doi：10.3969/j.issn.0490-6756.2018.04.003

空时分数阶 mBBM 方程的新精确解

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摘 要：为了构造空时分数阶 mBBM 方程的新显示解，本文首先利用分数阶复变换技巧将分数阶偏微分方程转化为常微分方程，然后应用扩展的 $(\frac{G'}{G})$- 展开法求解该常微分方程. 新精确解包括分别带有负幂次项的三角函数解，双曲函数解及有理函数解.

关键词：扩展的 $(\frac{G'}{G})$- 展开法；空时分数阶 mBBM 方程；精确解

中图分类号：O175.29 **文献标识码**：A **文章编号**：0490-6756(2018)04-0673-05

New explicit solutions for space-time fractional mBBM equation

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Abstract: In order to construct new explicit solutions of the space-time fractional mBBM equation, the fractional complex transformation is applied to turn the fractional partial differential equation into an ordinary differential equation (ODE). Then the extended $(\frac{G'}{G})$- expansion method is used to solve the ODE. The new exact solutions contain trigonometric function solutions, hyperbolic function solutions, and rational function solutions with negative power exponent.

Keywords: Extended $(\frac{G'}{G})$- expansion method; Space-time fractional mBBM equation; Explicit solutions
(2010 MSC 35R11, 83C15)

1 Introduction

In this paper, we consider the following space-time fractional mBBM equation[1-5]:

$$D_t^\alpha u + D_x^\alpha u - \upsilon u^2 D_x^\alpha u + D_x^{3\alpha} u = 0,\quad 0<\alpha\leqslant 1, t>0 \tag{1}$$

where u is the function of (x,t), υ is a non-zero positive constant, α is a parameter describing the order of the fractional derivative, D_t^α is a fractional differential operator in the sense of Jumarie's modified Riemann-Liouville derivative[6], which is defined as

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\xi)^{-\alpha}(f(\xi) - f(0))\mathrm{d}\xi,\ 0<\alpha<1, \\ (f^n(x))^{(\alpha-n)}, n\leqslant\alpha<n+1, n\geqslant 1 \end{cases} \tag{2}$$

Some important properties of the modified Riemann-Liouville derivative are:

$$D_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha} \tag{3}$$

收稿日期：2017-09-27
基金项目：国家自然科学基金(11371267)；四川省教育厅重点项目基金(2012ZA135)
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$$D_x^\alpha(f(x)g(x)) = g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x) \quad (4)$$

$$D_x^\alpha f[g(x)] = f_g'[g(x)]D_x^\alpha g(x) = D_g^\alpha f[g(x)](g'(x))^\alpha \quad (5)$$

The space-time fractional mBBM equation is used to model surface long waves in nonlinear dispersive media, the hydromagnetic waves in cold plasma and acoustic gravity waves in compressible fluids. The equation is an important mathematical model attracted the attention of many researchers[1-5]. For instance, Ref. [2] used ansatz method to solve (1) and obtained bright and dark soliton solutions, Ref. [3] applied $(\frac{G'}{G})$ and $(\frac{G'}{G},\frac{1}{G})$ -expansion methods and obtained some exact solutions.

There are many reliable methods to construct the exact solutions, such as the fractional first integral method[7,8], the exp-function method[9,10], and the dynamical system method[11-13]. Lately, Yin[14] based on the auxiliary equation $G''G = \alpha G'^2 + \beta GG' + \gamma G^2$, introduced a new method, called extended $(\frac{G'}{G})$- expansion method[15], which can seek more general explicit solutions. In this paper, we will apply extended $(\frac{G'}{G})$- expansion method in the sense of the modified Riemann-Liouville derivative to seek explicit solutions of (1).

2 The extended $(\frac{G'}{G})$-expansion method

Considering the following fractional partial differential equation:

$$P(u, D_t^\alpha u, D_x^\theta u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\theta u, D_x^\theta D_x^\theta u, \cdots) = 0, 0<\alpha,\theta\leqslant 1 \quad (6)$$

where u is an unknown function, $D_t^\alpha u$ and $D_x^\theta u$ are Jumarie's modified Riemann-Liouville derivatives of $u(x,t)$, P is a polynomial in $u(x,t)$ and its various partial derivatives including fractional derivatives in which the higher order derivatives and nonlinear terms are involved.

Step 1. By using the fractional complex transformation[16]

$$u(x,t) = u(\xi), \xi = \frac{kx^\theta}{\Gamma(1+\theta)} - \frac{ct^\alpha}{\Gamma(1+\alpha)} \quad (7)$$

where k and c are non-zero arbitrary constants, (6) reduces to an ordinary differential equation of the form

$$P(u, u', u'', u''', \cdots) = 0 \quad (8)$$

Step 2. Assume that the solution of (8) has the following form

$$u(\xi) = \sum_{i=0}^{m} a_i\left(\frac{G'}{G}\right)^i + \sum_{i=1}^{m} b_i\left(\frac{G'}{G}\right)^{-i} \quad (9)$$

where $a_i(i=0,1,2,3\cdots m)$, $b_i(i=1,2,3\cdots m)$ are constants and $G(\xi)$ satisfies the following auxiliary equation

$$G''G = BGG' + CG'^2 + EG^2 \quad (10)$$

with B, C, E are real parameters. Solving (10), we obtain

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{B}{2(1-C)} + \frac{\sqrt{\Omega}}{2(1-C)} \frac{C_1\sinh\frac{\sqrt{\Omega}}{2}\xi + C_2\cosh\frac{\sqrt{\Omega}}{2}\xi}{C_1\cosh\frac{\sqrt{\Omega}}{2}\xi + C_2\sinh\frac{\sqrt{\Omega}}{2}\xi}, \\ \Omega = B^2+4E-4CE>0, C\neq 1, \\ \frac{B}{2(1-C)} + \frac{\sqrt{-\Omega}}{2(1-C)} \frac{-C_1\sin\frac{\sqrt{-\Omega}}{2}\xi + C_2\cos\frac{\sqrt{-\Omega}}{2}\xi}{C_1\cos\frac{\sqrt{-\Omega}}{2}\xi + C_2\sin\frac{\sqrt{-\Omega}}{2}\xi}, \\ \Omega = B^2+4E-4CE<0, C\neq 1, \\ \frac{1}{1-C}\left(\frac{C_1}{C_1\xi+C_2} + \frac{B}{2}\right), \Omega = B^2+4E-4CE=0, C\neq 1 \end{cases} \quad (11)$$

where C_1, C_2 are arbitrary constants.

Step 3. Considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in (8), we can get the positive integer m.

Step 4. Substituting (9) and (10) into (8), we collect all the terms with the same order of $(\frac{G'}{G})$ together. Equating each coefficient of the obtained polynomials to zero yields a set of algebraic equations for $k, c, a_i (i=0,1,2,3\cdots m), b_i (i=1,2,3\cdots m)$. Solving the algebraic equations, we get more general type and new exact solutions of (6).

3 Applications

Applying the fractional complex transformation to (1), after integrating and letting the integral constant be zero, we obtain

$$(k-c)u - \frac{1}{3}k\upsilon u^3 + k^3 u'' = 0 \tag{12}$$

By applying the homogeneous balance between the highest order derivative term $u''(\xi)$ and nonlinear term $u^3(\xi)$ in (12), we have

$$u(\xi) = a_1\left(\frac{G'}{G}\right) + a_0 + b_1\left(\frac{G'}{G}\right)^{-1} \tag{13}$$

Calculating $u^3(\xi), u''(\xi)$ and then substituting $u^3(\xi), u''(\xi)$ and $u(\xi)$ into (12), collecting each coefficient of polynomials in $\left(\frac{G'}{G}\right)^i (i=-3,\cdots,0,\cdots 3)$ to zero yields a set of simultaneous algebraic equation for b_1, a_0, a_1, k and c:

$$\begin{cases}\left(\frac{G'}{G}\right)^3: -\frac{1}{3}k\upsilon a_1^3 + 2(C-1)^2 k^3 a_1 = 0, \\ \left(\frac{G'}{G}\right)^2: -k\upsilon a_0 a_1^2 + 3B(C-1)k^3 a_1 = 0, \\ \left(\frac{G'}{G}\right)^1: (k-c)a_1 - k\upsilon(a_1^2 b_1 + a_0^2 a_1) + k^3[B^2 + 2E(C-1]a_1 = 0, \\ \left(\frac{G'}{G}\right)^0: (k-c)a_0 - \frac{1}{3}k\upsilon a_0^3 - 2k\upsilon a_0 a_1 b_1 + k^3 BEa_1 + k^3 B(C-1)b_1 = 0, \\ \left(\frac{G'}{G}\right)^{-1}: (k-c)b_1 - k\upsilon(b_1^2 a_1 + a_0^2 b_1) + k^3[B^2 + 2E(C-1)]b_1 = 0, \\ \left(\frac{G'}{G}\right)^{-2}: -k\upsilon a_0 b_1^2 + 3BEk^3 b_1 = 0, \\ \left(\frac{G'}{G}\right)^{-3}: -\frac{1}{3}k\upsilon b_1^3 + 2E^2 k^3 b_1 = 0\end{cases} \tag{14}$$

Solving these algebraic equations, we have

$$\begin{cases}a_0 = \pm\frac{B|k|}{2}\frac{(C-1)}{|C-1|}\sqrt{\frac{6}{\upsilon}}, \\ a_1 = \pm|k\|C-1|\sqrt{\frac{6}{\upsilon}}, \\ b_1 = 0, c = k - \frac{1}{2}k^3(B^2 + 4E - 4CE)\end{cases} \tag{15}$$

$$\begin{cases}a_0 = \pm\frac{B|k|}{2}\frac{E}{|E|}\sqrt{\frac{6}{\upsilon}}, \\ b_1 = \pm|k\|E|\sqrt{\frac{6}{\upsilon}}, \\ a_1 = 0, c = k - \frac{1}{2}k^3(B^2 + 4E - 4CE)\end{cases} \tag{16}$$

Family 1. If $\Omega = B^2 + 4E - 4CE > 0, C \neq 1$, we have the hyperbolic function traveling wave solutions

$$u_1(\xi) = \pm\frac{|k|\sqrt{\Omega}}{2}\frac{|C-1|}{(1-C)}\sqrt{\frac{6}{\upsilon}}H_1 \tag{17}$$

$$u_2(\xi) = \pm\frac{B|k|}{2}\frac{E}{|E|}\sqrt{\frac{6}{\upsilon}} \pm |k\|E|\sqrt{\frac{6}{\upsilon}} \cdot \left[\frac{B}{2(1-C)} + \frac{\sqrt{\Omega}}{2(1-C)}H_1\right]^{-1} \tag{18}$$

where

$$H_1 = \frac{C_1\sinh\frac{\sqrt{\Omega}}{2}\xi + C_2\cosh\frac{\sqrt{\Omega}}{2}\xi}{C_1\cosh\frac{\sqrt{\Omega}}{2}\xi + C_2\sinh\frac{\sqrt{\Omega}}{2}\xi},$$

$\Omega = B^2 + 4E - 4CE, C_1, C_2$ are arbitrary constants.

On the other hand, assuming $C_2 = 0, C_1 \neq 0$ and $C > 1$, the solution of (17) can be written as

$$u_{1_1}(\xi) = \mp \frac{|k|\sqrt{\Omega}}{2}\sqrt{\frac{6}{\upsilon}}\tanh\frac{\sqrt{\Omega}}{2}\xi \tag{19}$$

Family 2. If $\Omega = B^2 + 4E - 4CE < 0, C \neq 1$, we have the trigonometric function traveling wave solutions

$$u_3(\xi) = \pm \frac{|k|\sqrt{-\Omega}}{2}\frac{|C-1|}{(1-C)}\sqrt{\frac{6}{\upsilon}}H_2 \tag{20}$$

$$u_4(\xi) = \pm\frac{B|k|}{2}\frac{E}{|E|}\sqrt{\frac{6}{\upsilon}} \pm |k \| E|\sqrt{\frac{6}{\upsilon}}\left[\frac{B}{2(1-C)} + \frac{\sqrt{-\Omega}}{2(1-C)}H_2\right]^{-1} \tag{21}$$

where

$$H_2 = \frac{-C_1\sin\frac{\sqrt{-\Omega}}{2}\xi + C_2\cos\frac{\sqrt{-\Omega}}{2}\xi}{C_1\cos\frac{\sqrt{-\Omega}}{2}\xi + C_2\sin\frac{\sqrt{-\Omega}}{2}\xi},$$

$\Omega = B^2 + 4E - 4CE, C_1, C_2$ are arbitrary constants.

On the other hand, assuming $C_1 = 0, C_2 \neq 0$ and $C > 1$, the solution of (20) can be written as

$$u_{3_1}(\xi) = \mp \frac{|k|\sqrt{-\Omega}}{2}\sqrt{\frac{6}{\upsilon}}\cot\frac{\sqrt{-\Omega}}{2}\xi \tag{22}$$

Family 3. If $\Omega = B^2 + 4E - 4CE = 0, C \neq 1$, we have the rational function traveling wave solutions

$$u_5(\xi) = \pm \frac{C_1|k|}{(C_1\xi + C_2)}\frac{|C-1|}{(1-C)}\sqrt{\frac{6}{\upsilon}} \tag{23}$$

$$u_6(\xi) = \pm\frac{B|k|}{2}\frac{E}{|E|}\sqrt{\frac{6}{\upsilon}} \pm |k \| E|(1-C)\sqrt{\frac{6}{\upsilon}}\left(\frac{C_1}{C_1\xi + C_2} + \frac{B}{2}\right)^{-1} \tag{24}$$

4 Conclusions

In this paper, the space-time fractional mBBM equation is converted to an ordinary differential equation by the fractional complex transformation. Then the extended $(\frac{G'}{G})$- expansion method is applied to introduce a new auxiliary equation (10) and add negative power exponent to the expansion of $(\frac{G'}{G})$. Comparing our results with the known results[1-5], $u_2(\xi), u_4(\xi), u_6(\xi)$ are new results. More over, the used technique is effective and productive.

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引用本文格式：

中　文：康丽，孙峪怀，廖红梅，等. 空时分数阶 mBBM 方程的新精确解[J]. 四川大学学报：自然科学版，2018，55：673.

英　文：Kang L, Sun Y H, Liao H M, *et al*. New explicit solutions for space-time fractional mBBM equation [J]. J Sichuan Univ: Nat Sci Ed, 2018, 55: 673.