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Bloch 空间上的微分复合算子差分的有界性及紧性的新刻画

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摘　要：令 D 为一维复平面上的单位圆盘，φ 和 ψ 是定义在 D 上的解析自映射. 将解析函数 f 映射成 $f^{(n)}\circ\varphi$ 的算子 $C_\varphi D^n$ 称为微分复合算子. 本文研究了 Bloch 空间上的微分复合算子的差分 $C_\varphi D^n - C_\psi D^n$，运用一种新的方式刻画了 $C_\varphi D^n - C_\psi D^n$ 的有界性和紧性. 此外，本文还给出了 $C_\varphi D^n - C_\psi D^n$ 本性范数的一些估计.

关键词：微分复合算子；差分；有界性；紧性；Bloch 空间
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A new characterization of boundedness and compactness for differences of differentiation composition operators between Bloch spaces

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Abstract: Let φ and ψ be analytic self-maps of the open unit disk D in the complex plane $\mathbf{C}$. The operator $C_\varphi D^n$, which maps an analytic function f to $f^{(n)}\circ\varphi$ is called differentiation composition operator, where $f^{(n)}$ denote the n-th derivative of f. In this paper, we give some new characterizations of the boundedness and compactness for the differences of differentiation composition operators $C_\varphi D^n - C_\psi D^n$ from the Bloch space to the Bloch space in the open unit disk D. Some estimates for the essential norm of differences of differentiation composition operators $C_\varphi D^n - C_\psi D^n$ between Bloch spaces in the open unit disk D are also considered.

Keywords: Differentiation composition operator; Difference; Boundedness; Compactness; Bloch space
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1　Introduction

Let $\mathbf{N}$ denote the set of non-negative integers. Let D be the open unit disk of complex plane $\mathbf{C}$ and $H(D)$ be the space of all analytic functions on D. Denoting by $S(D)$ the collection of all the analytic self-maps of D. If $\varphi\in S(D)$, then the composition operator C_φ is defined as

$$(C_\varphi f)(z)=f(\varphi(z)),\quad z\in D,\quad f\in H(D).$$

For related books about composition operators see Refs. [1] and [2].

Let $n\in\mathbf{N}$, $f^{(n)}$ denote the n-th derivative of f and $f^{(0)}=f$. A linear operator $C_\varphi D^n$ is defined by

$$(C_\varphi D^n f)(z)=f^{(n)}(\varphi(z)),\ z\in D,\ f\in H(D).$$

The operator is called the differentiation composi-

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tion operator. In fact, if $n=0$, it is the composition operator C_φ. For more recent research of $C_\varphi D^n$, we refer to Refs. [3-5].

Recall that the Bloch-type space $B\alpha(0<\alpha<\infty)$ consists of all $f\in H(D)$ such that

$$\|f\|_{B^\alpha}=|f(0)|+\sup_{z\in D}(1-|z|^2)^\alpha|f'(z)|<\infty.$$

As we all know, B^α is a Banach space under the norm $\|\cdot\|_{B^\alpha}$. When $\alpha=1$, we write B for B^1. Then $\|\cdot\|=\sup_{z\in D}(1-|z|^2)^\alpha|f'(z)|$, $\|f\|_B=|f(0)|+\|\cdot\|$ define the semi-norm and the norm on B respectively.

Recently, there has been an increase interest in characterizing the boundedness and compactness of differences of composition operators, or more generally differences of differentiation composition operators acting on spaces of analytic functions. In 2018, Hu and Zhu[6] gave a new characterization for the boundedness and compactness of the differences of generalized weighted composition operators $D^n_{\varphi,u}-D^n_{\psi,v}:B\to H^\infty_\alpha$. Among others, they showed that $D^n_{\varphi,u}-D^n_{\psi,v}:B\to H^\infty_\alpha$ is bounded if and only if

$$\sup_{j\in\mathbf{N}}\|D^n_{\varphi,u}-D^n_{\psi,v})P_j\|_{H^\infty_\alpha}<\infty.$$

Moreover, the estimate for the essential norms of the differences of these operators was given: if $D^n_{\varphi,u}-D^n_{\varphi,u}:B\to H^\infty_\alpha$ and $D^n_{\psi,v}-D^n_{\varphi,u}:B\to H^\infty_\alpha$ are bounded, then

$$\|(D^n_{\varphi,u}-D^n_{\psi,v})P_j\|_{e,B\to H^\infty_\alpha}\approx \limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})P_j\|_{H^\infty_\alpha},$$

where $P_j=z^j$, $z\in D$. Shi and Li[7] gave a new characterization for the compactness of the differences of two composition operators on the Bloch space. To be more specific, they proved that $C_\varphi-C_\psi$ is compact if and only if

$$\lim_{n\to\infty}\|\varphi^n-\psi^n\|_B=0.$$

Motivated by the study in Refs. [6, 7], we are concerned about the boundedness and compactness of the differences of two differentiation composition operators on Bloch space. More precisely, we will show that$C_\varphi D^n-C_\psi D^n:B\to B$ is bounded if and only if

$$\sup_{j\in\mathbf{N}}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B<\infty,\ z\in D.$$

And if $C_\varphi D^n:B\to B$ and $C_\psi D^n:B\to B$ are bounded, then $C_\varphi D^n-C_\psi D^n:B\to B$ is compact if and only if

$$\limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B=0,\ z\in D.$$

Throughout this paper, we will use the same positive constant C to denote various positive constant, the exact value of which may be different. The nation $a\lesssim b$, $a\gtrsim b$ mean that there may be different positive constant C such that $a\le Cb$, $a\ge Cb$. We say $a\approx b$ if both $a\lesssim b$ and $a\gtrsim b$ hold.

2 Notions and lemmas

In this section, we will recall some basic facts which are helpful in the later sections.

For any $a\in D$, let σ_a be the Möbius transformation on D defined by $\sigma_a(z)=\dfrac{a-z}{1-\bar{a}z}$. The pseudo-hyperbolic distance on D is defined by

$$\rho(z,w)=|\sigma_z(w)|=\left|\frac{z-w}{1-\bar{z}w}\right|,\ z,\ w\in D.$$

In the rest of the paper, we set $\rho(z):=\rho(\varphi(z),\psi(z))$ for the pseudo-hyperbolic distance $\varphi(z)$ and $\psi(z)$.

After some simple calculations, we can easily get the following lemma. We omitted details.

Lemma 2.1[8] For every positive integer n, if $f\in B$, then

$$|(1-|z|^2)^n f^{(n)}(z)-(1-|w|^2)^n f^{(n)}(w)|\le C\|f\|_B\rho(z,w),$$

for all $z,w\in D$.

Lemma 2.2[1] Let $\varphi,\psi\in S(D)$, then $C_\varphi D^n-C_\psi D^n:B\to B$ is compact if and only if $C_\varphi D^n-C_\psi D^n:B\to B$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbf{N}}$ in B which converges to zero uniformly on compact subsets of D, then $\|(C_\varphi D^n-C_\psi D^n)f_k\|_B\to 0$ as $k\to\infty$.

Lemma 2.3[9] For every bounded sequence $\{f_n\}_{n\in\mathbf{N}}$ in B which converges to 0 uniformly on compacts, if $K:B\to B$ is a compact operator, we have $\lim_{n\to\infty}\|Kf_n\|_B=0$.

Proof Suppose that there is a subsequence $\{f_{n_k}\}$ such that $\|Kf_{n_k}\|_B\ge\delta$, for every $k\in\mathbf{N}$ and some $\delta>0$. Since K is compact and $\{f_{n_k}\}$ is bounded in B, we obtain $\{Kf_{n_k}\}$ has a subsequence$\{Kf_{n_{k_l}}\}$and $f\in B$ such that

$$\lim_{l\to\infty}\|Kf_{n_{k_l}}-f\|_B=0.$$

From Lemma 2.3 in Ref. [10], we see that for any $f\in B$, we have

$$|f(z)|\le\frac{1}{\log 2}\|f\|_B\log\frac{2}{1-|z|^2},\ z\in D.$$

Hence for arbitrary $r\in(0,1)$ and any $z\in r\overline{D}$, we get

$$|(Kf_{n_{k_l}}-f)(z)|\le \frac{1}{\log 2}\|f\|_B\log\frac{2}{1-r^2}\|Kf_{n_{k_l}}-f\|_B.$$

Therefore, $Kf_{n_{k_l}}-f\to 0$ on compact subsets of D, as $l\to\infty$. And $f_{n_{k_l}}$ converges to 0 on compacts implies $Kf_{n_{k_l}}$ converges to 0 on compacts. Then we obtain $f\equiv 0$. Thus $\lim_{l\to\infty}\|Kf_{n_{k_l}}\|_B=0$, which is contradictory with $\|Kf_{n_k}\|_B\ge\delta$.

To further study the differences of differentiation composition operators, we define

$$M_\varphi(z):=\frac{1-|z|^2}{(1-|\varphi(z)|^2)^{n+1}}\varphi'(z),$$

$$M_\psi(z):=\frac{1-|z|^2}{(1-|\psi(z)|^2)^{n+1}}\psi'(z),$$

where $\varphi,\psi\in S(D), z\in D$. For any $a\in D$, we consider two test functions defined as follows:

$$E_a(z)=\int_0^z\frac{1-|a|^2}{(1-\bar{a}u)^{n+2}}\mathrm{d}u,$$

and

$$F_a(z)=\int_0^z\frac{1-|a|^2}{(1-\bar{a}u)^{n+2}}\frac{a-u}{1-\bar{a}u}\mathrm{d}u.$$

Some simple calculations show that $E_a, F_a\in B^{n+1}$. Thus, there exist $f_a, g_a\in B$ such that $f_a^{(n)}=E_a$, $g_a^{(n)}=F_a$.

3 Boundedness of $C_\varphi D^n-C_\psi D^n$

In this section, we begin to prove one of our main results in this paper. Hence, we first state some lemmas which will be used in the proofs of the main results.

Lemma 3.1 Let $n\in\mathbf{N}$, φ and ψ be analytic self-maps of D. Then the following inequalities hold.

(i)
$$\sup_{z\in D}|M_\varphi(z)|\rho(z)\lesssim\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)g_a\|_B;$$

(ii)
$$\sup_{z\in D}|M_\psi(z)|\rho(z)\lesssim\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)g_a\|_B;$$

(iii)
$$\sup_{z\in D}|M_\varphi(z)-M_\psi(z)|\lesssim\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)g_a\|_B.$$

Proof First, for any $z\in D$, we choose the test function f_a. Setting $a=\varphi(z)$, we see

$$\begin{aligned}&\|(C_\varphi D^n-C_\psi D^n)f_{\varphi(z)}\|_B\ge(1-|z|^2)|(f_{\varphi(z)}^{(n)}(\varphi(z))-f_{\varphi(z)}^{(n)}(\psi(z)))'|=\\&(1-|z|^2)|E_{\varphi(z)}(\varphi(z))\varphi'(z)-E_{\varphi(z)}(\psi(z))\psi'(z)|=\\&(1-|z|^2)\left|\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}\varphi(z))^{n+2}}\varphi'(z)-\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}\psi(z))^{n+2}}\psi'(z)\right|=\\&\left|M_\varphi(z)-\frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^{n+1}}{(1-\overline{\varphi(z)}\psi(z))^{n+2}}M_\psi(z)\right|\ge\\&|M_\varphi(z)|-\frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^{n+1}}{|1-\overline{\varphi(z)}\psi(z)|^{n+2}}|M_\psi(z)|\end{aligned}\qquad(1)$$

Similar to the discussion of (1), we prove

$$\begin{aligned}&\|(C_\varphi D^n-C_\psi D^n)g_{\varphi(z)}\|_B\ge(1-|z|^2)|F_{\varphi(z)}(\varphi(z))\varphi'(z)-F_{\varphi(z)}(\psi(z))\psi'(z)|\ge\\&(1-|z|^2)\left|\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}\psi(z))^{n+2}}\frac{\varphi(z)-\psi(z)}{1-\overline{\varphi(z)}\psi(z)}\right||\psi'(z)|=\\&|M_\psi(z)|\frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^{n+1}}{|1-\overline{\varphi(z)}\psi(z)|^{n+2}}\rho(z)\end{aligned}\qquad(2)$$

Multiplying (1) by $\rho(z)$ and (2), we have

$$|M_\varphi(z)|\rho(z)\le$$

$$\|(C_\varphi D^n-C_\psi D^n)f_{\varphi(z)}\|_B\rho(z)+$$

$$\|(C_\varphi D^n-C_\psi D^n)g_{\varphi(z)}\|_B\le$$

$$\|(C_\varphi D^n - C_\psi D^n) f_{\varphi(z)}\|_B + \|(C_\varphi D^n - C_\psi D^n) g_{\varphi(z)}\|_B \quad (3)$$

Similarly,

$$|M_\psi(z)|\rho(z) \le \|(C_\varphi D^n - C_\psi D^n) f_{\psi(z)}\|_B + \|(C_\varphi D^n - C_\psi D^n) g_{\psi(z)}\|_B \quad (4)$$

Taking supremum in (3) over D, we obtain

$$\sup_{z\in D}|M_\varphi(z)|\rho(z) \le \sup_{z\in D}\|(C_\varphi D^n - C_\psi D^n) f_{\varphi(z)}\|_B + \sup_{z\in D}\|(C_\varphi D^n - C_\psi D^n) g_{\varphi(z)}\|_B \le \sup_{a\in D}\|(C_\varphi D^n - C_\psi D^n) f_a\|_B + \sup_{a\in D}\|(C_\varphi D^n - C_\psi D^n) g_a\|_B.$$

Similarly, it holds that

$$\sup_{z\in D}|M_\psi(z)|\rho(z) \le \sup_{a\in D}\|(C_\varphi D^n - C_\psi D^n) f_a\|_B + \sup_{a\in D}\|(C_\varphi D^n - C_\psi D^n) g_a\|_B.$$

On the other hand, we use function $f_{\varphi(z)}$ again, it follows that

$$\begin{aligned}
&\|(C_\varphi D^n - C_\psi D^n) f_{\varphi(z)}\|_B \ge \\
&\left|M_\varphi(z) - \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^{n+1}}{(1-\overline{\varphi(z)}\psi(z))^{n+2}} M_\psi(z)\right| \ge |M_\varphi(z) - M_\psi(z)| - |M_\psi(z)| \cdot \\
&\left|1 - \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^{n+1}}{(1-\overline{\varphi(z)}\psi(z))^{n+2}}\right| \ge |M_\varphi(z) - M_\psi(z)| - |M_\psi(z)| \cdot \\
&|(1-|\varphi(z)|^2)^{n+1} E_{\varphi(z)}(\varphi(z)) - (1-|\psi(z)|^2)^{n+1} E_{\varphi(z)}(\psi(z))| \ge \\
&|M_\varphi(z) - M_\psi(z)| - |M_\psi(z)|\rho(z).
\end{aligned}$$

The last inequality follows by Lemma 2.1. Hence by (4), it yields that

$$\begin{aligned}
|M_\varphi(z) - M_\psi(z)| &\lesssim \|(C_\varphi D^n - C_\psi D^n) f_{\varphi(z)}\|_B + |M_\psi(z)|\rho(z) \lesssim \\
&\|(C_\varphi D^n - C_\psi D^n) f_{\varphi(z)}\|_B + \|(C_\varphi D^n - C_\psi D^n) f_{\psi(z)}\|_B + \|(C_\varphi D^n - C_\psi D^n) g_{\psi(z)}\|_B.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sup_{z\in D}|M_\varphi(z) - M_\psi(z)| &\lesssim \sup_{z\in D}\|(C_\varphi D^n - C_\psi D^n) f_{\varphi(z)}\|_B + \sup_{z\in D}\|(C_\varphi D^n - C_\psi D^n) f_{\psi(z)}\|_B + \sup_{z\in D}\|(C_\varphi D^n - C_\psi D^n) g_{\psi(z)}\|_B \lesssim \\
&\sup_{a\in D}\|(C_\varphi D^n - C_\psi D^n) f_a\|_B + \sup_{a\in D}\|(C_\varphi D^n - C_\psi D^n) g_a\|_B.
\end{aligned}$$

This completes the proof of the lemma.

Lemma 3.2　Let $n\in\mathbf{N}$, φ and ψ be analytic self-maps of D. Then the following inequalities hold.

(i)

$$\sup_{a\in D}\|(C_\varphi D^n - C_\psi D^n) f_a\|_B \lesssim \sup_{j\in\mathbf{N}}\|(C_\varphi D^n - C_\psi D^n) z^j\|_B;$$

(ii)

$$\sup_{a\in D}\|(C_\varphi D^n - C_\psi D^n) g_a\|_B \lesssim \sup_{j\in\mathbf{N}}\|(C_\varphi D^n - C_\psi D^n) z^j\|_B.$$

Proof　Let us write

$$E_a(z) = (1-|a|^2)\int_0^z \frac{1}{(1-\bar{a}u)^{n+2}}\mathrm{d}u = (1-|a|^2)\int_0^z \sum_{k=0}^\infty \frac{\Gamma(k+n+2)}{k!\ \Gamma(n+2)}\bar{a}^k u^k \mathrm{d}u = (1-|a|^2)\sum_{k=0}^\infty \frac{\Gamma(k+n+2)}{(k+1)!\ \Gamma(n+2)}\bar{a}^k z^{k+1}.$$

Then we can immediately get the upper bound of $\|(C_\varphi D^n - C_\psi D^n) f_a\|_B$ and $\|(C_\varphi D^n - C_\psi D^n) g_a\|_B'$,

$$\begin{aligned}
&\|(C_\varphi D^n - C_\psi D^n) f_a\|_B = \\
&\|E_a(\varphi(z)) - E_a(\psi(z))\|_B \le (1-|a|^2)\sum_{k=0}^\infty \frac{\Gamma(k+n+2)}{(k+1)!\ \Gamma(n+2)}|a|^k \|\varphi^{k+1} - \psi^{k+1}\|_B \le \\
&(1-|a|^2)\sum_{k=0}^\infty \frac{\Gamma(k+n+2)}{(k+1)!\ \Gamma(n+2)}|a|^k (k+1)^{-n} \sup_{k\in\mathbf{N}}(k+1)^n \|\varphi^{k+1} - \psi^{k+1}\|_B \lesssim \\
&(1-|a|^2)\sum_{k=0}^\infty |a|^k \sup_{k\in\mathbf{N}}(k+1)^n \|\varphi^{k+1} - \psi^{k+1}\|_B \le \\
&\sup_{j\ge n+1}(j-n)^n \|\varphi^{j-n} - \psi^{j-n}\|_B \lesssim \sup_{j\in\mathbf{N}}\|(C_\varphi D^n - C_\psi D^n) z^j\|_B \quad (5)
\end{aligned}$$

Hence

$$\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B\lesssim \sup_{j\in\mathbf{N}}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

Note that

$$\frac{a-u}{1-\bar a u}=a-(1-|a|^2)\sum_{k=0}^{\infty}\bar a^k u^{k+1}.$$

Analogous to the above discussion, we can write $F_a(z)$ as follows,

$$F_a(z)=(1-|a|^2)\int_0^z\sum_{k=0}^{\infty}\frac{\Gamma(k+n+2)}{k!\ \Gamma(n+2)}\bar a^k u^k(a-(1-|a|^2)\sum_{k=0}^{\infty}\bar a^k u^{k+1})\mathrm{d}u=$$
$$aE_a(z)-(1-|a|^2)^2\int_0^z\sum_{k=1}^{\infty}\Big(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\Big)\bar a^{k-1}u^k\mathrm{d}u=$$
$$aE_a(z)-(1-|a|^2)^2\sum_{k=1}^{\infty}\frac{1}{k+1}\Big(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\Big)\bar a^{k-1}z^{k+1}.$$

Thus

$$\|(C_\varphi D^n-C_\psi D^n)g_a\|\le\|af_a^{(n)}(\varphi(z))-af_a^{(n)}(\psi(z))\|_B+$$
$$(1-|a|^2)^2\sum_{k=1}^{\infty}\frac{1}{k+1}\Big(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\Big)|a|^{k-1}\|\varphi^{k+1}-\psi^{k+1}\|_B\le$$
$$\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+(1-|a|^2)^2\sum_{k=1}^{\infty}\frac{1}{k+1}\Big(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\Big)|a|^{k-1}\|\varphi^{k+1}-\psi^{k+1}\|_B \tag{6}$$

Since $\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\approx k^{n+2}, k\to\infty$, then

$$(1-|a|^2)^2\sum_{k=1}^{\infty}\frac{1}{k+1}(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)})|a|^{k-1}\|\varphi^{k+1}-\psi^{k+1}\|_B\lesssim$$
$$(1-|a|^2)^2\sum_{k=1}^{\infty}\frac{k^{n+2}}{(k+1)^{n+1}}|a|^{k-1}(k+1)^n\|\varphi^{k+1}-\psi^{k+1}\|_B\le$$
$$(1-|a|^2)^2\sum_{k=1}^{\infty}k\ |a|^{k-1}(k+1)^n\|\varphi^{k+1}-\psi^{k+1}\|_B\lesssim$$
$$(k+1)^n\|\varphi^{k+1}-\psi^{k+1}\|_B\lesssim\sup_{j\ge n+1}(j-n)^n\|\varphi^{j-n}-\psi^{j-n}\|_B\lesssim$$
$$\sup_{j\in\mathbf{N}}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

Combing last inequality with (5) and (6), we have

$$\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)g_a\|_B\lesssim \sup_{j\in\mathbf{N}}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

This completes the proof of Lemma 3.2.

After these lemmas, we are now ready to complete the proof of our first theorem.

Theorem 3.3 Let $n\in\mathbf{N}$, φ and ψ be analytic self-maps of D. Then $C_\varphi D^n-C_\psi D^n: B\to B$ is bounded if and only if

$$\sup_{j\in\mathbf{N}}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B<\infty,\ z\in D.$$

Proof We first prove the necessity. Assume that $C_\varphi D^n-C_\psi D^n:B\to B$ is bounded. When $j\ge 2$, we have

$$\|z^j\|_B=\max_{z\in D}j\ |z|^{j-1}(1-|z|^2)=\frac{2j}{j+1}\Big(\frac{j-1}{j+1}\Big)^{\frac{j-1}{2}}.$$

Obviously, these maxima form a decreasing sequence which tends to 2/e. Then for any $j\in\mathbf{N}$, we have $\|z^j\|_B\approx 1$. Thus

$$\|(C_\varphi D^n-C_\psi D^n)z^j\|_B\le \|C_\varphi D^n-C_\psi D^n\|_{B\to B}<\infty.$$

Taking supremum for any $j\in\mathbf{N}$, the necessity is complete.

Next, we show the sufficiency. Let $f\in B$ with $\|f\|_B=1$. Then we obtain

$$\|(C_\varphi D^n-C_\psi D^n)f=$$
$$\sup_{z\in D}(1-|z|^2)|f^{(n+1)}(\varphi(z))\varphi'(z)-f^{(n+1)}(\psi(z))\psi'(z)|=$$
$$\sup_{z\in D}|(1-|\varphi(z)|^2)^{n+1}f^{(n+1)}(\varphi(z))M_\varphi(z)-(1-|\psi(z)|^2)^{n+1}f^{(n+1)}(\psi(z))M_\psi(z)|\le$$

$$\sup_{z\in D}|M_\varphi(z)-M_\psi(z)||(1-|\varphi(z)|^2)^{n+1}f^{(n+1)}(\varphi(z))|+$$
$$\sup_{z\in D}|M_\psi(z)||(1-|\varphi(z)|^2)^{n+1}f^{(n+1)}(\varphi(z))-(1-|\psi(z)|^2)^{n+1}f^{(n+1)}(\psi(z))|\lesssim$$
$$\sup_{z\in D}|M_\varphi(z)-M_\psi(z)|+\sup_{z\in D}|M_\psi(z)|\rho(z)\lesssim\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+$$
$$\sup_{a\in D}\|(C_\varphi D^n-C_\psi D^n)g_a\|_B\lesssim\sup_{j\in\mathbf{N}}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

where we use the fact that

$$\sup_{z\in D}(1-|z|^2)^{n+1}|f^{(n+1)}(z)|<\infty,$$

Lemma 2.1, Lemma 3.1 and Lemma 3.2. By the assumption $\sup_{j\in\mathbf{N}}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B<\infty$, $C_\varphi D^n-C_\psi D^n: B\to B$ is bounded. The proof is complete.

4 Essential norm estimates of $C_\varphi D^n-C_\psi D^n$

In this section, we will introduce a sufficient and necessary condition for the compactness of $C_\varphi D^n-C_\psi D^n: B\to B$. For that, we will prove a few lemmas.

Lemma 4.1 Let $n\in\mathbf{N}$, φ and ψ be analytic self-maps of D. Then the following statements hold.

(i)

$$\lim_{r\to1}\sup_{|\varphi(z)|>r}|M_\varphi(z)|\rho(z)\lesssim \lim_{|a|\to1}\sup\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\lim_{|a|\to1}\sup\|(C_\varphi D^n-C_\psi D^n)g_a\|_B;$$

(ii)

$$\lim_{r\to1}\sup_{|\psi(z)|>r}|M_\psi(z)|\rho(z)\lesssim \lim_{|a|\to1}\sup\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\lim_{|a|\to1}\sup\|(C_\varphi D^n-C_\psi D^n)g_a\|_B;$$

(iii)

$$\lim_{r\to1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}|M_\varphi(z)-M_\psi(z)|\lesssim \lim_{|a|\to1}\sup\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\lim_{|a|\to1}\sup\|(C_\varphi D^n-C_\psi D^n)g_a\|_B.$$

Proof For any $z\in D$, from Lemma 3.1, we can obtain the following consequences,

$$|M_\varphi(z)|\rho(z)\le\|(C_\varphi D^n-C_\psi D^n)f_{\varphi(z)}\|_B+\|(C_\varphi D^n-C_\psi D^n)g_{\varphi(z)}\|_B,$$
$$|M_\psi(z)|\rho(z)\le\|(C_\varphi D^n-C_\psi D^n)f_{\psi(z)}\|_B+\|(C_\varphi D^n-C_\psi D^n)g_{\psi(z)}\|_B,$$

and

$$|M_\varphi(z)-M_\psi(z)|\le \|(C_\varphi D^n-C_\psi D^n)f_{\varphi(z)}\|_B+\|(C_\varphi D^n-C_\psi D^n)f_{\psi(z)}\|_B+\|(C_\varphi D^n-C_\psi D^n)g_{\psi(z)}\|_B.$$

Based on the above inequalities, we can get the assertion as desired.

Lemma 4.2 Let $n\in\mathbf{N}$, φ and ψ be analytic self-maps of D. Suppose that $C_\varphi D^n-C_\psi D^n: B\to B$ is bounded. Then the following equalities hold.

(i)

$$\lim_{|a|\to1}\sup\|(C_\varphi D^n-C_\psi D^n)f_a\|_B\lesssim\lim_{j\to\infty}\sup\|(C_\varphi D^n-C_\psi D^n)z^j\|_B;$$

(ii)

$$\lim_{|a|\to1}\sup\|(C_\varphi D^n-C_\psi D^n)g_a\|_B\lesssim\lim_{j\to\infty}\sup\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

Proof Since $C_\varphi D^n-C_\psi D^n$ is bounded, by the proof of Lemma 3.2, it is easy to see

$$\sup_{k\in\mathbf{N}}(k+1)^n\|\varphi^{k+1}-\psi^{k+1}\|_B<\infty.$$

Then for any $a\in D$ and each $N\in\mathbf{N}$, we have

$$\|(C_\varphi D^n-C_\psi D^n)f_a\|_B\le(1-|a|^2)\sum_{k=0}^{N}\frac{\Gamma(k+n+2)}{(k+1)!\ \Gamma(n+2)}|a|^k\|\varphi^{k+1}-\psi^{k+1}\|_B+$$
$$(1-|a|^2)\sum_{k=N+1}^{\infty}\frac{\Gamma(k+n+2)}{(k+1)!\ \Gamma(n+2)}|a|^k\|\varphi^{k+1}-\psi^{k+1}\|_B\lesssim$$
$$(1-|a|^2)\sum_{k=0}^{N}\frac{\Gamma(k+n+2)}{(k+1)!\ \Gamma(n+2)}|a|^k(k+1)^{-n}\sup_{k\in\mathbf{N}}(k+1)^n\|\varphi^{k+1}-\psi^{k+1}\|_B+$$
$$(1-|a|^2)\sum_{k=N+1}^{\infty}\frac{\Gamma(k+n+2)}{(k+1)!\ \Gamma(n+2)}|a|^k(k+1)^{-n}(k+1)^n\|\varphi^{k+1}-\psi^{k+1}\|_B \qquad (7)$$

Meanwhile,

$$(1-|a|^2)\sum_{k=N+1}^{\infty}\frac{\Gamma(k+n+2)}{(k+1)!\ \Gamma(n+2)}|a|^k(k+1)^{-n}(k+1)^n\|\varphi^{k+1}-\psi^{k+1}\|_B\le \sup_{j\ge N+n+2}(j-n)^n\|\varphi^{j-n}-\psi^{j-n}\|_B.$$

Letting $|a|\to1$ in (7) leads to

$$\limsup_{|a|\to1}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B\lesssim \sup_{j\ge N+n+2}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

Thus

$$\limsup_{|a|\to1}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B\lesssim \limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

Similar to (7), we can get the following inequality by the proof of Lemma 3.2:

$$\|(C_\varphi D^n-C_\psi D^n)g_a\|_B\le \|(C_\varphi D^n-C_\psi D^n)f_a\|_B+(1-|a|^2)^2\sum_{k=1}^{\infty}\frac{1}{k+1}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\right)|a|^{k-1}\|\varphi^{k+1}-\psi^{k+1}\|_B=$$
$$\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+(1-|a|^2)^2\sum_{k=1}^{N}\frac{1}{k+1}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\right)|a|^{k-1}\|\varphi^{k+1}-\psi^{k+1}\|_B+$$
$$(1-|a|^2)^2\sum_{k=N+1}^{\infty}\frac{1}{k+1}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\right)|a|^{k-1}\|\varphi^{k+1}-\psi^{k+1}\|_B \tag{8}$$

Since

$$(1-|a|^2)^2\sum_{k=N+1}^{\infty}\frac{1}{k+1}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+2)}{l!\ \Gamma(n+2)}\right)|a|^{k-1}\|\varphi^{k+1}-\psi^{k+1}\|_B\lesssim\sup_{k\ge N+2}k^n\|\varphi^k-\psi^k\|_B\lesssim \sup_{j\ge N+n+2}(j-n)^n\|\varphi^{j-n}-\psi^{j-n}\|_B.$$

Combining the last inequality with (8), and letting $|a|\to1$, we get

$$\limsup_{|a|\to1}\|(C_\varphi D^n-C_\psi D^n)g_a\|_B\le \limsup_{|a|\to1}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\sup_{j\ge N+n+2}(j-n)^n\|\varphi^{j-n}-\psi^{j-n}\|_B\lesssim$$
$$\limsup_{|a|\to1}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\sup_{j\ge N+n+2}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B\lesssim \limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

Based on the above results, we are now ready to estimate the essential norm of $C_\varphi D^n-C_\psi D^n: B\to B$.

Theorem 4.3 Let $n\in\mathbf{N}$, φ and ψ be analytic self-maps of D. Assume that $C_\varphi D^n, C_\psi D^n: B\to B$ are bounded, then

$$\|C_\varphi D^n-C_\psi D^n\|_{e,B\to B}\approx \lim_{r\to1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}|M_\varphi(z)-M_\psi(z)|+\lim_{r\to1}\sup_{|\varphi(z)|>r}|M_\varphi(z)|\rho(z)+\lim_{r\to1}\sup_{|\psi(z)|>r}|M_\psi(z)|\rho(z)\approx$$
$$\limsup_{|a|\to1}\|(C_\varphi D^n-C_\psi D^n)f_a\|_B+\limsup_{|a|\to1}\|(C_\varphi D^n-C_\psi D^n)g_a\|_B\approx \limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B, z\in D.$$

Proof First, we prove

$$\|C_\varphi D^n-C_\psi D^n\|_{e,B\to B}\gtrsim \limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

For any positive integer j, we have $\|z^j\|_B\approx1$. It is easy to see that $\|z^j\|_B$ is bounded and z^j converges uniformly to 0 on compact subsets of D in B, as $j\to\infty$. We assume that K is any compact operator from B to B. By Lemma 2.3, we obtain $\lim_{j\to\infty}\|Kz^j\|_B=0$. Thus

$$\|C_\varphi D^n-C_\psi D^n-K\|_B\gtrsim \limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n-K)z^j\|_B\ge \limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B.$$

Then we obtain

$$\|C_\varphi D^n-C_\psi D^n\|_{e,B\to B}\gtrsim \limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n)z^j\|_B \tag{9}$$

For $r\in[0,1)$, set $K_r: H(D)\to H(D)$. Then for any $f\in H(D)$, we define

$$(K_rf)(z)=f_r(z)=f(rz).$$

It is clear that $f-f_r\to0$ uniformly on compact subsets of D as $r\to1$. Moreover, K_r is compact on B and $\|K_r\|_{B\to B}\le1$. Let $r_j\subset(0,1)$ be a sequence such that $r_j\to1$as $j\to\infty$. Then for each positive integer j, the operator $(C_\varphi D^n-C_\psi D^n)K_{r_j}: B\to B$ is compact. Then we have

$$\|C_\varphi D^n-C_\psi D^n\|_{e,B\to B}\le$$

$$\limsup_{j\to\infty}\|C_\varphi D^n-C_\psi D^n-(C_\varphi D^n-C_\psi D^n)K_{r_j}\|_{B\to B}=\limsup_{j\to\infty}\|(C_\varphi D^n-C_\psi D^n)(I-K_{r_j})\|_{B\to B}\le\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\|(C_\varphi D^n-C_\psi D^n)(I-K_{r_j})f\|_B=\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{z\in D}\Omega_j^f(z),$$

where

$$\Omega_j^f(z):=(1-|z|^2)|(I-K_{r_j})f^{(n+1)}(\varphi(z))\varphi'(z)-(I-K_{r_j})f^{(n+1)}(\psi(z))\psi'(z)|.$$

For any $r\in(0,1)$, define

$$D_1:=\{z\in D:|\varphi(z)|\le r,|\psi(z)|\le r\},$$
$$D_2:=\{z\in D:|\varphi(z)|\le r,|\psi(z)|>r\},$$
$$D_3:=\{z\in D:|\varphi(z)|>r,|\psi(z)|\le r\},$$
$$D_4:=\{z\in D:|\varphi(z)|>r,|\psi(z)|>r\}.$$

Taking

$$J_i=\sup_{\|f\|_B\le1}\sup_{z\in D_i}\Omega_j^f(z),$$

we can write $\Omega_j^f(z)$ as follows

$$\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{z\in D}\Omega_j^f(z)=\max\{\limsup_{j\to\infty}J_1,\limsup_{j\to\infty}J_2,\limsup_{j\to\infty}J_3,\limsup_{j\to\infty}J_4\}.$$

Setting the function $f(z)=z^{n+1}\in B$. By the boundedness of $C_\varphi D^n$ and $C_\psi D^n$, we can easily obtain

$$\sup_{z\in D}(1-|z|^2)|\varphi'(z)|<\infty$$

and

$$\sup_{z\in D}(1-|z|^2)|\psi'(z)|<\infty.$$

Meanwhile, $(I-K_{r_j})f^{(n+1)}(z)\to0$ uniformly on compact subsets of D. Thus we get

$$\limsup_{j\to\infty}J_1=\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{z\in D_1}\Omega_j^f(z)\le\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{|\varphi(z)|\le r}(1-|z|^2)|(I-K_{r_j})f^{(n+1)}(\varphi(z))\varphi'(z)|+\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{|\psi(z)|\le r}(1-|z|^2)|(I-K_{r_j})f^{(n+1)}(\psi(z))\psi'(z)|=0.$$

On the other hand, we can get the following estimates by Lemma 2.1,

$$\Omega_j^f(z)\lesssim(1-|\varphi(z)|^2)^{n+1}|(I-K_{r_j})f^{(n+1)}(\varphi(z))||M_\varphi(z)-M_\psi(z)|+|M_\psi(z)||(I-K_{r_j})f^{(n+1)}(\varphi(z))(1-|\varphi(z)|^2)^{n+1}-(I-K_{r_j})f^{(n+1)}(\psi(z))(1-|\psi(z)|^2)^{n+1}|\lesssim(1-|\varphi(z)|^2)^{n+1}|(I-K_{r_j})f^{(n+1)}(\varphi(z))||M_\varphi(z)-M_\psi(z)|+|M_\psi(z)|\rho(z).$$

Meanwhile, we get

$$\Omega_j^f(z)\lesssim(1-|\psi(z)|^2)^{n+1}|(I-K_{r_j})f^{(n+1)}(\psi(z))||M_\varphi(z)-M_\psi(z)|+|M_\varphi(z)|\rho(z).$$

By assumption, $C_\varphi D^n-C_\psi D^n$ is bounded, we have $\sup_{z\in D_2}|M_\varphi(z)-M_\psi(z)|<\infty$. Hence

$$\limsup_{j\to\infty}J_2\lesssim\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{z\in D_2}((1-|\varphi(z)|^2)^{n+1}|(I-K_{r_j})f^{(n+1)}(\varphi(z))||M_\varphi(z)-M_\psi(z)|+|M_\psi(z)|\rho(z))\lesssim\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{|\varphi(z)|\le r}(1-|\varphi(z)|^2)^{n+1}|(I-K_{r_j})f^{(n+1)}(\varphi(z))||M_\varphi(z)-M_\psi(z)|+\sup_{|\psi(z)|>r}|M_\psi(z)|\rho(z)=\sup_{|\psi(z)|>r}|M_\psi(z)|\rho(z),$$

where we used $(I-K_{r_j})f^{(n+1)}(z)\to0$ uniformly on compact subsets of D as $j\to\infty$ again in the last inequality. Due to the arbitrary of r, we have

$$\limsup_{j\to\infty}J_2\lesssim\lim_{r\to1}\sup_{|\psi(z)|>r}|M_\psi(z)|\rho(z).$$

Similarly,

$$\limsup_{j\to\infty}J_3\lesssim\lim_{r\to1}\sup_{|\varphi(z)|>r}|M_\varphi(z)|\rho(z).$$

In addition,

$$\limsup_{j\to\infty}J_4\lesssim\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{z\in D_4}((1-|\psi(z)|^2)^{n+1}|(I-K_{r_j})f^{(n+1)}(\psi(z))||M_\varphi(z)-M_\psi(z)|+|M_\varphi(z)|\rho(z))\le\limsup_{j\to\infty}\sup_{\|f\|_B\le1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}\|(I-K_{r_j})f\|_B|M_\varphi(z)-M_\psi(z)|+\sup_{|\varphi(z)|>r}|M_\varphi(z)|\rho(z)\lesssim\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}|M_\varphi(z)-M_\psi(z)|+\sup_{|\varphi(z)|>r}|M_\varphi(z)|\rho(z).$$

Here we used the fact that $\limsup_{j\to\infty}\|(I-K_{r_j})f\|_B\le2$ in the last inequality. Thus

$$\lim_{j\to\infty}\sup J_4 \lesssim \lim_{r\to 1}\sup_{|\varphi(z)|>r\,|\psi(z)|>r} |M_\varphi(z)-M_\psi(z)| + \lim_{r\to 1}\sup_{|\varphi(z)|>r} |M_\varphi(z)|\rho(z).$$

Then we have

$$\lim_{j\to\infty}\sup \sup_{\|f\|_B\le 1}\sup_{z\in D}\Omega_j^f(z) =$$
$$\max_{1\le i\le 4}\lim_{j\to\infty}\sup \sup_{\|f\|_B\le 1}\sup_{z\in D_i}\Omega_j^f(z) =$$
$$\max\{\lim_{j\to\infty}\sup J_1, \lim_{j\to\infty}\sup J_2, \lim_{j\to\infty}\sup J_3, \lim_{j\to\infty}\sup J_4\} \lesssim$$
$$\lim_{r\to 1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}} |M_\varphi(z)-M_\psi(z)| + \lim_{r\to 1}\sup_{|\varphi(z)|>r}|M_\varphi(z)|\rho(z) + \lim_{r\to 1}\sup_{|\psi(z)|>r}|M_\psi(z)|\rho(z).$$

which together with Lemma 4. 1 and 4. 2 imply

$$\|C_\varphi D^n - C_\psi D^n\|_{e,B\to B} \lesssim$$
$$\lim_{r\to 1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}} |M_\varphi(z)-M_\psi(z)| + \lim_{r\to 1}\sup_{|\varphi(z)|>r}|M_\varphi(z)|\rho(z) + \lim_{r\to 1}\sup_{|\psi(z)|>r}|M_\psi(z)|\rho(z) \lesssim$$
$$\lim_{|a|\to 1}\sup \|(C_\varphi D^n - C_\psi D^n)f_a\|_B + \lim_{|a|\to 1}\sup \|(C_\varphi D^n - C_\psi D^n)g_a\|_B \lesssim$$
$$\lim_{j\to\infty}\sup \|(C_\varphi D^n - C_\psi D^n)z^j\|_B \qquad (10)$$

Combining (9) with (10), we complete the proof.

Corollary 4. 4 Let $n\in\mathbf{N}$, φ and ψ be analytic self-maps of D. Suppose that $C_\varphi D^n, C_\psi D^n: B\to B$ are bounded, then $C_\varphi D^n - C_\psi D^n$ is compact if and only if $\lim_{j\to\infty}\sup \|(C_\varphi D^n - C_\psi D^n)z^j\|_B = 0, z\in D$.

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