsilon(t)\|u\|_2^2]-\varepsilon'(t)\|u\|_2^2+$$
$$2\|u\|_2^2+2\lambda\|u\|_1^2=$$
$$(f(u),2Au)+(g(x),2Au) \tag{8}$$

Recalling the Sobolev embeding $H^{6/5}\to L^{2r}(r<5)$ and the interpolation inequality

$$\|u\|_{H^{6/5}}\leqslant C\|u\|_1^{4/5}\|u\|_2^{1/5},$$

for all $u\in I_t^1$ and combining with (5) and (7), we have

$$|(f(u),2Au)|\leqslant 2\int_\Omega C(1+|u|^r)Au\,dx\leqslant$$
$$C(1+\|u\|_2^{2r/5})+\frac{1}{2}\|u\|_2^2 \tag{9}$$

Together with (8)～(9), it follows that

$$\frac{d}{dt}[\|u\|_1^2+\varepsilon(t)\|u\|_2^2]+$$
$$(1-\varepsilon'(t))\|u\|_2^2+2\lambda\|u\|_1^2\leqslant$$
$$C+2\|g\|^2+C\|u\|_2^{2r/5}.$$

Due to the boundedness of $\varepsilon(t)$ and (2), there exists a positive constantLsuch that $\varepsilon(t)\leqslant L$, and then it leads to

$$(1-\varepsilon'(t))\|u\|_2^2\geqslant\|u\|_1^2\geqslant\frac{\varepsilon(t)}{L}\|u\|_1^2 \tag{10}$$

Let $y(t)=\|u\|_1^2+\varepsilon(t)\|u\|_2^2$ and choose $\alpha_1=\min\{\frac{1}{L},2\lambda\}$ with $L,\lambda>0$. Then we obtain

$$\frac{d}{dt}y(t)+\alpha_1y(t)\leqslant C+2\|g\|^2+$$
$$Cy^{2r/5}(t),t\geqslant t_0.$$

By Lemma 2.4, we conclude that

$$y(t)\leqslant\frac{5}{5-r}y(t_0)e^{-\alpha_1(t-t_0)}+K_1,t\geqslant t_0,$$

where

$$K_1=\frac{5}{5-r}(C+2\|g\|^2)m(\alpha_1)+$$
$$[Cm(\frac{\alpha_1(5-r)}{5})]5/(5-r).$$

Thus we have

$$\|u(t)\|_1^2+\varepsilon(t)\|u(t)\|_2^2\leqslant$$
$$\frac{5}{5-r}[\|u(t_0)\|_1^2+$$
$$\varepsilon(t_0)\|u(t_0)\|_2^2]e^{-\alpha_1(t-t_0)}+K_1.$$

So

$$B=\{u\in B_t(\rho_1):\|u(t)\|_1^2+\varepsilon(t)\|u(t)\|_2^2\leqslant\rho_1\}$$

is a time-dependent absorbing set in I_t^1.

Theorem 3.4 Assume that (2)～(3) and (5) hold, then the process $U(t,\tau)$: $I_\tau^1\to I_t^1$ generated by the problem (1) satisfies Condition (C_t) in I_t^1.

Proof Let $\{\omega_k\}_{k=1}^\infty$ be a orthogonal basis of I_t^1 which consists of eigenvalues of $A=-\Delta$. The corresponding eigenvalues are denoted by $0<\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\cdots\lambda_j\leqslant\cdots,\lambda_j\to\infty$ with $A\omega_k=\lambda_k\omega_k$, $\forall k\in\mathbf{N}$. Let $V_m=\text{span}\{\omega_1,\cdots,\omega_m\}$ in V and let $P_m:V\to V_m$ be an orthogonal projector. We write

$$u=P_mu+(I-P_m)u=u_1+u_2.$$

Multiplying the equation (1) by $2Au_2$ and integrating on Ω, we find that

$$\frac{d}{dt}[\|u_2\|_1^2+\varepsilon(t)\|u_2\|_2^2]-$$
$$\varepsilon'(t)\|u_2\|_2^2+2\|u_2\|_2^2+2\lambda\|u_2\|_1^2=$$
$$(f(u),2Au_2)+(g(x),2Au_2) \tag{11}$$

Since $g\in L^2(\Omega)$ and $f:D(A)\to$ V is compact, from Lemma 2.3, there exists some m such that for any $\eta>0$,

$$\|(I-P_m)g\|\leqslant\frac{\eta}{2},\ \|(I-P_m)f\|\leqslant\frac{\eta}{2}.$$

Combining with (10), (11), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|u_2\|_1^2+\varepsilon(t)\|u_2\|_2^2]+\alpha_2(\|u_2\|_1^2+\varepsilon(t)\|u_2\|_2^2)\leqslant C\eta^2,$$

where $\alpha_2=\min\{\frac{1}{L},2\lambda\}$ with $L,\lambda>0$. By the Gronwall Lemma, it follows that

$$\|u_2\|_1^2+\varepsilon(t)\|u_2\|_2^2\leqslant (\|u_2(t_1)\|_1^2+\varepsilon(t_1)\|u_2(t_1)\|_2^2)e^{-\alpha_2(t-t_1)}+K_2, t\geqslant t_1 \quad (12)$$

where $K_2=\frac{C\eta^2}{\alpha_2}$. Taking

$$t_2=t_1+\frac{1}{\alpha_2}ln(\alpha_2\rho_1/C\eta^2),$$

from Lemma 3.2 and (12), we conclude that

$$\|u\|_1^2+\varepsilon(t)\|u\|_2^2\leqslant C(1+\frac{1}{\alpha_2})\eta^2, \forall t\geqslant t_2.$$

Thus we obtain that the process $U(t,\tau)$ satisfies Condition (C_t).

Theorem 3.5 Assume that the conditions (2)~(5) hold, then the process $U(t,\tau): I_\tau^1\rightarrow I_t^1$ generated by the problem (1) has a invariant time-dependent global attractor $A=\{A_t\}_{t\in\mathbf{R}}$.

Proof It follows from lemma 3.3 and Theorem 3.4 that the problem (1) exists a unique time-dependent global attractor $A=\{A_t\}_{t\in\mathbf{R}}$. The proof is complete.

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引用本文格式:

中 文: 王静, 马巧珍, 姚晓斌. 非经典扩散方程的时间依赖强全局吸引子的存在性 [J]. 四川大学学报: 自然科学版, 2018, 55: 1133.

英 文: Wang J, Ma Q Z, Yao X B. Existence of time-dependent strong global attractors for nonclassical diffusion equation [J]. J Sichuan Univ: Nat Sci Ed, 2018, 55: 1133.