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一类带 logistic 源项的趋化方程组解的整体存在性和有界性

林静秋，何　璞[2]，侯智博[3]
(西华大学理学院，成都 610039)

摘　要：本文研究了一类具有 logistic 源项的趋化方程组解的性质.利用先验估计并结合 Neumann 热半群的衰减性质，本文证明：当 logistic 源项中的二次项系数足够大时，方程组的齐次 Neumann 初边值问题的经典解在边界光滑的三维有界区域上整体存在且一致有界.

关键词：趋化方程组；Logistic 源；整体存在；一致有界
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Global existence and boundedness of solutions of a chemotaxis system with logistic source

LIN Jing-Qiu[1], *HE Pu*[2], *HOU Zhi-Bo*[3]
(School of Science, Xihua University, Chengdu 610039, China)

Abstract: The properties of solutions of a class of chemotaxis systems with logistic source are considered. By using prior estimates and decay properties of Neumann heat semi-group, it is proved that there exists a unique global classical solution for the homogenous Neumann initial value problem in three-dimensional bounded domain with smooth boundary if the quadratic coefficients of the logistic source is sufficiently large.

Keywords: Chemotaxis system; Logistic source; Global existence; Boundedness
(2010 MSC 35K55, 35Q92, 92C17)

1 Introduction

In past decades, the nonlinear parabolic system have been widely studied[1-6]. Particularly, in 1970, Keller and Segel proposed the following chemotaxis Keller-Segel model [5]:

$$\begin{cases} u_t = \nabla \cdot (D_u(u,v)\nabla u - \chi u \nabla v) + \\ \quad H(u,v), x\in\Omega, t>0, \\ v_t = D_v(u,v)\Delta v + K(u,v), x\in\Omega, t>0, \end{cases}$$

where u is the cell density, v is the density of the chemoattrctant, $H(u,v)$ and $K(u,v)$ are model source terms related to interactions, $D_u(u,v)$ and $D_v(u,v)$ are the diffusivity of the cells and chemoattactant, respectively. When $D_u(u,v) = D_v(u,v) = 1, \chi = 1$ and $H(u,v)=0, K(u,v) = -v+u$, the model recovers the classical minimal model[6]:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), x\in\Omega, t>0, \\ v_t = \Delta v - v + u, x\in\Omega, t>0. \end{cases}$$

The solution of the Neumann boundary value

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作者简介：林静秋(1994－)，女，四川泸州人，硕士研究生，主要研究方向为偏微分方程.E-mail：18215617187@163.com
通讯作者：侯智博.E-mail：hzb_303@126.com

problem of this system in bounded domain $\Omega\subset\mathbf{R}^N$ will blow up when $N\geq 3$ or $N=2$ and $\int_\Omega u_0$ is large[7]. When $N=1$, Osaki and Yagi[8] established the existence of global bounded classical solutions for any sufficiently smooth initial value. When $N=2$, Nagai[9] proved the solution is bounded if $\int_\Omega u_0 < 4\pi$. When $N\geq 3$, Winkler[10] obtained the same conclusion if $\|u_0\|_{L^{\frac{N}{2}+\delta}(\Omega)}$, $\|\nabla v_0\|_{L^{N+\delta}(\Omega)}<\varepsilon$.

In view of various biological phenomena and environment for cells, many variants of Keller-Segel model have been developed and investigated (see Refs. [6, 11] and references therein). Among them, some recent works qualitatively study the effects of interplay between self-diffusion and cross-diffusion[12,13], between self-diffusion and logistic damping[14], or between nonlinear signal production and logistic growth[15]. In order to address the dependence of dynamical behaviors of solutions on the interactions between nonlinear cross-diffusion and logistic source, the following model:

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\nabla v)+f(u), x\in\Omega, t>0, \\ \tau v_t=\Delta v-v+u, x\in\Omega, t>0 \end{cases}$$

are exentisively considered. When $N\geq 2$, solutions with the logistic term $f(u)=0$ may blow up in finite time[16,17]. If $\tau=1, N=2$ and $f(u)=\gamma u-\mu u^2$, where $\mu>0$ is arbitrarily small, all of solutions are global and bounded[8]. In the case $N\leqslant 2$, even for arbitrarily small $\mu>0$ are sufficient to rule out any explosion by guaranteeing global existence of bounded classical solutions for all reasonably smooth initial data[18]. Whereas in the case $N\geq 3$, the same conclusion holds provided that $\mu>0$ is suitably large[19]. Note that the additional logistic term destroys the energy structure of corresponding free Keller-Segel system obtained in the limit case $\gamma=\mu=0$ [9] apparently.

Another common type is to consider $\tau=0$ that reflects and takes to a limit the physically reasonable model assumption that chemicals diffuse much faster than cells move, we can accordingly obtained initial-boundary value problem for the parabolic-elliptic system. The solutions are global and bounded whenever $\mu>0$ satisfies $\mu>\frac{N-2}{N}$, while for any $\mu>0$ one can at least construct globally existing weak solutions. The existence of weak solutions and a bounded absorbing set in $L^\infty(\Omega)$ are proved under more general conditions. In Ref. [20], it is shown that in another related model:

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\nabla v)+\lambda u-\mu u^\alpha, x\in\Omega, t>0, \\ 0=\Delta v-m(t)+u,\ x\in\Omega, t>0, \end{cases}$$

the blow-up may occur for space-dimension $N\geq 5$ and $1<\alpha<\frac{3}{2}+\frac{1}{2N-2}$.

In short, the logistic source exerts a certain growth-inhibiting influence which may keep the solution bounded and rule out blow-up. Currently, most scholars have studied $K(u,v)=-v+u$ in Keller-Segel model variants, and relatively few in terms of $K(u,v)=-uv$. Therefore, in this paper, we assume that $K(u,v)=-uv$. Particularily, we consider the following parabolic-parabolic chemotaxis-growth system with cross diffusion and consumption terms:

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\nabla v)+ru-\mu u^2, \\ \quad x\in\Omega, t>0, \\ v_t=\Delta v-uv,\ x\in\Omega, t>0, \\ \nabla u\cdot n=\nabla v\cdot n=0, x\in\partial\Omega, t>0, \\ u(x,0)=u_0(x), v(x,0)=v_0(x), x\in\Omega \end{cases} \tag{1}$$

where $\Omega\subset\mathbf{R}^3$ is a bounded domain with smooth boundary, n denotes the outward normal vector field on $\partial\Omega$, $r\in\mathbf{R}$, $\mu>0$, u and v represent the density of cells and the concentration of chemical substance. In order to specify the framework for our analysis, let us assume throughout the paper that the initial data satisfy

$$\begin{cases} u_0\in C^0(\overline{\Omega}), u_0\geq 0 \text{ and } u_0\equiv 0 \text{ in } \overline{\Omega}, \\ v_0\in W^{1,\infty}(\Omega), v_0\geq 0 \text{ and } v_0\equiv 0 \text{ in } \overline{\Omega} \end{cases} \tag{2}$$

The goal of this paper is to build the existence of global bounded classical solutions for suitably large μ under the influence of logistic term in three dimensional convex bounded domain. The main theorem of this paper can be stated as follows:

Theorem 1.1　Suppose that (2) holds. Then whenever $\mu \geq 3+\frac{23}{3}\|v_0\|^2_{L^\infty(\Omega)}$, System (1) possesses a unique global classical solution (u,v) which is uniformly bounded in the sense that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\leqslant C \text{ for all } t\in(0,\infty) \tag{3}$$

with some positive constant C.

2　Global existence and some preliminaries

We first state the local solvability of System (1), which can by proved by a straightforward adaptation of the corresponding procedure in Lemma 3.1 of Ref. [6] to our current setting.

Lemma 2.1　Suppose that (2) holds. Then there exist $T_{\max}\in(0,\infty]$ and unique classical solution (u,v) of System (1) in $\Omega\times(0,T_{\max})$ such that

$$u\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times[0,T_{\max})),$$
$$v\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times[0,T_{\max})).$$

Moreover, we have $u\geq 0$ and $v\geq 0$ in $\overline{\Omega}\times[0,T_{\max})$, and if $T_{\max}<\infty$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\to\infty \text{ as } t\to T_{\max}.$$

The following lemma is easily obtained but will be frequently used in the sequel.

Lemma 2.2　If (2) holds, then the solution of (1) satisfies

$$\int_\Omega u(x,t)\mathrm{d}x\leqslant \max\left\{\int_\Omega u_0,\frac{r_+}{\mu}|\Omega|\right\}=:m \tag{4}$$

for all $t\in(0,T_{\max})$.

Proof　The conclusion directly results from an integration of the first equation in (1) over Ω.

As the consequence of the maximum principle and the nonnegativity of the solution, we have the following result.

Lemma 2.3　If (2) holds, then the solution of (1) satisfies

$$\|v(\cdot,t)\|_{L^\infty(\Omega)}\leqslant\|v_0\|_{L^\infty(\Omega)} \tag{5}$$

for all $t\in(0,T_{\max})$.

Lemma 2.4　There exists a positive constant M such that

$$\int_\Omega|\nabla v|^2+\frac{\|v_0\|^2_{L^\infty(\Omega)}}{\mu}\int_\Omega u\leqslant M \tag{6}$$

for all $t\in(0,T_{\max})$.

Proof　Integration by parts and the Young inequality results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^2&=2\int_\Omega\nabla v\cdot\nabla(\Delta v-uv)\leqslant\\&-2\int_\Omega|\Delta v|^2-2\int_\Omega|\nabla v|^2+\\&2\int_\Omega v(u-1)\Delta v\leqslant\\&-\int_\Omega|\Delta v|^2-2\int_\Omega|\nabla v|^2+\\&\int_\Omega v^2(u-1)^2\leqslant\\&-\int_\Omega|\Delta v|^2-2\int_\Omega|\nabla v|^2+\\&\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u^2+2\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u+\\&\|v_0\|^2_{L^\infty(\Omega)}\end{aligned} \tag{7}$$

On the other hand,

$$\frac{\|v_0\|^2_{L^\infty(\Omega)}}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\leqslant\frac{r_+\|v_0\|^2_{L^\infty(\Omega)}}{\mu}\int_\Omega u-\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u^2 \tag{8}$$

To sum up (7), (8), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega|\nabla v|^2+\frac{\|v_0\|^2_{L^\infty(\Omega)}}{\mu}\int_\Omega u\right\}&\leqslant\\-\left(\int_\Omega|\nabla v|^2+\frac{\|v_0\|^2_{L^\infty(\Omega)}}{\mu}\int_\Omega u\right)&+\\\left(2\|v_0\|^2_{L^\infty(\Omega)}+\frac{r_++1}{\mu}\|v_0\|^2_{L^\infty(\Omega)}\right)\int_\Omega u&+\\\|v_0\|^2_{L^\infty(\Omega)}.&\end{aligned}$$

Since Lemma 2.2 shows that $\int_\Omega u(x,t)\mathrm{d}x\leqslant\max\left\{\int_\Omega u_0,\frac{r_+}{\mu}|\Omega|\right\}$ for all $t\in(0,T_{\max})$, a comparison argument leads to

$$\begin{aligned}\int_\Omega|\nabla v|^2+\frac{\|v_0\|^2_{L^\infty(\Omega)}}{\mu}\int_\Omega u\leqslant&\\\max\Bigg\{|\nabla v_0|^2+\frac{\|v_0\|^2_{L^\infty(\Omega)}}{\mu}\int_\Omega u_0,\ \|v_0\|^2_{L^\infty(\Omega)}&+\\2\|v_0\|^2_{L^\infty(\Omega)}+\frac{r_++1}{\mu}\|v_0\|^2_{L^\infty(\Omega)},&\\\max\left\{\int_\Omega u_0,\frac{r_+}{\mu}|\Omega|\right\}\Bigg\}.&\end{aligned}$$

3　Boundedness of u,v

This section contains the main step of our analysis by establishing an estimate for a combination of $\int_\Omega u^2$, $\int_\Omega |\nabla v|^4$ and $\int_\Omega u|\nabla v|^2$.

Let us first derive the following differential inequality for $\int_\Omega u^2$.

Lemma 3.1

$$\frac{d}{dt}\int_\Omega u^2+\int_\Omega|\nabla u|^2\leqslant\int_\Omega u^2|\nabla v|^2+2r\int_\Omega u^2-2\mu\int_\Omega u^3 \tag{9}$$

for all $t\in(0,T_{\max})$.

Proof Testing the first equation in (1) against u, we can obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2=-\int_\Omega|\nabla u|^2+\int_\Omega u\nabla u\cdot\nabla v+r\int_\Omega u^2-\mu\int_\Omega u^3$$

for all $t\in(0,T_{\max})$, which directly results in (9) by using the Young inequality to yield

$$\int_\Omega u\nabla u\cdot\nabla v\leqslant\frac{1}{2}\int_\Omega|\nabla u|^2+\frac{1}{2}\int_\Omega u^2|\nabla v|^2 \text{ for all } t\in(0,T_{\max}).$$

Lemma 3.2

$$\frac{d}{dt}\int_\Omega|\nabla v|^4+\int_\Omega|\nabla|\nabla v|^2|^2\leqslant 7\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u^2|\nabla v|^2+2\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial n} \tag{10}$$

for all $t\in(0,T_{\max})$.

Proof Using the facts

$$\nabla v\cdot\nabla\Delta v=\frac{1}{2}\Delta|\nabla v|^2-|D^2v|^2$$

and $|\Delta v|\leqslant\sqrt{3}|D^2v|$, we can derive that

$$\frac{1}{4}\frac{d}{dt}\int_\Omega|\nabla v|^4=\int_\Omega|\nabla v|^2\nabla v\cdot\nabla(\Delta v-uv)=$$
$$\frac{1}{2}\int_\Omega|\nabla v|^2\Delta|\nabla v|^2-\int_\Omega|\nabla v|^2|D^2v|^2+\int_\Omega uv|\nabla v|^2\Delta v+\int_\Omega uv\nabla v\cdot\nabla|\nabla v|^2\leqslant$$
$$\frac{1}{2}\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial n}-\frac{1}{2}\int_\Omega\left|\nabla|\nabla v|^2\right|^2-\int_\Omega|\nabla v|^2|D^2v|^2+$$
$$\sqrt{3}\|v_0\|_{L^\infty(\Omega)}\int_\Omega u|\nabla v|^2|D^2v|+\|v_0\|_{L^\infty(\Omega)}\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2 \tag{11}$$

Then an application of Young′s inequality to the last two integralyields

$$\sqrt{3}\|v_0\|_{L^\infty(\Omega)}\int_\Omega u|\nabla v|^2|D^2v|+\|v_0\|_{L^\infty(\Omega)}\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2\leqslant$$
$$\int_\Omega|\nabla v|^2|D^2v|^2+\frac{3}{4}\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u^2|\nabla v|^2+\frac{1}{4}\int_\Omega|\nabla|\nabla v|^2|^2+\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u^2|\nabla v|^2.$$

Inserting this inequality into (11) and rearranging it, we arrive at our conclusion.

Lemma 3.3

$$\frac{d}{dt}\int_\Omega u|\nabla v|^2+\left(\mu+1-\frac{2}{3}\|v_0\|^2_{L^\infty(\Omega)}\right)\int_\Omega u^2|\nabla v|^2\leqslant 3\int_\Omega|\nabla u|^2+\frac{11}{12}\int_\Omega\left|\nabla|\nabla v|^2\right|^2+r\int_\Omega u|\nabla v|^2+\int_{\partial\Omega}u\frac{\partial|\nabla v|^2}{\partial n} \tag{12}$$

for all $t\in(0,T_{\max})$.

Proof A direct calculation shows that

$$\frac{d}{dt}\int_\Omega u|\nabla v|^2=\int_\Omega|\nabla v|^2\{\Delta u-\nabla\cdot(u\nabla v)+ru-\mu u^2\}+2\int_\Omega u\nabla v\cdot\nabla(\Delta v-uv) \tag{13}$$

We can thereupon derive from integrating by parts and employing the identity

$$\nabla v\cdot\nabla\Delta v=\frac{1}{2}\Delta|\nabla v|^2-|D^2v|^2$$

that

$$\frac{d}{dt}\int_\Omega u|\nabla v|^2=-\int_\Omega\nabla u\cdot\nabla|\nabla v|^2+\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2+r\int_\Omega u|\nabla v|^2-\mu\int_\Omega u^2|\nabla v|^2+\int_\Omega u\Delta|\nabla v|^2-2\int_\Omega u|D^2v|^2-$$

$$2\int_\Omega u^2|\nabla v|^2-2\int_\Omega uv\nabla u\cdot\nabla v=$$
$$-2\int_\Omega\nabla u\cdot\nabla|\nabla v|^2+\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2+$$
$$r\int_\Omega u|\nabla v|^2-(\mu+2)\int_\Omega u^2|\nabla v|^2+$$
$$\int_{\partial\Omega}u\frac{\partial|\nabla v|^2}{\partial n}-2\int_\Omega u|D^2v|^2-$$
$$2\int_\Omega uv\nabla u\cdot\nabla v \tag{14}$$

for all $t\in(0,T_{\max})$. Then one can estimate by the Young inequality that

$$-2\int_\Omega\nabla u\cdot\nabla|\nabla v|^2\leqslant$$
$$\frac{3}{2}\int_\Omega|\nabla u|^2+\frac{2}{3}\int_\Omega|\nabla|\nabla v|^2|^2 \tag{15}$$
$$\int_\Omega u\nabla v\cdot\nabla|\nabla v|^2\leqslant$$
$$\int_\Omega u^2|\nabla v|^2+\frac{1}{4}\int_\Omega|\nabla|\nabla v|^2|^2 \tag{16}$$

and

$$-2\int_\Omega uv\nabla u\cdot\nabla v\leqslant\frac{3}{2}\int_\Omega|\nabla u|^2+$$
$$\frac{2}{3}\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u^2|\nabla v|^2 \tag{17}$$

for all $t\in(0,T_{\max})$. In view of (15)~(17), the identity (14) readily implies (12).

In view of Lemma 3.1~3.3, we can easily see the following result.

Lemma 3.4

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{4\int_\Omega u^2+\int_\Omega|\nabla v|^4+\int_\Omega u|\nabla v|^2\right\}+$$
$$\int_\Omega|\nabla u|^2+\frac{1}{12}\int_\Omega|\nabla|\nabla v|^2|^2+8\mu\int_\Omega u^3+$$
$$\left(\mu-3-\frac{23}{3}\|v_0\|^2_{L^\infty(\Omega)}\right)\int_\Omega u^2|\nabla v|^2\leqslant$$
$$8r\int_\Omega u^2+r\int_\Omega u|\nabla v|^2+$$
$$2\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial n}+\int_{\partial\Omega}u\frac{\partial|\nabla v|^2}{\partial n} \tag{18}$$

for all $t\in(0,T_{\max})$.

Next we will show that if μ is suitably large, then all integrals on the right side in (18) can adequately be estimated in terms of the respective dissipated quantities on the left, in consequence implying the L^2 estimate of u and the boundedness estimate for $|\nabla v|$.

Lemma 3.5　Suppose that

$$\mu\geq 3+\frac{23}{3}\|v_0\|^2_{L^\infty(\Omega)}.$$

Then there exists positive constant C such that

$$\int_\Omega u^2(\cdot,t)\leqslant C \tag{19}$$

and

$$\int_\Omega|\nabla v(\cdot,t)|^4\leqslant C \tag{20}$$

for all $t\in(0,T_{\max})$.

Proof　Let

$$y(t):=4\int_\Omega u^2+\int_\Omega|\nabla v|^4+\int_\Omega u|\nabla v|^2.$$

Since $\mu\geq 3+\frac{23}{3}\|v_0\|^2_{L^\infty(\Omega)}$, Lemma 3.4 implies that

$$y'(t)+y(t)+\int_\Omega|\nabla u|^2+$$
$$\frac{1}{12}\int_\Omega|\nabla|\nabla v|^2|^2+8\mu\int_\Omega u^3\leqslant$$
$$(8r+4)\int_\Omega u^2+(r+1)\int_\Omega u|\nabla v|^2+$$
$$\int_\Omega|\nabla v|^4+2\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial n}+$$
$$\int_{\partial\Omega}u\frac{\partial|\nabla v|^2}{\partial n} \tag{21}$$

Using the Young inequality, we can assert that for any $\delta>0$, there exists some $C_1>0$ such that the first three terms on the right hand side fulfilling

$$(8r+4)\int_\Omega u^2+(r+1)\int_\Omega u|\nabla v|^2+$$
$$\int_\Omega|\nabla v|^4\leqslant\left\{8r+4+\frac{(r+1)^2}{4}\right\}\int_\Omega u^2+$$
$$2\int_\Omega|\nabla v|^4\leqslant$$
$$8\mu\int_\Omega u^3+\delta\int_\Omega|\nabla v|^{\frac{16}{3}}+C_1 \tag{22}$$

for all $t\in(0,T_{\max})$. Recalling the boundedness of $\int_\Omega|\nabla v|^2$ asserted in Lemma 2.4, we can apply the Gagliardo-Nirenberg inequality to estimate

$$\int_\Omega|\nabla v|^{\frac{16}{3}}=\||\nabla v|^2\|^{\frac{8}{3}}_{L^{\frac{8}{3}}(\Omega)}\leqslant$$
$$C_2\|\nabla|\nabla v|^2\|^2_{L^2(\Omega)}\||\nabla v|^2\|^{\frac{2}{3}}_{L^1(\Omega)}+$$
$$C_2\||\nabla v|^2\|^{\frac{8}{3}}_{L^1(\Omega)}\leqslant$$
$$C_3\int_\Omega|\nabla|\nabla v|^2|^2+C_3$$

for all $t\in(0,T_{\max})$ and some positive constants

C_2, C_3. Taking $\delta = \frac{1}{24C_3}$ in (22), we arrive at

$$(8r+4)\int_\Omega u^2 + (r+1)\int_\Omega u\,|\nabla v|^2 + \int_\Omega |\nabla v|^4 \leqslant 8\mu\int_\Omega u^3 + \frac{1}{24}\int_\Omega \left|\nabla |\nabla v|^2\right|^2 + C_1 + \frac{1}{24} \tag{23}$$

The boundary trace embedding[21] $W^{\frac{1}{2},2}(\Omega)$ is continuously embedded into $L^2(\partial\Omega)$, which guarantees the existence of $C_4 > 0$ such that

$$\|\varphi\|_{L^2(\partial\Omega)} \leqslant c_4 \|\varphi\|_{W^{\frac{1}{2},2}(\Omega)}$$

for all $\varphi \in W^{\frac{1}{2},2}(\Omega)$.

On the other hand, since $W^{1,2}(\Omega)$ is compactly embedded into $W^{\frac{1}{2},2}(\Omega)$ and $W^{\frac{1}{2},2}(\Omega)$ is continuously embedded into L^1, then the Ehrling lemma entails that for any $\varepsilon > 0$ we can pick $C_5(\varepsilon) > 0$ such that

$$\int_{\partial\Omega}\varphi^2 \leqslant C_4^2 \|\varphi\|^2_{W^{\frac{1}{2},2}(\Omega)} \leqslant \varepsilon\int_\Omega |\nabla\varphi|^2 + C_5(\varepsilon)\left(\int_\Omega|\varphi|\right)^2$$

for all $\varphi \in W^{1,2}(\Omega)$. This estimate and the one-sided pointwise inequality[22]

$$\frac{\partial |\nabla v|^2}{\partial n} \leqslant C_6 |\nabla v|^2$$

for all $x \in \partial\Omega$ and $t \in (0, T_{\max})$ with some $C_6 > 0$ enables us to estimate the two rightmost summands in (21) as

$$\begin{aligned}
&2\int_{\partial\Omega}|\nabla v|^2\frac{\partial |\nabla v|^2}{\partial n} + \int_{\partial\Omega}u\frac{\partial|\nabla v|^2}{\partial n} \leqslant \\
&3C_6\int_{\partial\Omega}|\nabla v|^4 + \frac{C_6}{4}\int_{\partial\Omega}u^2 \leqslant \\
&\frac{1}{24}\int_\Omega \left|\nabla |\nabla v|^2\right|^2 + \\
&3C_5(\varepsilon)C_6\left(\int_\Omega|\nabla v|^2\right)^2 + \int_\Omega|\nabla u|^2 + \\
&\frac{C_5(\varepsilon)C_6}{4}\left(\int_\Omega|u|\right)^2 \leqslant \\
&\frac{1}{24}\int_\Omega\left|\nabla|\nabla v|^2\right|^2 + \int_\Omega|\nabla u|^2 + C_7
\end{aligned} \tag{24}$$

for all $t \in (0, T_{\max})$, where $C_7 = 3C_5(\varepsilon)C_6M^2 + \frac{C_5(\varepsilon)C_6m^2}{4}$ with M, m as in Lemma 2.2 and Lemma 2.4, respectively. Substituting (23), (24) into (21), we conclude that

$$y'(t) + y(t) \leqslant C_8 \text{ for all } t \in (0, T_{\max})$$

with $C_8 = C_1 + C_7 + \frac{1}{24}$, which immediately leads to

$$y(t) \equiv 4\int_\Omega u^2 + \int_\Omega|\nabla v|^4 + \int_\Omega u\,|\nabla v|^2 \leqslant C_9$$

for all $t \in (0, T_{\max})$ with some positive constant C_9 from the comparison argument.

With the boundedness of $\int_\Omega u^2$ and $\int_\Omega |\nabla v|^4$ at hand, we can derive the L^∞ estimate of u by using the variation-of-constants formula now.

Lemma 3.6 If $\mu \geq 3 + \frac{23}{3}\|v_0\|^2_{L^\infty(\Omega)}$ then there exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C \tag{25}$$

for all $t \in (0, T_{\max})$

Proof We first use the variation-of-constants formula to represent $u(\cdot, t)$ for each $t \in (0, T_{\max})$ as

$$\begin{aligned}
u(\cdot,t) = {} & e^{(t-t_0)\Delta}u(\cdot,t_0) - \int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot \\
& (u(\cdot,s)\nabla v(\cdot,s))ds + \\
& \int_{t_0}^t e^{(t-s)\Delta}(ru(\cdot,s) - \mu u^2(\cdot,s))ds
\end{aligned} \tag{26}$$

where $t_0 = (t-1)_+$. Then one can easily infer from the maximum principle that

$$\|e^{(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} \tag{27}$$

if $t \in (0, 1]$ and if $t > 1$ then the $L^p - L^q$ estimate for the Neumann heat semigroup yields $C_1 > 0$ such that

$$\begin{aligned}
&\|e^{(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty(\Omega)} \leqslant \\
&C_1(t-t_0)^{-\frac{3}{2}}\|u(\cdot,t_0)\|_{L^\infty(\Omega)} = \\
&C_2\|u(\cdot,t_0)\|_{L^\infty(\Omega)} \leqslant C_2 m
\end{aligned} \tag{28}$$

Noticing $r\xi - \mu\xi^2 \leqslant \frac{r^2}{4\mu}$ for all $\xi \in \mathbf{R}$, we can estimate

$$\begin{aligned}
&\int_{t_0}^t e^{(t-s)\Delta}(ru(\cdot,s) - \mu u^2(\cdot,s))ds \leqslant \\
&\int_{t_0}^t e^{(t-s)\Delta}\frac{r^2}{4\mu}ds = \frac{r^2}{4\mu}(t-t_0) \leqslant C_3
\end{aligned} \tag{29}$$

where $C_3 = \frac{r^2}{4\mu}$.

Finally, we will estimate the second integral on the right hand of (26). Fix an arbitrary $p \in (3, 4)$. The known smoothing properties of the

Neumann heat semigroup (Ref. [10], Lemma 1.3) provides $C_3 > 0$ fulfilling

$$\left\| \int_{t_0}^{t} e^{(t-s)\Delta} \nabla \cdot (u(\cdot,s)\nabla v(\cdot,s)) ds \right\|_{L^\infty(\Omega)} \leqslant C_4 \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \| u(\cdot,s)\nabla v(\cdot,s) \|_{L^p(\Omega)} ds \tag{30}$$

The Hölder inequality implies that

$$\| u\nabla v \|_{L^p(\Omega)} \leqslant \| u \|_{L^{\frac{4p}{4-p}}(\Omega)} \| \nabla v \|_{L^4(\Omega)} \leqslant \| u \|_{L^\infty(\Omega)}^{\frac{5p-4}{4p}} \| u \|_{L^1(\Omega)}^{\frac{4-p}{4p}} \| \nabla v \|_{L^4(\Omega)},$$

which yields positive constant C_5 fulfilling

$$\left\| \int_{t_0}^{t} e^{(t-s)\Delta} \nabla \cdot (u(\cdot,s)\nabla v(\cdot,s)) ds \right\|_{L^\infty(\Omega)} \leqslant C_5 \int_{t_0}^{t} (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \| u(\cdot,s) \|_{L^\infty(\Omega)}^{\frac{5P-4}{4P}} ds \tag{31}$$

according to (4),(20) and (30).

Denote $U(T) := \sup_{t\in(0,T)} \| u(\cdot,t) \|_{L^\infty(\Omega)}$ for any $T \in (0,T_{\max})$. Noticing that $t-t_0 \leqslant 1$, equality (31) shows that

$$\left\| \int_{t_0}^{t} e^{(t-s)\Delta} \nabla \cdot (u(\cdot,s)\nabla v(\cdot,s)) ds \right\|_{L^\infty(\Omega)} \leqslant C_5 U^{\frac{5p-4}{4p}}(T) \int_0^1 \sigma^{-\frac{1}{2}-\frac{3}{2p}} d\sigma \tag{32}$$

In view of (26)~(30),(32), we can obtain $C_6 > 0$ such that

$$U(T) \leqslant C_6 + C_6 U^{\frac{5p-4}{4p}}(T)$$

for all $T \in (0,T_{\max})$, which directly yields

$$U(T) \leqslant \max\{1, (2C_6)^{\frac{4p}{4-p}}\}$$

for all $T \in (0,T_{\max})$. This is our desired conclusion.

Since the boundedness of $\| u \|_{L^\infty(\Omega)}$ has been verified, we can deduce the boundedness of ∇v as follows.

Lemma 3.7 If the initial data condition (2) holds and if $\mu \geq 3 + \frac{23}{3} \| v_0 \|_{L^\infty(\Omega)}^2$, then there exists a positive constant C such that the solution of (1) satisfies

$$\| \nabla v(\cdot,t) \|_{L^\infty(\Omega)} \leqslant C \tag{33}$$

for all $t \in (0,T_{\max})$

Proof We use the standard estimate for Neumann heat semigroup to conclude that

$$\| \nabla v(\cdot,t) \|_{L^\infty(\Omega)} \leqslant \| \nabla e^{t\Delta} v(\cdot,0) \|_{L^\infty(\Omega)} + \int_0^t \| \nabla e^{(t-s)\Delta} u(\cdot,s) v(\cdot,s) \|_{L^\infty(\Omega)} \leqslant C_1 + \int_0^t C_2 (1+(t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} \cdot \| u(\cdot,s) v(\cdot,s) \|_{L^\infty(\Omega)},$$

where λ_1 denotes the first nonzero eigenvalue of $-\Delta$ in Ω under the homogeneous Neumann boundary conditions. Since $\| u(\cdot,t) \|_{L^\infty(\Omega)} \leqslant C$ and (5) is valid, we can obtain our result immediately.

Proof of Theorem 1.1 In combination with Lemma 2.3, Lemma 3.6, and Lemma 3.7, we can draw our conclusion.

Remark The result of Theorem 1.1 can be extended to any $\alpha \geq 2$, that is to say, if the first equaion is

$$u_t = \Delta u - \nabla \cdot (u \nabla v) + ru - \mu u^\alpha (\alpha \geq 2),$$

then we still have Theorem 1.1. The proof for that is just a trival modification of the present one by making use of Young's inequality.

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