doi：10.3969/j.issn.0490-6756.2018.02.005

$\mathbf{C}^3$ 中曲面 Kähler 角的刚性定理

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摘　要：浸入到近复 Hermit 流形的曲面的 Kähler 角是一个重要的不变量，可以用于刻画曲面偏离拟全纯曲线的程度. 近年来，具有常 Kähler 角的曲面仍是很有意义的研究对象. 对于 3 维复欧氏空间 $\mathbf{C}^3$ 中具有常 Kähler 角的曲面收缩子，本文证明了两个刚性定理. 这些定理是有关 $\mathbf{C}^3$ 中曲面自收缩子的相应定理的直接拓展.

关键词：刚性定理；浸入曲面；Kähler 角；自收缩子

中图分类号：O186.1　　**文献标识码**：A　　**文章编号**：0490-6756(2018)02-0243-08

Some rigidity theorems of the Kähler angle of surfaces in $\mathbf{C}^3$

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Abstract: The Kähler angle of a surface immersed in an almost Hermitian manifold is an important invariant which can be used to measure the deviation of the surface from being a complex (or pseudo-holomorphic) one and, in particular, the surface with a constant Kähler angle has been an interesting object in the study of submanifolds for years. In this paper, we prove two rigidity theorems for complete self-shrinkers of mean curvature flow with constant Kähler angle, which are immersed in the complex Euclidean space $\mathbf{C}^3$ of dimension 3. These are direct extensions of some known theorems for self-shrinkers immersed in $\mathbf{C}^2$

Keywords: Rigidity theorem; Immersed surfaces; Kähler angle; Self-shrinker

(2010 MSC 53C44, 53D12)

1　Introduction

Let $(\mathbf{N}, \boldsymbol{J}, \langle\cdot,\cdot\rangle)$ be an almost Hermitian manifold with ω its Kähler form, and $x: M^m \to \mathbf{N}$ be an isometrically immersed submanifold of dimension m. Denote, accordingly, by TM^m, $T^\perp M^m$ and dV_M the tangent space, the normal space and the volume form of x. Then x is called totally real if $\boldsymbol{J}(x_* TM^m) \subset T^\perp M^m$. In particular, in case that $m = \frac{1}{2}\dim_{\mathbf{R}} \mathbf{N}$, the totally real submanifolds are also called Lagrangian. Furthermore, when m=2, we can define the Kähler angle θ of x by[1]

$$x^*\omega = \cos\theta d V_M,\ \theta \subset [0, \pi] \tag{1}$$

Then, x is totally real if and only if the Kähler angle $\theta \equiv \frac{\pi}{2}$. Furthermore, if $\cos\theta > 0$ everywhere then x is called symplectic. Symplectic surfaces

收稿日期：2017-10-23

基金项目：国家自然科学基金(11671221,11371018)

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are also of great interest to study. For example, if the initial surface of a mean curvature flow in a Kähler-Einstein surface is symplectic, then the surface M_t at every time t is also symplectic[2,3]. So this kind flow is reasonably referred to as a symplectic mean curvature flow.

As we know, the concept of Kähler angle has been effectively used to study conformal minimal surfaces that are immersed in the complex projective space $\mathbf{C}P^n$. For example, each map appeared in the Veronese sequence must be a minimal immersion of 2-sphere into $\mathbf{C}P^n$. With a constant Kähler angle[4-6]. On the other hand, the Kähler angle is successfully used in Refs. [7]and [8] to study minimal surfaces immersed in the nearly Kähler manifold $\boldsymbol{S}^6$.

Denote by $\mathbf{R}^{m+p}$ the real Euclidean space of dimension $m+p$ with $m\geq 2$ and $p\geq 1$. Then, an isometric immersion $x:M^m\to\mathbf{R}^{m+p}$ with the mean curvature vector field H is called a self-shrinker (of the mean curvature flow) if $H=-x^{\perp}$, where $^{\perp}$ is the orthogonal projection of $\mathbf{R}^{m+p}$ on to the normal space $T^{\perp}M$ of x.

To introduce our main theorems in this paper, we first cite some of the relevant results. By using the self-adjoint property of a stability operator, Arezzo-Sun proved the following two rigidity theorems:

Theorem 1. 1[9] Let x: $M^2\to\mathbf{C}^2$ be a complete symplectic self-shrinker with polynomial volume growth. If the second fundamental form h of x satisfies $|h|^2\leqslant 2$, then $|h|^2\equiv 0$ and $x(M^2)$ must be a plane.

Theorem 1. 2[9] Let x: $M^2\to\mathbf{C}^2$ be a complete symplectic self-shrinker with Kähler angle θ and polynomial volume growth. lf $|h|^2$ is bounded and $\cos\theta\geq\delta>0$, then $x(M^2)$must be a plane.

***Remark* 1** Before the appearance of Theorem 1. 2, Han-Sun proved in Ref. [10] that a translating soliton $x:M^2\to\mathbf{C}^2$ to the symplectic mean curuature flow in $\mathbf{C}^2$ with polynomial volume growth, non-positive normal curvature and bounded second fundamental form must be minimal if the Kähler angle θ of x satisfies $\cos\theta\geq\delta\geq 0$.

Complementary to the above symplectic case, Li-Wang considered the Lagrangian case and proved the following theorem:

Theorem 1. 3[11] Let x: $M^2\to\mathbf{C}^2$ be a compact orientable Lagrangian self-shrinker. If the second fundamental form h of x satisfies $|h|^2\leq 2$, then $|h|^2\equiv 0$ and $x(M^2)$ must be the Clifford torus $S^1(1)\times S^1(1)$.

***Remark* 2** Castro and Lerma also proved Theorem 1. 3 in Ref. [12] under the additional condition that the Gauss curvature K of M^2 is either non-negative or non-positive.

Recently Li Xingxiao and Li Xiao proved the following theorems which are generalizations of the above theorems in some different direction:

Theorem 1. 4[13] Let x: $M^2\to\mathbf{C}^2$ be a complete self-shrinker with Kähler angle θ satisfying $\cos\theta\geq 0$. If the second fundamental form h of x is square integrable with the weight $e^{-\frac{|x|^2}{2}}$, that is,

$$\int_M |h|^2 e^{-\frac{|x|^2}{2}}\,dV_M<\infty \tag{2}$$

then θ is constant and $x(M^2)$is either a Lagrangian surface or a plane.

Theorem 1. 5[13] Let $x:M^2\to\mathbf{C}^2$ be a complete self-shrinker with Kähler angle θ. If the second fundamental form h of x is square integrable with the weigh $e^{-\frac{|x|^2}{2}}$, and there exists a number $\lambda\in[0,1)$ such that

$$|\nabla\theta|^2\leq\frac{\lambda\cos^2\theta\,|\bar{D}\boldsymbol{J}_M|^2}{4(1-\lambda\cos^2\theta)},$$

then θ is constant and $x(M^2)$is either a Lagrangian surface or a plane.

***Remark* 3** The complex structure $\boldsymbol{J}_M$ appeared in Theorem 1. 5, which is well-defined on $\mathbf{C}^2$ along x, was also used in Refs. [2,14].

The aim of the present paper is to extend the above last two theorems to higher codimensions. To this end, the natural, simpler but important step is to consider surfaces in $\mathbf{C}^3$ of the next higher dimension to $\mathbf{C}^2$, and the basic idea of doing so comes from Ref. [13]. Now the main theorems here can be stated as follows.

Theorem 1.6 Let $x: M^2 \to \mathbf{C}^3$ be a complete self- shrinker with the Kähler angle satisfying $\sin\theta \neq 0$. Then the trivial holomorphic bundle $M^2 \times \mathbf{C}^3$ is decomposed into the orthogonal direct sum of two $\boldsymbol{J}$-invariant subbundles V and N: $M^2 \times \mathbf{C}^3 = V \oplus \mathbf{N}$, where V is of rank 4 and generated by the tangent bundle $x_*(TM)$. Furthermore, if

(1) $\cos\theta \geqslant 0$;

(2) $\mathbf{N}$ is flat with respect to the induced connection;

(3) The second fundamental form h of x is square integrable with the weight $e^{-\frac{|x|^2}{2}}$, then θ is constant, and $x(M^2)$ is either a totally real surface, or a real plane contained in $\mathbf{C}^2$ up to a holomorphic isometry of $\mathbf{C}^3$.

Theorem 1.7 Let $x: M^2 \to \mathbf{C}^3$ be a complete self-shriker with Kähler angle satisfying $\sin\theta \neq 0$. Denote by $\mathbf{N}_0 := \mathbf{N}^\perp \subset T^\perp M$, the sub-normal bundle complementary to $\mathbf{N}$ in the normal bundle $T^\perp M$ of x. If

(1) $\mathbf{N}$ is flat with resprct to the induced connection;

(2) The second fundamental form h of x is square integrable with the weigh $e^{-\frac{|x|^2}{2}}$;

(3) There exists some $\lambda \in (0,1)$ such that

$$|\nabla\theta|^2 \leq \frac{(1-\lambda)\cos^2\theta(|h|^2 - 2K_{\mathbf{N}_0})}{1-\lambda\cos\theta} \tag{3}$$

where $K_{\mathbf{N}_0}$ is the (Gaussian) curuature of $\mathbf{N}_0$, then θ is constant, and $x(M^2)$ is either a totally real surface, or a real plane contained in $\mathbf{C}^2$ up to a holomorphic isometry of $\mathbf{C}^3$.

***Remark* 4** In case that $\sin\theta \neq 0$, the orientation of M^2 and the complex structure $\boldsymbol{J}$ naturally define an orientation of $\mathbf{N}_0$ and $\mathbf{N}$ by formulas (6)~(8) (see Remark 5). Moreover, the complex structure $\boldsymbol{J}$ induces an orientation-preserving bundle map $\tilde{\boldsymbol{J}}: TM \to \mathbf{N}_0$. In particular, the (Gaussian) curvatures $K_{\mathbf{N}_0}, K_{\mathbf{N}}$ on $\mathbf{N}_0$ and $\mathbf{N}$, respectively, are well-defined. So the condition in the above theorems that $\mathbf{N}$ is flat is the same that $K_{\mathbf{N}} \equiv 0$. Furthermore, for any local oriented orthonormal tangent frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and oriented orthonormal frame $\{\boldsymbol{e}_3, \boldsymbol{e}_4\}$ of $\mathbf{N}_0$, the Ricci equation implies that

$$|h|^2 - 2K_{\mathbf{N}_0} \equiv |h|^2 - 2R^{\perp}_{3412} = \sum_i((h^4_{2i} + h^3_{1i})^2 + (h^4_{1i} - h^3_{2i})^2 + (h^5_{1i})^2 + (h^6_{1i})^2 + (h^5_{2i})^2 + (h^6_{2i})^2) \geq 0 \tag{4}$$

It can be easily shown by (14)~(16) that, when $\sin 2\theta \neq 0$, the equality in (4) holds if and only if x is of constant Kähler angle, the bundle map $\tilde{\boldsymbol{J}}$ is connection-preserving and, up to a holomorphic isometry of $\mathbf{C}^3$, $x(M^2) \subset \mathbf{C}^2$.

2 Kähler angles of surfaces and self-shrhrinker of mean curvature flow

We shall recall some necessary facts for the Kä hler angle and self-shrinkers. First the following convention of the ranges of indices are to be agreed with throughout this paper, if on other is specified: $1 \leqslant i,j,k \leqslant 2$, $3 \leqslant \alpha,\beta,\gamma \leqslant 6$, $1 \leqslant A, B, C \leqslant 6$.

Let $x: M^2 \to \mathbf{C}^3$ be an immersed surface and $\omega, \boldsymbol{J}, \langle\cdot,\cdot\rangle$ be the standard Kähler form, complex structure and the corresponding metric on $\mathbf{C}^3$, accordingly. The Kähler θ of x in $\mathbf{C}^2$ defined by (1) is equivalent to be given by

$$\cos\theta = \langle \boldsymbol{J}(x_*\boldsymbol{e}_1), x_*\boldsymbol{e}_2\rangle \tag{5}$$

where $\{e_1, e_2\}$ is an oriented orthonormal frame field on M^2, Clearly, the left hand side of (5) is independent of the choice of $\{e_1, e_2\}$.

Suppose that $\sin\theta \neq 0$. Then, starting from any oriented orthonormal frame field $\{e_1, e_2\}$ on M^2, we can find along x an orthonormal frame field

$$\{x_*e_1, x_*e_2, e_3, e_4, e_5, e_6\}$$

of $\mathbf{C}^3$, such that following are satisfied[7]:

$$\boldsymbol{J}(x_*e_1) = (x_*e_2)\cos\theta + e_3\sin\theta,\ \boldsymbol{J}(x_*e_2) = -(x_*e_1)\cos\theta + e_4\sin\theta \tag{6}$$

$$\boldsymbol{J}(e_3) = -(x_*e_1)\sin\theta - e_4\cos\theta,\ \boldsymbol{J}(e_4) =$$

$$-(x_* e_2)\cos\theta + e_3 \sin\theta \qquad (7)$$

$$J(e_5) = -e_6,\ J(e_6) = e_5 \qquad (8)$$

***Remark* 5** Clearly, $\{e_3, e_4\}$ is uniquely determined by $\{e_1, e_2\}$ and J, but $\{e_5, e_6\}$ is not. Define $\mathbf{N}_0 := \mathrm{Span}_{\mathbf{R}}\{e_3, e_4\}$ Then $\mathbf{N}_0$ is independent of the choice of $\{e_1, e_2\}$ and thus is a well-defined sub-normal bundle. Note that the orientation of $\mathbf{N}_0$ given by $\{e_3, e_4\}$ is also well-defined and uniquely determined by the orientation of TM. Let $\mathbf{N}$ be the sub-normal bundle orthogonally complementary to $\mathbf{N}_0$ in the normal bundle $T^{\perp}M$, which is J invariant and has the orientation by $\{e_5, e_6\}$. Write $V := x_*(TM) \oplus \mathbf{N}_0$. Then V is also J invariant and the trivial bundle $M \times \mathbf{C}^3$ can be decomposed as follows:

$$M \times \mathbf{C}^3 = V \oplus \mathbf{N} \equiv x_*(TM) \oplus \mathbf{N}_0 \oplus \mathbf{N} \qquad (9)$$

Denote by D, $\bar{D}$ and $D^{\perp}$ the Levi-Civita connections on M^2, $\mathbf{C}^3$ and the normal bundle, accordingly. Then the formulas of Gauss and Weingarten are give by

$$\bar{D}_X(x_* Y) = x_*(D_X Y) + h(X, Y),$$

$$\bar{D}_X \xi = -x_*(A_\xi X) + D_X^{\perp}\xi,$$

where X and Y are tangent vector fields, ξ is a normal vector field on x, h denotes the second fundamental form and A is the shape operator.

In what follows we often identify M^2 with $x(M^2)$ and omit x_* from some formulas and equations.

Let $\{\omega^A\}$ be the dual frame field of $\{e_A\}$, and $\{\omega_A^B\}$ the components of the Levi-Civita connection of $\mathbf{C}^3$ with respect to $\{e_A\}$. Since the Levi-Civita connection $\bar{D}$ is a comples one, that is,

$$\bar{D}(J e_A) = J(\bar{D} e_A),$$

a direct computation similar to that in Ref. [7] using (6)~(8) proves the following lemma:

Lemma 2.1[7] If $\sin\theta \neq 0$, then the following identities for an immersed surface $x: M^2 \to \mathbf{C}^3$:

$$d\theta = \omega_1^4 - \omega_2^3 \qquad (10)$$

$$(\omega_1^3 + \omega_2^4)\cos\theta + (\omega_3^4 - \omega_1^2)\sin\theta = 0 \qquad (11)$$

$$\omega_1^6 = \omega_2^5 \cos\theta + \omega_3^5 \sin\theta,\ \omega_2^6 = -\omega_1^5 \cos\theta + \omega_4^5 \sin\theta \qquad (12)$$

$$\omega_3^6 = -\omega_1^5 \sin\theta - \omega_4^5 \cos\theta,\ \omega_4^6 = -\omega_2^5 \sin\theta + \omega_3^5 \cos\theta \qquad (13)$$

Write $\omega_A^C = \sum_B \Gamma_{AB}^C \omega^B$. Then $\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j$ with h_{ij}^α being the components of the second fundamental form h of x. Thus, by (10), the Kähler angle θ of x is constant if and only if

$$h_{11}^4 = h_{12}^3,\ h_{12}^4 = h_{22}^3 \qquad (14)$$

Moreover, by (12) and (13), we have

$$\Gamma_{3i}^5 = \frac{1}{\sin\theta} h_{1i}^6 - \cot\theta h_{2i}^5,$$

$$\Gamma_{4i}^5 = \frac{1}{\sin\theta} h_{2i}^6 + \cot\theta h_{1i}^5 \qquad (15)$$

$$\Gamma_{3i}^6 = -\frac{1}{\sin\theta} h_{1i}^5 - \cot\theta h_{2i}^5,\ \Gamma_{4i}^6 = -\frac{1}{\sin\theta} h_{2i}^5 + \cot\theta h_{1i}^6 \qquad (16)$$

Now let $x: M^m \to \mathbf{R}^{m+p}$ be a self-shrinker (of the mean curvature flow) then the mcan curvature is by definition $H = -x^{\perp}$. In this case, there is an important operator L acting on smooth functions, which was first introduced and used by Colding and Minicozzi[15]. This operator is defined as follows:

$$L = \Delta - \langle x, \nabla \rangle = e^{\frac{|x|^2}{2}} \mathrm{div}(e^{-\frac{|x^2|}{2}} \nabla \cdot) \qquad (17)$$

where Δ, ∇ and div denote the Laplacian, gradient and divergence on M^m, accordingly.

Given an orthonormal tangent frame field $\{e_i; 1 \leqslant i \leqslant m\}$ on M^m with the dual $\{\omega^i\}$ and a normal frame field $\{e_\alpha; m+1 \leqslant \alpha \leqslant m+p\}$, write $h = \sum_{\alpha,i,j} h_{ij}^\alpha w^i w^j e_\alpha$. Then the $H = \sum_\alpha H^\alpha e_\alpha$ with $H^\alpha = \sum_i h_{ii}^\alpha$.

Lemma 2.2[1] Let $x: M^m \to R^{m+p}$ be a self-shrinker. Then

$$H_{,i}^\alpha = \sum_j h_{ij}^\alpha \langle x, e_j \rangle \qquad (18)$$

We end this section with another lemma which is essential to our argument.

Lemma 2.3[17] Let $x: M^m \to \mathbf{R}^{m+p}$ be a complete immersed submanifold. If u and v are $\mathbf{C}^2$-smooth functions with

$$\int_M (|u \nabla v| + |\nabla u \nabla v| + |uLv|) e^{-\frac{|x|^2}{2}} \mathrm{d}V_M < \infty,$$

then it holds that

$$\int_M u\, Lv e^{-\frac{|x|^2}{2}} \mathrm{d}V_M = -\int_M \langle \nabla u, \nabla v \rangle e^{-\frac{|x|^2}{2}} \mathrm{d}V_M.$$

3 Examples

In this section, we provide two examples of surfaces in $\mathbf{C}^2$ of constant Kähler angle, which are of course surfaces in $\mathbf{C}^3$ with constant Kähler angles.

Example 3.1 The Clifford torus

$$T^2:=S^1(1)\times S^1(1)\subset \mathbf{C}\times\mathbf{C}=\mathbf{C}^2\subset \mathbf{C}^3.$$

Clearly, T^2 is flat, Lagrangian in $\mathbf{C}^2$ and totally real in $\mathbf{C}^3$.

Example 3.2 [13] 2-planes in $\mathbf{C}^2$ with constant Kähler angles.

For any given two real constants θ_1 and θ_2, denote $\theta:=\theta_1+\theta_2$. Let $x:R^2\to\mathbf{C}^3$ be defined by $(z_1,z_2,z_3)=x(u_1,u_2)$ with

$$z_1=u_1\cos\theta_1+\sqrt{-1}u_2\cos\theta_2,$$

$$z_2=-u_2\sin\theta_2-\sqrt{-1}u_1\sin\theta_1,$$

$$z_3\equiv 0.$$

For $(u_1,u_2)\in\mathbf{R}^2$. Choose $\boldsymbol{e}_1=\dfrac{\partial}{\partial u_1}$ and $\boldsymbol{e}_2=\dfrac{\partial}{\partial u_2}$. Then

$$x_*\boldsymbol{e}_1=(\cos\theta_1,0,0,-\sin\theta_1),$$
$$x_*\boldsymbol{e}_2=(0,-\sin\theta_2,\cos\theta_2,0) \tag{19}$$

Thus

$$\boldsymbol{J}(x_*\boldsymbol{e}_1)=(0,\sin\theta_1,\cos\theta_1,0),$$
$$\boldsymbol{J}(x_*\boldsymbol{e}_2)=(-\cos\theta_2,0,0,-\sin\theta_2) \tag{20}$$

It follows that

$$\langle \boldsymbol{J}(x_*\boldsymbol{e}_1),x_*\boldsymbol{e}_2\rangle=\cos(\theta_1+\theta_2)=\cos\theta,$$

implying that x is of constant Kähler angle θ.

4 Proof of the main theorems

Let $x:M^2\to\mathbf{C}^3$ be a self-shrinker with Kähler angle θ. Suppose that $\sin\theta\neq 0$. Then, by virtue of Remark 5, there are two well-defined sub-normal bundies $\boldsymbol{N}_0$ and $\boldsymbol{N}$, such that $V:=x_*(TM)\oplus\boldsymbol{N}_0$ and $\boldsymbol{N}$ are both $\boldsymbol{J}$-inviant, and the decomposition (9) holds. Thus, to prove Theorem 1.6 and Theorem 1.7, we only need to prove the rigidity part of the conclusions. For this, the following lemma is needed:

Lemma 4.1 Let $K_{\boldsymbol{N}_0}$ be the Gaussian curvature of the sub-normal bundle $K_{\boldsymbol{N}_0}$. If $\boldsymbol{N}$ is flat, then it holds that

$$L\cos\theta=-\cos\theta(|h|^2-2K_{\boldsymbol{N}_0}) \tag{21}$$

$$\frac{1}{2}L\cos^2\theta=\sin^2\theta|\nabla\theta|^2-\cos^2\theta(|h|^2-2K_{\boldsymbol{N}_0}) \tag{22}$$

Proof As done in the Section 2, we choose an oriented orthonormal frame field $\{e_1,e_2\}$ on M^2 which, via (6) and (8), makes it possible for us to find a normal frame field $\{e_3,e_4,e_5,e_6\}$. Then by (10) and (11)

$$\nabla\theta=\sum_i(h^4_{1i}-h^3_{2i})e_i,\Gamma^2_{1i}-\Gamma^4_{3i}=\cot\theta(h^3_{1i}+h^4_{2i})\text{for } i=1,2 \tag{23}$$

From (15),(16) and the Codazzi equation, it follows that

$$\begin{aligned}\Delta\theta=&\sum_i(h^4_{1i}-h^3_{2i})_{,i}=\\&\sum_i(e_i(h^4_{1i}-h^3_{2i})-(h^4_{1j}-h^3_{2j})\Gamma^j_{ii})=\\&\sum_i(h^4_{1ii}-h^3_{2ii}+h^4_{2i}\Gamma^2_{1i}-h^3_{1i}\Gamma^1_{2i}-h^3_{1i}\Gamma^4_{3i}+\\&h^4_{2i}\Gamma^3_{4i})+\sum_i(h^5_{1i}\Gamma^5_{4i}+h^6_{1i}\Gamma^6_{4i}-h^5_{2i}\Gamma^5_{3i}-\\&h^6_{2i}\Gamma^6_{3i})=\sum_i(H^4_{,1}-H_{,2}^3+(h^4_{2i}+h^3_{1i})\Gamma^2_{1i}-\\&(h^4_{2i}+h^3_{1i})\Gamma^4_{3i}+h^5_{1i}(\frac{1}{\sin\theta}h^6_{2i}+\cot\theta h^5_{1i})+\\&h^6_{1i}(-\frac{1}{\sin\theta}h^5_{2i}+\cot\theta h^6_{1i})-\\&h^5_{2i}(\frac{1}{\sin\theta}h^6_{1i}-\cot\theta h^5_{2i})-\\&h^6_{2i}(-\frac{1}{\sin\theta}h^5_{1i}-\cot\theta h^6_{2i}))=\\&\sum_i(H_{,1}^4-H_{,2}^3+\cot\theta((h^4_{2i}+h^3_{1i})^2+\\&(h^5_{1i})^2+(h^6_{1i})^2+(h^5_{2i})^2+(h^6_{2i})^2)+\\&\frac{2}{\sin\theta}(h^5_{1i}h^6_{2i}-h^6_{1i}h^5_{2i}))\end{aligned} \tag{24}$$

By the Ricci equation and the flatness assumption of $\boldsymbol{N}$,

$$0=K_N\equiv R^{\perp}_{5621}=h^5_{1i}h^6_{2i}-h^6_{1i}h^5_{2i}.$$

Therefore

$$\begin{aligned}\Delta\theta=&\sum_i(H_{,1}^4-H^3_{,2}+\cot\theta((h^4_{2i}+h^3_{1i})^2+\\&(h^5_{1i})^2+(h^6_{2i})^2+(h^5_{2i})^2+(h^6_{2i})^2))\end{aligned} \tag{25}$$

By using this and (23), we find

$$\Delta\cos\theta=-\cos\theta|\nabla\theta|^2-\sin\theta\Delta\theta=$$

$$-\cos\theta\sum_i(h_{1i}^4-h_{2i}^3)^2-\sin\theta(H,_1^4-H,_2^3)-$$
$$\cos\theta\sum_i((h_{2i}^4+h_{1i}^3)^2+(h_{1i}^5)^2+(h_{2i}^6)^2+$$
$$(h_{2i}^5)^2+(h_{2i}^6)^2)=-\cos\theta\sum_i((h_{1i}^4-h_{2i}^3)^2+$$
$$(h_{2i}^4+h_{1i}^3)^2+(h_{1i}^5)^2+(h_{1i}^6)^2+(h_{2i}^5)+$$
$$(h_{2i}^6)^2)-\sin\theta(H,_1^4-H,_2^3)=$$
$$-\cos\theta(|h|^2-2R_{3421}^{\perp})-$$
$$\sin\theta(H,_1^4-H,_2^3)=-\cos\theta(|h|^2-2K_{N_0})-$$
$$\sin\theta(H,_1^4-H,_2^3) \tag{26}$$

From this, we also find

$$\frac{1}{2}\Delta(\cos^2\theta)=|\nabla\cos\theta|^2+$$
$$\cos\theta(\Delta\cos\theta)=\sin^2\theta|\nabla\theta|^2-$$
$$\cos^2\theta(|h|^2-2K_{N_0})-$$
$$\cos\theta\sin\theta(H,_1^4-H,_2^3) \tag{27}$$

By Lemma 2.2, we have

$$H,_1^4-H,_2^3=\sum_i h_{1i}^4\langle x,e_i\rangle-$$
$$\sum_i h_{2i}^3\langle x,e_i\rangle=\sum_i(h_{1i}^4-h_{2i}^3)\langle x,e_i\rangle \tag{28}$$

Inserting (28) into (26) and (27), respectively, we obtain

$$\Delta\cos\theta=-\cos\theta(|h|^2-2K_{N_0})-$$
$$\sin\theta\sum_i(h_{1i}^4-h_{2i}^3)\langle x,e_i\rangle=$$
$$-\cos\theta(|h|^2-2K_{N_0})-$$
$$\langle x,\sin\theta\sum_i(h_{1i}^4-h_{2i}^3)e_i\rangle=$$
$$-\cos\theta(|h|^2-2K_{N_0})+\langle x,\nabla\cos\theta\rangle \tag{29}$$

and

$$\frac{1}{2}\Delta\cos^2\theta=|\Delta\cos\theta|^2+\cos\theta(\Delta\cos\theta)=$$
$$\sin^2\theta|\nabla\theta|^2-\cos^2\theta(|h|^2-2K_{N_0})+$$
$$\frac{1}{2}\cos\theta\langle x,\nabla\cos\theta\rangle=\sin^2\theta|\nabla\theta|^2-$$
$$\cos^2\theta(|h|^2-2K_{N_0})+\frac{1}{2}\langle x,\nabla\cos^2\theta\rangle \tag{30}$$

By the definition of L, (29) and (30) become, respectively,

$$L\cos\theta=\Delta\cos\theta-\langle x,\nabla\cos\theta\rangle=$$
$$-\cos\theta(|h|^2-2K_{N_0})$$
$$\frac{1}{2}L\cos^2\theta=\frac{1}{2}\Delta\cos^2\theta-\frac{1}{2}\langle x,\nabla\cos^2\theta\rangle=$$
$$\sin^2\theta|\nabla\theta|^2-\cos^2\theta(|h|^2-2K_{N_0}).$$

Thus Lemma 4.1 is proved.

The proof of the rigidity part of Theorem 1.6 Take $\upsilon=\cos\theta$. Then

$$|\nabla\upsilon|^2=\sin^2\theta|\nabla\theta|^2=$$
$$\sin^2\theta\sum_i|h_{1i}^4-h_{2i}^3|^2\leqslant$$
$$2\sin^2\theta\sum_i((h_{1i}^4)^2+(h_{2i}^3)^2)\leqslant 2|h|^2 \tag{31}$$

$$|L\upsilon|=(|L\cos\theta|=$$
$$|\cos\theta(|h|^2-2K_{N_0})|\leqslant 4|h|^2 \tag{32}$$

Since $\int_M|h|^2 e^{-\frac{|x|^2}{2}}dV_M<\infty$, (31) and (32) show that, for $u\equiv 1$,

$$\int_M(|u\nabla\upsilon|+|\nabla u\nabla\upsilon|+|uL\upsilon|)e^{-\frac{|x|^2}{2}}dV_M<\infty.$$

Thus by Lemma 2.4, we obtain

$$\int_M L\cos\theta e^{-\frac{|x|^2}{2}}dV_M=0 \tag{33}$$

Since, by the assumption, $\cos\theta\geq 0$, it follows from (21) and (33) that

$$0=\int_M L\cos\theta e^{-\frac{|x|^2}{2}}dV_M=-\int_M\cos\theta(|h|^2-$$
$$2K_{N_0})e^{-\frac{|x|^2}{2}}dV_M\leq 0 \tag{34}$$

where we have used the inequality (4). Consequently, we obtain

$$\cos\theta(|h|^2-2K_{N_0})\equiv 0 \tag{35}$$

Consider the function $\varphi=|\nabla\theta|^2$. If φ is not identically zero on M^2, then there exists a point p_0 where $\varphi\neq 0$. So by the continuity, there is a connected domain U containing p_0 such that $\varphi>0$ at every point of U. Denote

$$U_0=\{p\in U\mid\cos\theta(p)=0\}.$$

The U_0 obviously contains no interior points. Therefore $U\backslash U_0$ is dense in U_0 In particular, $U\backslash U_0\neq\varnothing$. By (35), $|h|^2-2K_{N_0}\equiv 0$ in $U\backslash U_0$. So by (4)

$$h_{1i}^4=h_{2i}^3,\ h_{2i}^4=-h_{1i}^3,\ h_{1i}^5=h_{2i}^5=$$
$$h_{1i}^6=h_{2i}^6=0,\ i=1,2$$

hold identically on $U\backslash U_0$. Thus, by (14), we have $\nabla\theta\equiv 0$ on $U\backslash U_0$ which contradicts the definition of U. This shows that $\varphi\equiv 0$ on M^2, that is, θ is constant.

If $\cos\theta=0$, then x is totally real.

If $\cos\theta\neq 0$, then $|h|^2-2K_{\mathbf{N}_0}\equiv 0$. It follows that

$$h_{11}^4=h_{21}^3=-h_{22}^4,\ h_{22}^3=h_{21}^4=-h_{11}^3,$$
$$h_{1i}^5=h_{2i}^5=h_{1i}^6=h_{2i}^6=0,\ i=1,2$$

implying that $H\equiv 0$. Since x is a self-shrinker, we obtain

$$\langle x,e_3\rangle=\langle x,e_4\rangle=\langle x,e_5\rangle=\langle x,e_6\rangle=0 \tag{36}$$

Hence, for $\alpha=3,4,5,6$,

$$\langle x,\bar{D}_{e_i}e_\alpha\rangle=e_i\langle x,e_\alpha\rangle-\langle\bar{D}_{e_i}x,e_\alpha\rangle=$$
$$0-\langle e_i,e_\alpha\rangle=0,$$

namely

$$\langle x,-A_{e^\alpha}e_i+\sum\Gamma_{\alpha i}^\beta e_\beta\rangle=0.$$

So that

$$\sum_j h_{ij}^\alpha\langle x,e_j\rangle=0,\ \alpha=3,4 \tag{37}$$

From (37), it is easily seen that

$$\begin{vmatrix}h_{11}^3 & h_{21}^3\\ h_{12}^3 & h_{22}^3\end{vmatrix}=\begin{vmatrix}h_{11}^4 & h_{21}^4\\ h_{12}^4 & h_{22}^4\end{vmatrix}=0,$$

that is,

$$h_{11}^3h_{22}^3-(h_{21}^3)^2=h_{11}^4h_{22}^4-(h_{21}^4)^2=0,$$

which with $H=0$ shows that

$$h_{11}^3=h_{21}^3=h_{11}^4=h_{21}^4=0.$$

We already know that $h_{1i}^5=h_{2i}^5=h_{1i}^6=h_{2i}^6=0$ for $i=1,2$. So M^2 is totally geodesic and, by the completeness, M^2 must be a plane, Finally, by (15) and (16), $\mathbf{N}$ is parallel in $\mathbf{C}^3$, This with the fact that $\mathbf{N}$ is $\boldsymbol{J}$-invariant means that, up to a holomorphic isometry of $\mathbf{C}^2$, $x(M^2)$ is contained in $\mathbf{C}^2$.

The proof of the rigidity part of Theorem 1.7

Similar to those in (1), we take $u\equiv 1$ and $\upsilon=\frac{1}{2}\cos^2\theta$. Since

$$|\nabla\upsilon|^2=\cos^2\theta\sin^2\theta|\nabla\theta|^2=$$
$$\frac{1}{4}\sin^2 2\theta\sum_i|h_{1i}^4-h_{2i}^3|^2\leq$$
$$\frac{1}{2}\sin^2 2\theta\sum_i((h_{1i}^4)+(h_{2i}^3)^2)\leq\frac{1}{2}|h|^2 \tag{38}$$

$$|L\upsilon|=\frac{1}{2}|L\cos^2\theta|=\sin^2\theta|\nabla\theta|^2-$$
$$\cos^2\theta(|h|^2-2K_{N_0})\leq 6|h|^2 \tag{39}$$

we can use Lemma 2.3 to get

$$\frac{1}{2}\int_M(L\cos^2\theta)e^{-\frac{|x|^2}{2}}dV_M=0 \tag{40}$$

On the other hand, by (3) and (22), we find

$$\frac{1}{2}L\cos^2\theta=\sin^2\theta|\nabla\theta|^2-$$
$$\cos^2\theta(|h|^2-2K_{\mathbf{N}_0})\leq$$
$$\sin^2\theta\frac{\lambda\cos^2\theta(|h|^2-2K_{\mathbf{N}_0})}{1-\lambda\cos^2\theta}-$$
$$\cos^2\theta(|h|^2-2K_{\mathbf{N}_0})=$$
$$-\frac{(1-\lambda)\cos^2\theta(|h|^2-2K_{\mathbf{N}_0})}{1-\lambda\cos^2\theta}\leq 0.$$

Take the integration of the above inequality, we find easily that

$$\cos^2\theta(|h|^2-2K_{\mathbf{N}_0})=0.$$

By using an argument similar to that in (1), we can conclude that either $\cos\theta\equiv 0$ or $|h|^2-2K_{\mathbf{N}_0}\equiv 0$. Then the rest of the proof is omitted here since it is the same as that of the proof of Theorem 1.6.

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