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具随机生成元的受控随机发展方程的 Pontryagin 型最大值原理

张　宇

(四川大学数学学院，成都 610064)

摘　要：本文研究了当控制区域是凸集时带有随机生成元的受控正向随机发展方程的 Pontryagin 型最大值原理. 运用 Malliavin 分析方法,本文给出了当 $p \geq 2$ 时控制系统温和解的存在唯一性,运用转置方法获得了当 $1 < q \leq 2$ 时对偶系统的适定性,并运用凸变分方法推导了相应的最大值原理.

关键词：随机发展方程；随机生成元；最大值原理

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Pontryagin-type stochastic maximum principle of stochastic evolution equation with a random generator

ZHANG Yu

(School of Mathematics, Sichuan University, Chengdu 610064, China)

Abstract: In this paper, we establish a Pontryagin-type maximum principle for a control stochastic evolution equation with a random generator and a convex control domain. Given $p \geq 2$, the existence and uniqueness of mild solution to the control system are obtained by using the Malliavin calculus. To study the well-posedness of the adjoint system when $1 < q \leq 2$, the transposition method is used. The well-posedness results for these systems are established. The desired Pontryagin-type maximum principle is deduced by a standard convex perturbation technique.

Keywords: Stochastic evolution equation; Random generator; Maximum principle

1 Introduction

Let $T>0$, V be a real and separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Let H be another real and separable Hilbert space. Let (Ω, F, F, P) be a complete filtered probability space with the filtration $F=\{F_t\}_{t\in[0,T]}$ being generated by $B(\cdot)$, a V-cylindrical Brownian motion on the time interval $[0,T]$. For any $r\in[1,\infty)$, denote by $L_{F_t}^r(\Omega;H)$ the Banach space of all F_t-measurable random variables $\eta:\Omega\to H$ such that $E|\eta|_H^r<\infty$ with the canonical norm. Write $D_F([0,T];L^r(\Omega;H))$ for the set of all H-valued F-adapted processes $\varphi(\cdot):[0,T]\to L_{F_T}^r(\Omega;H)$ being càdlàg, i. e., right continuous with left limits. Clearly, $D_F([0,T];L^r(\Omega;H))$ is a Banach space with the norm

$$|\varphi(\cdot)|_{D_F([0,T];L^r(\Omega;H))}=\sup_{t\in[0,T]}(E|\varphi(t)|_H^r)^{\frac{1}{r}}.$$

Denote by $C_F([0,T];L^r(\Omega;H))$ the Banach space of all H-valued F-adapted processes $\varphi(\cdot):[0,T]$

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作者简介：张宇(1992－)，男，湖北荆门人，硕士，主要研究方向为分布参数系统理论. E-mail:yu_zang@163.com

$\to L^r_{F_T}(\Omega;H)$ being continuous with the norm inherited from $D_F([0,T];L^r(\Omega;H))$. Write $L(X,Y)$ for the (Banach) space of all bounded linear operators from a Banach X to another Banach space Y. We simply write $L(X,X)$ for $L(X)$. For any fixed $p,q\in[1,\infty]$, write $L^p_F(0,T;L^q(\Omega;H))$ for the set of all H-valued F-adapted processes such that

$$\left[\int_0^T (\mathrm{E}\mid X(t)\mid_H^q)^{\frac{p}{q}}\mathrm{d}t\right]^{\frac{1}{p}}<\infty$$

with the canonical norm. It is denoted by $L^p_F(0,T;H)$ if $p=q$. For any $t\in[0,T]$, one can define the spaces $D_F([t,T];L^r(\Omega;H))$, $C_F([t,T];L^r(\Omega;H))$ and $L^p_F(t,T;L^q(\Omega;H))$ in a similar way. Denote by $L_2(V;H)$ the space of all Hilbert-Schmidt operators from V into H with the inner product

$$<F,G>_{L_2(V;H)}=\sum_{j=1}^{\infty}<Fe_j,Ge_j>_H,$$

$\forall F,G\in L_2(V;H)$. Write $L_2^0=L_2(V;H)$. Let U be a convex subset of a real and separable Hilbert space H_1, for which the metric is endowed with the norm of H_1. Define

$U[0,T]=\{u(\cdot):[0,T]\to U\mid u(\cdot)$ is F-adapted$\}$.

In this paper, we consider the following controlled stochastic evolution equation (SEE):

$$\begin{cases}\mathrm{d}x(t)=A(t)x(t)+a(t,x(t),u(t))\mathrm{d}t+b(t,x(t),u(t))\mathrm{d}B(t),t\in[0,T],\\ x(0)=x_0\end{cases}\tag{1}$$

with a cost functional

$$J(u(\cdot))=\mathrm{E}[\int_0^T g(t,x(t),u(t))\mathrm{d}t+h(x(T))]\tag{2}$$

where $x_0\in L^p_{F_0}(\Omega;H)$ for a given $p\geq 2$. $u(\cdot)\in U[0,T]$ is a control variable, $x(\cdot)$ is the corresponding state variable, $A(t,\omega)$ is a family of unbounded random operators on H, $a(\cdot,\cdot,\cdot):[0,T]\times H\times U\to H$ and $b(\cdot,\cdot,\cdot):[0,T]\times H\times U\to L_2^0$, $g(\cdot,\cdot,\cdot):[0,T]\times H\times U\to R$ and $h(\cdot):H\to R$ are given functions satisfying some conditions to be given later. We are concerned with the following optimal control problem:

Problem(OP) Find a control $\bar{u}(\cdot)\in U[0,T]$ such that

$$J(\bar{u}(\cdot))=\inf_{u(\cdot)\in U[0,T]}J(u(\cdot))\tag{3}$$

Any $\bar{u}(\cdot)$ satisfying (3) is called optimal control, the corresponding state process $\bar{x}(\cdot)$ is called an optimal state, and $(\bar{x}(\cdot),\bar{u}(\cdot))$ is called an optimal pair.

Let us introduce the following conditions:

(A_1) Suppose that $a(\cdot,\cdot,\cdot):[0,T]\times H\times U\to H$, $b(\cdot,\cdot,\cdot):[0,T]\times H\times U\to L_2^0$ are two maps satisfying

(i) For any $(x,u)\in H\times U$, $a(\cdot,x,u)$ and $b(\cdot,x,u)$ are Lebesgue measurable,

(ii) For any $(t,x)\in[0,T]\times H$, $a(t,x,\cdot)$ and $b(t,x,\cdot)$ are continuous,

(iii) There exists a constant $C>0$ such that for any $(t,x_1,x_2,u)\in[0,T]\times H\times H\times U$,

$$\begin{cases}|a(t,x_1,u)-a(t,x_2,u)|_H+|b(t,x_1,u)-b(t,x_2,u)|_{L_2^0}\leq C|x_1-x_2|_H,\\ |a(t,0,u)|_H+|b(t,0,u)|_{L_2^0}\leq C\end{cases}\tag{4}$$

(A_2) Suppose that $g(\cdot,\cdot,\cdot):[0,T]\times H\times U\to R$ and $h(\cdot):H\to R$ are two functions satisfying

(i) For any $(x,u)\in H\times U$, $g(\cdot,x,u)$ is Lebesgue measurable,

(ii) For any $(t,x)\in[0,T]\times H$, $g(t,x,\cdot)$ is continuous, and for any $(t,x_1,x_2,u)\in[0,T]\times H\times H\times U$,

$$\begin{cases}|g(t,x_1,u)-g(t,x_2,u)|+|h(x_1-h(x_2))|\leq C\,|x_1-x_2|_H,\\ |g(t,0,u)+|h(0)||\leq C\end{cases}\tag{5}$$

(A_3) The maps $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$, the functional $g(\cdot,\cdot,\cdot)$ and $h(\cdot)$ are C^1 with respect to x and u, and for any $(t,x,u)\in[0,T]\times H\times U$,

$$\begin{cases}|a_x(t,x,u)|_{L(H)}+|b_x(t,x,u)|_{L(H)}+|g_x(t,x,u)|_H+|h_x(x)|_H\leq C,\\ |a_u(t,x,u)|_{L(H_1;H)}+|b_u(t,x,u)|_{L(H_1;H)}+|g_u(t,x,u)|_{H_1}\leq C\end{cases}\tag{6}$$

When $\{A(t)\}_{t\in[0,T]}$ is a family of deterministic operators, the well-posedness of (1) in the sense of mild solution is well-understood[1].

When $\{A(t)\}_{t\in[0,T]}$ is a family of random operators, the corresponding random evolution system $S(t,s)$ is also random and F-adapted with respect to t. In this case, the well-posedness of (1) is usually understood in the sense of weak solution[2]. Note that the usual mild solution based on Itô integral does not make sense. Indeed, the stochastic process $S(t,s)b(s,x(s),u(s))$ may not F_s-measurable, and therefore the stochastic integral $\int_0^t S(t,s)b(s,x(s),u(s))\mathrm{d}B(s)$ is anticipative and the stochastic integral is interpreted as a Skorohod integral[3]. From Ref. [4], we know that the corresponding mild solution is not necessarily the weak solution of the SEEs, because a new complementary term appears. Hence, one introduces a new stochastic integral called "forward integral". The "forward integral" is defined as the limit of Riemann sums taking values of the process on the left point of each interval (see Ref. [5]) and one can show that the corresponding mild solution is the weak solution to (1). When $\{A(t)\}_{t\in[0,T]}$ is a family of random operators, we can get the mild solution to (1) for $p\geq 2$ (see Ref. [6] for the estimation for the stochastic integral).

In our paper, we will show the well-posedness of the corresponding BSEE, and the desired maximum principle, which is first order necessary conditions for the optimal control of the above Problem (OP).

2　Preliminaries

For $\varphi=a,b,g$, put $\varphi_1(t)=\varphi_x(t,\bar{x},\bar{u}),\varphi_2(t)=\varphi_u(t,\bar{x},\bar{u})$.

First, we need the following backward stochastic evolution equation (BSEE):

$$\begin{cases}\mathrm{d}y(t)=-A^*(t)y(t)\mathrm{d}t-(a_1^*(t)y(t)+b_1^*(t)Y(t))\mathrm{d}t+g_1(t)\mathrm{d}t+Y(t)\mathrm{d}B(t),t\in[0,T],\\ y(T)=-h_x(\bar{x}(T))\end{cases}\tag{7}$$

Here $y(T)\in L^q_{F_0}(\Omega;H)$ for $q\in(1,2]$. The study of BSEEs is stimulated by the classical works (see Refs. [7～9]), and it plays an important role in stochastic controls (see Refs. [10～15]). When A is an unbounded operator and the filtration is natural, one can get the well-posedness of the BSEEs by using the Martingale Representation Theorem (see Ref. [9]). When the filtration is the general filtration, we also get the well-posedness of BSEEs with random generators in the sense of mild solution is an unsolved problem at the moment, we adopt the method introduced in Ref. [16] to get the well-posedness of (7) with natural filtration.

Now, we define the transposition solution to (7). Consider the following SEE:

$$\begin{cases}\mathrm{d}z(s)=(A(s)z(s)v_1(s))\mathrm{d}s+v_2(s)\mathrm{d}B(s),s\in[t,T],\\ z(t)=\eta\end{cases}\tag{8}$$

where $\eta\in L^p_{F_t}(\Omega;H)$, $v_1\in L^1_F(t,T;L^p(\Omega;H))$, $v_2\in L^p_F(t,T;L^p(\Omega;\mathrm{L}_2^0))$. The solution to (8) is understood in the weak sense.

Definition 2.1　Let $p\geq 2$, $1<q\leq 2$, and $\frac{1}{p}+\frac{1}{q}=1$. We call $(y(\cdot),Y(\cdot))\in D_F([0,T];L^q(\Omega;H))\times L^q_F(0,T;\mathrm{L}_2^0)$ a transposition solution to (3) if for any $t\in[0,T]$, $\eta\in L^p_{F_t}(\Omega;H)$, $v_1\in L^1_F(t,T;L^p(\Omega;H))$, $v_2\in L^p_F(t,T;L^p(\Omega;\mathrm{L}_2^0))$, then

$$E\int_t^T\left\langle v_1(\tau),y(\tau)\right\rangle_H\mathrm{d}\tau+E\int_t^T\left\langle v_2(\tau),Y(\tau)\right\rangle_{\mathrm{L}_2^0}\mathrm{d}\tau=$$

$$E<z(T),y_T>_H - E<\eta,y(t)>_H + E\int_t^T <z(\tau),f(\tau)>_H \mathrm{d}\tau \qquad (9)$$

In this paper we have assumed that the filtration is natural. When $\{S(t)\}_{t\in[0,T]}$ is a C_0-semigroup generated by an unbounded operator A, it is deterministic and therefore the well-posedness of the corresponding BSEE follows from the Martingale Representation Theorem. However, when $\{A(s,\omega)\}$ is a family of random operators, the corresponding random evolution system $\{S(t,s), 0\le s\le t\le T\}$ is a family of random processes. Although $F=\{F_t\}_{t\in[0,T]}$ is a natural filtration generated by B, we cannot simply use the Martingale Representation Theorem obtain the mild solution to (7). This is why we use the transposition method (introduced in Ref. [16]), which avoids the use of the Martingale Representation Theorem.

In this paper, we will prove the well-posedness of the SEEs only for $p\in[2,\infty)$. We cannot get BDG-type inequality with respect to Skorohod integral for $p\in(1,2)$, the main reason is that the Skorohod integral is not the martingale. So according to duality, we only get the well-posedness of linear BSEEs for $1<q\le 2$.

We begin with some knowledge on Malliavin Calculus (see Ref. [17]).

Definition 2.2 An H-isonormal process on Ω is a mapping $W: H\to L^2(\Omega)$ with the following two properties:

(i) For all $h\in H$, the random variable $W(h)$ is Gaussian;

(ii) For all $h_1, h_2\in H$, we have $\mathrm{E}(W(h_1)W(h_2))=<h_1,h_2>_H$.

Definition 2.3 An $L^2(0,T;H)$-isonormal process is called an H-cylindrical Brownian motion on $[0,T]$.

If V_1 and V_2 are two real and separable Hilbert spaces, we will denote its tensor product by $V_1\otimes V_2$ which is isometric to the space $L_2(V_2;V_1)$ of Hilbert-Schmidt operators from V_2 to V_1.

Let K be a real and separable Hilbert space and $W(\cdot)$ be a V-cylindrical Brownian motion on $[0,T]$. For any $p\ge 2$ we can introduce the Sobolev space $D^{1,p}(K)$ of K-valued random variables in the following way. If F is a smooth K-valued random variable of the form

$$F=\sum_{i=1}^m f_i(W(v_1),\cdots W(v_m))b_i \qquad (10)$$

where $v_i\in L^2(0,T;V)$, $b_i\in K$ and $f_i\in C_b^\infty(\mathbf{R}^m)$ (f_i is an infinitely differentiable function such that f_i is bounded together with all its partial derivatives), then the derivative of the F is defined as

$$DF=\sum_{i=1}^m\sum_{j=1}^m \frac{\partial f_i}{\partial x_j}(W(v_1),\cdots W(v_m))b_i\otimes v_j \qquad (11)$$

So DF is a smooth random variable with values in $L^2(0,T;L_2^0(V;K))$. Then $D^{1,p}(K)$ is the completion of the class of smooth K-valued random variables, denoted by S_K, with respect to the norm

$$|F|_{1,p}^p = E|F|_K^p + E\left(\int_0^T |D_tF|^2_{L_2(V;K)}\mathrm{d}t\right)^{\frac{p}{2}} \qquad (12)$$

The derivative operator D is closable from $S_K\in L^p(\Omega;K)$ into the space $L^p(\Omega;L^2(0,T;L_2(V;K)))$ for each $p\ge 1$.

For any $n\ge 1$, the Sobolev space $\mathrm{D}^{n,p}(K)$ is defined as the completion of S_K by the norm

$$|F|_{n,p}^p = \sum_{i=1}^n E\left(\int_{[0,T]^i} |D_{t_1}\cdots D_{t_i}F|^2_{L_2(V^{\otimes i};K)}\mathrm{d}t_1\cdots\mathrm{d}t_i\right)^{\frac{p}{2}} + E|F|_K^p \quad (13)$$

Given two real and separable Hilbert spaces H and G, we can consider $K=L_2(H;G)$, and in this case, for any F in the space $D^{1,p}(L_2(H;G))$, we have

$$DF\in L^p(\Omega;L^2(0,T;L_2(H;L_2(V;G)))),$$

since

$$L_2(V;L_2(H;G))=L_2(H;L_2(V;G)).$$

Definition 2.4 Let $F\in L^2(\Omega;L(H;G))$, we say that F belongs to the Sobolev space $D^{1,2}(L_2(H;G))$ if the following conditions hold:

(i) For any $h\in H$, $F(h)$ belongs to $D^{1,2}(G)$;

(ii) There exists an element $DF\in L^2([0,T]\times\Omega;L(H;L_2(V;G)))$, such that for every $h\in H$, we have $D_t(F(h))=(D_tF)(h)$ for almost all

$(t,\omega)\in[0,T]\times\Omega$.

We use the notation $\Delta=\{(t,s)\in[0,T]^2: t\geq s\}$. Let us recall the notion of random evolution system[6].

Definition 2.5　A random evolution system is a family of random operators $\{S(s,t), 0\leq s\leq t\leq T\}$ on H verifying the following properties:

(i) $S(t,s)$ is F-adapted with respect to t for each $t\geq s$;

(ii) For each $\omega\in\Omega$, $\{S(t,s),(t,s)\in\Delta\}$ is an evolution system in the following sense:

(a) $S(s,s)=I$ and $S(t,r)=S(t,s)S(s,r)$ for any $0\leq r\leq s\leq t\leq T$,

(b) For any $h\in H$, $(t,s)\to S(t,s)h$ is continuous from Δ into H.

Let us introduce the following hypothesis on a given random evolution system:

(H_1) For each $(t,s)\in\Delta$, $S(t,s)\in D^{2,2}(L(H;H))$ and $\int_0^t |S(t,s)|_{2,p}^p ds<\infty$ for all $p\geq 2$;

(H_2) There is a version of $D_r S(t,s)$ such that for all $\omega\in\Omega$ and $h\in H$, the limit

$$D_s^- S(t,s)(h)=\lim_{\varepsilon\to 0^+} D_s S(t,s-\varepsilon)(h) \tag{14}$$

exists in L_2^0 and $D_s^- S(t,s)$ belongs to $D^{1,2}(L(H;L_2^0))$;

(H_3) There is a constant $M>0$ such that the following estimates hold for all $t\geq s\geq r$:

(H_{3a}) $|S(t,s)|_{L(H;H)}\leq M$,

(H_{3b}) $|D_s S(t,r)|_{L(H;L_2^0)}\leq M$,

(H_{3c}) $\sum_{i=1}^{\infty}|D_r(D_s^- S(t,s))e_i|^2_{L(H;L_2^0)}\leq M^2$.

Definition 2.6　We denote by δ_H the adjoint of the derivative operator D acting on $D^{1,2}(H)$. That is, the domain of δ_H is the space of processes u in $L^2([0,T]\times\Omega;L_2^0)$ such that

$$\left| E\int_0^T \langle D_t F, u_t\rangle_{L_2^0} dt \right| \leq C\,|F|_{L^2(\Omega;H)},\quad \forall F\in S_H$$

and

$$E\int_0^T \langle D_t F, u_t\rangle_{L_2^0} dt = E\langle F,\delta_H(u)\rangle_H,\quad \forall F\in D^{1,2}(H) \tag{15}$$

The operator δ_H is called the H-Skorohod integral, write $\delta_H(u)=\int_0^T u(s)dB(s)$.

Definition 2.7　Let $Y:[0,T]\times\Omega\to L_2^0$ be a measurable process such that $Y(v)\in L^1(0,T;H)$ a.s. for each $v\in V$. We say that Y elongs to $\mathrm{Dom}\delta^-$ if

$$Y^n := n\int_0^T \sum_{i=1}^{\infty} Y(s)(e_i)\left(B\left(\left(s+\frac{1}{n}\right)\wedge T\right)(e_i)-B(s)(e_i)\right)ds \tag{16}$$

converges in probability as n tends to infinity. The limit of the sequence $\{Y^n\}_{n=1}^{\infty}$ is denoted by $\int_0^T Y(s)dB(s)^-$ and is called the forward integrals of Y with respect to B.

Form Ref.[6], the relationship between Skorohod and forward integrals are as follows.

Lemma 2.8　Fix $p\geq 2$. Let $\Phi=\{\Phi(t), t\in[0,T]\}$ be a L_2^0-valued adapted process such that $E\int_0^T |\Phi(s)|_{L_2^0}^p ds<\infty$. Let $S(t,s)$ be a random evolution system satisfying the hypothesis (H_1), (H_2) and (H_3). Then for each $t\in[0,T]$, $\{S(t,s)\Phi(s)I_{[0,t]}(s), s\in[0,T]\}$ belongs to $\mathrm{Dom}\delta^-$ and

$$\int_0^t S(t,r)\Phi(r)dB(r)^- = \int_0^t S(t,r)dB(r) + \int_0^t \sum_{i=1}^{\infty}(D_r^- S(t,r))(e_i)\Phi(e_i)dr \tag{17}$$

Next, we need the following result.

Lemma 2.9[13]　Let H be a separable Hilbert space. Then, for any $\xi\in L_{F_T}^r(\Omega;H)$, $r\geq 1$ and $t\in[0,T)$, it holds that

$$\lim_{s\to t^+}|E(\xi|F_s)-E(\xi|F_t)|_{L_{F_T}^r(\Omega;H)}=0 \tag{18}$$

Remark　From Ref.[13], we not only get the right continuity of the conditional expectation with respect to the filtration, but also its left limit.

Let us recall the Itô formulas about the anticipating H-valued processes[6]. We use the notation $L^{k,p}(H)=L^p(0,T;\mathrm{D}^{k,p}(H))$ for any $k,p\geq 1$.

Lemma 2.10　Let $F\in C^2(H)$ and $X=\{X(t), t\in[0,T]\}$ be the stochastic process defined by

$$X(t)=X_0+\int_0^t \varphi(s)ds+\int_0^t \Phi(s)dB(s) \tag{19}$$

where we have the following conditions:

(i) $X_0\in \mathrm{D}^{1,2}(H)$;

(ii) $\varphi\in \mathrm{L}^{1,2}(H)$;

(iii) $\Phi\in \mathrm{L}^{2,4}(L_2^0)$,

then

$$F(X(t)) = F(X_0)+\int_0^t \langle F'(X(s)),\varphi(s)\rangle_H \mathrm{d}s + \int_0^t F'(X(s))\mathrm{d}B(s) + \frac{1}{2}\int_0^t \langle F''(X(s))(\nabla X)_s,\Phi(s)\rangle_{L_2^0}\mathrm{d}s \quad (20)$$

with

$$(\nabla X)_t = 2D_tX_0 + 2\int_0^t D_t\varphi(s)\mathrm{d}s + 2\int_0^t D_t\Phi(s)\mathrm{d}B(s)+\Phi(t) \quad (21)$$

3 Well-posedness of the vector-valued SEEs with random generators

In this section, we present the well-posedness result for the semi-linear SEEs.

Condition 3.1 Suppose that $F:[0,T]\times\Omega\times H\to H$ and $\widetilde{F}:[0,T]\times\Omega\times H\to L_2^0$ are two given functions satisfying

(i) Both $F(\cdot,x)$ *and* $\widetilde{F}(\cdot,x)$ are F-adapted for any $x\in H$;

(ii) There exist a $C\geq 0$ and for any $x,y\in H$ such that

$$|F(t,x)-F(t,y)|_H\leq C|x-y|_H,$$
$$|\widetilde{F}(t,x)-\widetilde{F}(t,y)|_{L_2^0}\leq C|x-y|_H \quad (22)$$

$\{A(s,\omega),s\in[0,T],\omega\in\Omega\}$ is a family of unbounded random operators on H such that $H_0\in \mathrm{Dom}\,A^*(s)$ where H_0 is a dense subset of H. Then exists a random evolution system $S(t,s)$ satisfying the hypotheses (H_1), (H_2) and (H_3) such that

$$S^*(t,s)A^*(t)y=\frac{\mathrm{d}}{\mathrm{d}t}S^*(t,s)y$$

for all $y\in H_0$.

Consider the following semi-linear SEEs:

$$\begin{cases}\mathrm{d}X(t)=[A(t)X(t)+F(t,X(t))]\mathrm{d}t+F(t,X(t))\mathrm{d}B(t),t\in[0,T],\\ X(0)=X_0\end{cases} \quad (23)$$

Definition 3.2 An adapted and continuous H-valued process $X=\{X(t),t\in[0,T]\}$ is called a mild solution to (23), if for any $t\in[0,T]$,

$$X(t) = S(t,0)X_0+\int_0^t S(t,s)F(s,X(s))\mathrm{d}s + \int_0^t S(t,s)\widetilde{F}(s,X(s))\mathrm{d}B(s)^-,\mathrm{P}-\mathrm{a.s} \quad (24)$$

where $\int_0^t(\cdot)\mathrm{d}B(s)^-$ denotes the forward integral.

We recall the following known result (see Ref. [11]).

Lemma 3.3 Fix $p\geq 2$. Let $\Phi=\{\Phi(t),t\in[0,T]\}$ be a L_2^0-valued adapted process such that $E\int_0^T|\Phi(s)|_{L_2^0}^p\mathrm{d}s<\infty$. Let $S(t,s)$ be a random evolution system satisfying the hypotheses (H_1), (H_2) and (H_3). Then for each $t\in[0,T]$, $\{S(t,s)\Phi(s)I_{[0,t]}(s),s\in[0,T]\}$ belongs to $\mathrm{Dom}\delta^-$ and

$$\sup_{t\in[0,T]}E\left|\int_0^t S(t,r)\Phi(r)\mathrm{d}B(r)^-\right|_H^p\leq CE\int_0^T|\Phi(s)|_{L_2^0}^p\mathrm{d}s \quad (25)$$

where $C>0$, which depends on T and the random evolution system $S(t,s)$.

The main result in this section is as follows.

Theorem 3.4 Fix $p\geq 2$. Let $S(t,s)$ be a random evolution system satisfying (H_1), (H_2) and (H_3), Condition 1 hold, $X_0\in L_{F_0}^p(\Omega;H)$, $F(\cdot,0)\in L_F^1(0,T;L^p(\Omega;H))$ and $\widetilde{F}(\cdot,0)\in L_F^p(0,T;L_2^0)$. Then the equation (23) admits an unique mild solution, and $X(\cdot)\in C_F([0,T];L^p(\Omega;H))$. Moreover,

$$|X(\cdot)|_{C_F([0,T];L^p(\Omega;H))}\leq C(|X_0|_{L_{F_0}^p(\Omega;H)}+|F(\cdot,0)|_{L_F^1(0,T;L^p(\Omega;H))}+|\widetilde{F}(\cdot,0)|_{L_F^p(0,T;L_2^0)}) \quad (26)$$

Proof The proof will be divided into two steps.

Step 1. We claim that the equation (23) exists a mild solution when $F(t,X(t))=f(t)\in L_F^1(0,T;L^p(\Omega;H))$, $\widetilde{F}(t,X(t))=\widetilde{f}(t)\in L_F^p(0,T;L_2^0)$. Clearly,

$$X(t)=S(t,0)X_0+\int_0^t S(t,s)f(s)\mathrm{d}s+\int_0^t S(t,s)\widetilde{f}(s)\mathrm{d}B(s)^- \quad (27)$$

is a mild solution to the equation (23). Now, we prove that $X(\cdot)\in C_F([0,T];L^p(\Omega;H))$. Indeed, some computations can yield that

$$|X(\cdot)|_{L_F^\infty([0,T];L^p(\Omega;H))}\le C(|X_0|_{L^p_{F_0}(\Omega;H)}+|f(\cdot)|_{L^1_F(0,T;L^p(\Omega;H))}+|\widetilde{F}(\cdot)|_{L^p_F(0,T;L_2^0)}) \tag{28}$$

Since $S(t,0)X_0+\int_0^t S(t,s)f(s)\,\mathrm{d}s$ is H-valued continuous with respect to t, and it suffices to prove the continuity of $\int_0^t S(t,s)\widetilde{f}(s)\,\mathrm{d}B(s)^-$ in $L^p_{F_T}(\Omega;H)$. We can obtain that

$$\lim_{t\to t_0}|X(t)-X(t_0)|_{L^p_{F_T}(\Omega;H)}=0,t_0\in[0,T] \tag{29}$$

Hence, there have $X(\cdot)\in C_F([0,T];L^p(\Omega;H))$.

Step 2. Choose $T_1\in(0,T]$. For any $Y(\cdot)\in C_F([0,T];L^p(\Omega;H))$, we consider the following equation:

$$\begin{cases}\mathrm{d}X(t)=[A(t)X(t)+F(t,X(t))]\mathrm{d}t+F(t,X(t))\mathrm{d}B(t),t\in[0,T_1],\\ X(0)=X_0\end{cases} \tag{30}$$

Let $J:C_F([0,T];L^p(\Omega;H))\to C_F([0,T];L^p(\Omega;H))$, and $J(Y)=X$. By some lengthy and technique computations, one can show that J is a contractive map when T_1 is small enough. By means of the Banach fixed point theorem, there exists a unique $X(\cdot)\in C_F([0,T_1];L^p(\Omega;H))$ such that $J(X)=X$. So we can see that $X(\cdot)$ is a mid solution to the equation to (23). The uniqueness of such solution to (23), and (26) holds when $T=T_1$.

Repeating the above argument, we obtain a mild solution to the equation (23). The uniqueness of such solution to (23) and (26) are obvious, The poof of this theorem is complete.

4　Well-posedness of the vector-valued BSEEs with random generators

In this section, we present the well-posedness result for the BSEEs.

Before proving the well-posedness of the BSEEs, we recall the following known result (see Ref. [10]).

Lemma 4.1　Assume that $r\in(1,+\infty)$,

$$r'=\frac{r}{r-1},\ \alpha'\in[1,+\infty),$$

$$\alpha'=\begin{cases}\dfrac{\alpha}{\alpha-1} & if\quad \alpha\in(1,+\infty],\\ \infty & if\quad \alpha=1,\end{cases}$$

$f_1\in L_F^r(0,T;L^\alpha(\Omega;H))$, $f_2\in L_F^{r'}(0,T;L^{\alpha'}(\Omega;H))$. Then there exists a monotonic sequence $\{h_n\}_{n=1}^\infty$ of positive numbers such that $\lim_{n\to\infty}h_n=0$, and for almost all $t\in[0,T]$,

$$\lim_{n\to\infty}\frac{1}{h_n}\int t+h_n\,{}_tE<f_1(s),f_2(s)>_H\mathrm{d}s=E<f_1(s),f_2(s)>_H \tag{31}$$

Let us consider the following linear BSEE. The equation is as follows:

Theorem 4.2　Assume $q\in(1,2]$. Then the equation (8) admits one and only one transposition solution $(y(\cdot),Y(\cdot))\in D_F([0,T];L^q(\Omega;H))\times L_F^q(0,T;L_2^0)$. Further,

$$|(y(\cdot),Y(\cdot))|_{D_F([0,T];L^q(\Omega;H))\times L_F^q(0,T;L_2^0)}\le C(|y_T|_{L^q_{L_T}(\Omega;H)}+|f(\cdot)|_{L^1_F(0,T;L^q(\Omega;H))}) \tag{32}$$

Proof　We borrow some ideas from Ref. [8]. The proof is divided into four steps.

Step 1. For any $t\in[0,T]$, we define a linear functional l on the Banach space $L_F^1(t,T;L^p(\Omega;H))\times L_F^p(t,T;L_2^0)\times L^p_{F_t}(\Omega;H)$ as follows:

$$\ell(v_1(\cdot),v_2(\cdot),\eta)=E<z(T),y_T>_H+E\int_t^T<z(s),f(s)>_H\mathrm{d}s,$$

where $z(\cdot)\in C_F([t,T];L^p(\Omega;H))$ is a mild solution to (9). Then, some lengthy computations yield that ℓ is a bounded linear functional on $L_F^1(t,T;L^p(\Omega;H))\times L_F^p(t,T;L_2^0)\times L^p_{F_t}(\Omega;H)$. According to a representation theorem in Ref. [7], there exist $(y^t(\cdot),Y^t(\cdot),\xi^t)\in L_F^\infty(0,T;L^q(\Omega;H))\times L_F^q(0,T;L_2^0)\times L^q_{F_t}(\Omega;H)$ such that

$$E<z(T),y_T>_H+E\int_t^T<z(s),f(s)>_H\mathrm{d}s=E\int_t^T<v_1(\tau),y^t(\tau)>_H\mathrm{d}\tau+E\int_t^T<v_2(\tau),Y^t(\tau)>_{L_2^0}\mathrm{d}\tau+E<\eta,\xi^t>_H \tag{33}$$

Further, there is a positive constant $C=C(T)$, independent of t, such that

$$|(y^t(\cdot),Y^t(\cdot),\xi^t)|_{L_F^\infty(0,T;L^q(\Omega;H))\times L_F^q(0,T;L_2^0)\times L_{F_t}^q(\Omega;H)}\leq C(|y_T|_{L_{F_T}^q(\Omega;H)}+|f(\cdot)|_{L_F^1(0,T;L^q(\Omega;H))}),\ t\in[0,T] \quad (34)$$

Step 2. According to the Step 1, $(y^t(\cdot),Y^t(\cdot))$ may depend on t. We further show that $(y^t(\cdot),Y^t(\cdot))$ is independent on t by some lengthy computations, that is, for any t_1 and t_2 with $0\leq t_2\leq t_1\leq T$, and a. e. $(t,\omega)\in[t_1,T]\times\Omega$, it holds that

$$(y^{t_1}(\cdot),Y^{t_1}(\cdot))=(y^{t_2}(\cdot),Y^{t_2}(\cdot)).$$

Step 3. In this step, we need to prove that ξ^t is càdlàg with respect to t in $L_{F_T}^q(\Omega;H)$. We can show that

$$\xi^t=E(S^*(T,t)y_T+\int_t^T S^*(s,t)f(s)\mathrm{d}s|F_t) \quad (35)$$

Firstly, we start to prove the right continuity of (35) with respect to t. For any fixed $t\in[0,T)$ with $t\leq s$. According to Lemma 2.7, we can get that

$$\lim_{s\to t^+}|\xi^{t_1}-\xi^{t_2}|_{L_{F_T}^q(\Omega;H)}=0 \quad (36)$$

Similarly, we can get that $\forall\varepsilon>0$, there exists a $\delta>0$ such that $\forall t_1,t_2\in(t-\delta,t)$,

$$|\xi^{t_1}-\xi^{t_2}|_{L_{F_T}^q(\Omega;H)}\leq\varepsilon \quad (37)$$

Hence, ξ^t is càdlàg with respect to t in $L_{F_T}^q(\Omega;H)$.

Step 4. Let $0\leq t_1\leq t_2\leq T$, we have

$$E<\gamma,\xi^{t_2}>_H=E<S(T,t_2)\gamma,y_T>_H+E\int_{t_2}^T<S(\tau,t_2)\gamma,f(\tau)>_H\mathrm{d}\tau+\frac{1}{t_1-t_2}E\int_{t_2}^T<\chi_{[t_2,t_1]}(\tau)\gamma,y(\tau)>_H\mathrm{d}\tau-\frac{1}{t_1-t_2}E<\int_{t_2}^T S(T,\tau)\chi_{[t_2,t_1]}(\tau)\gamma\mathrm{d}\tau,y_T>_H-\frac{1}{t_1-t_2}E\int_{t_2}^T<\int_{t_2}^\tau S(\tau,r)\chi_{[t_2,t_1]}(r)\gamma\mathrm{d}r,f(\tau)>_H\mathrm{d}\tau \quad (38)$$

Then, $\xi^{t_2}=y(t_2)$, P−a. s. Furthermore,

$$E<z(T),y_T>_H+E\int_t^T<z(s),f(s)>_H\mathrm{d}s=E\int_t^T<v_1(\tau),y(\tau)>_H\mathrm{d}\tau+E\int_t^T<v_2(\tau),Y(\tau)>_{L_2^0}\mathrm{d}\tau+E<\eta,y(t)>_H \quad (39)$$

Finally, we get that $(y(\cdot),Y(\cdot))$ is a transposition solution to (8), and (32) holds. The proof of this theorem is complete.

5 Necessary condition of optimal controls for the case of convex control domain

In this section, we shall give a necessary condition for optimal control problems. The main methods come from the Ref. [8].

For the optimal pair $(\bar{x}(\cdot),\bar{u}(\cdot))$, fix a $u(\cdot)\in U[0,T]$ with

$$\delta u(\cdot)\triangleq u(\cdot)-\bar{u}(\cdot)\in L_F^p(0,T;H_1) \quad (40)$$

Consider the following equation:

$$\begin{cases}\mathrm{d}x_2(t)=[A(t)x_2(t)+a_1(t)x_2(t)+a_2(t)\delta u(t)]\mathrm{d}t+[b_1(t)x_2(t)+b_2(t)\delta u(t)]\mathrm{d}B(t),t\in U[0,T],\\ x_2(0)=0\end{cases} \quad (41)$$

In order to get the pointwise-type maximum principle, similar to Ref. [8], we need the following result.

Lemma 5.1 Assume that H_1 is a Hilbert space, $p\geq2$ and $1<q\leq2$ meet $\frac{1}{p}+\frac{1}{q}=1$, and U is a nonempty subset of H_1. If $F(\cdot)\in L_F^q(t,T;H_1)$, and $\bar{u}(\cdot)\in U[0,T]$ such that

$$E\int_0^T<F(\cdot),u(t,\cdot)-\bar{u}(t,\cdot)>_{H_1}\mathrm{d}t\leq0 \quad (42)$$

holds for any $u(\cdot)\in U[0,T]$ satisfying (40), then for any point $u\in U$, the following pointwise inequality holds:

$$<F(t,\omega),u-\bar{u}(t,\omega)>_{H_1}\leq0,$$
$$\text{a. e. } (t,\omega)\in[0,T]\times\Omega \quad (43)$$

Theorem 5.2 Suppose that $p\geq2$, $x_0\in L_{F_0}^p(\Omega;H)$, and U is convex. Let the assumptions (A_1), (A_2) and (A_3) hold. If $(\bar{x}(\cdot),\bar{u}(\cdot))$ is an optimal pair, then

$$\langle a_u(t,\bar{x}(\cdot),\bar{u}(\cdot))^* y(t) + b_u(t,\bar{x}(\cdot),\bar{u}(\cdot))^* Y(t) - g_u(t,\bar{x}(\cdot),\bar{u}(\cdot)), u-\bar{u}(t)\rangle_{H_1} \le 0,$$
$$\text{a. e. } (t,\omega)\in[0,T]\times\Omega,\ \forall u\in U \qquad (44)$$

Proof We use the convex perturbation technique and divide the proof into two steps.

Step 1. For $u(\cdot)$ given by (40), since U is convex, we see that $u^\varepsilon(\cdot)=(1-\varepsilon)\bar{u}(\cdot)+\varepsilon u(\cdot)\in U[0,T]$.

Let $x^\varepsilon(\cdot)$ be the state of (1) with the control being $u^\varepsilon(\cdot)$, and

$$x_1^\varepsilon(\cdot)=\frac{1}{\varepsilon}(x^\varepsilon(\cdot)-\bar{x}(\cdot)).$$

Put $x_3^\varepsilon(\cdot)=x_1^\varepsilon(\cdot)-x_2(\cdot)$. Then, $x_3^\varepsilon(\cdot)$ solves the following equation:

$$\begin{cases} dx_3^\varepsilon(t)=[A(t)x_3^\varepsilon(t)+a_1^\varepsilon(t)x_3^\varepsilon(t)+(a_1^\varepsilon(t)-a_1(t))x_2(t)+a_2^\varepsilon(t)-a_2(t)\delta u(t)]dt+ \\ \qquad [b_1^\varepsilon(t)x_3^\varepsilon(t)+(b_1^\varepsilon(t)-b_1(t))x_2(t)+(b_2^\varepsilon(t)-b_2(t))\delta u(t)]dB(t),t\in[0,T], \\ x_3^\varepsilon(0)=0 \end{cases} \qquad (45)$$

where

$$\begin{cases} \varphi_1^\varepsilon(t)=\int_0^1 \varphi_x(t,\bar{x}(t)+\sigma\varepsilon x_1^\varepsilon(t),u^\varepsilon(t))d\sigma, \\ \varphi_2^\varepsilon(t)=\int_0^1 \varphi_u(t,\bar{x}(t),\bar{u}+\sigma\varepsilon\delta u(t))d\sigma \end{cases} \qquad (46)$$

Further,

$$\lim_{\varepsilon\to 0^+} |x_1^\varepsilon(\cdot)-x_2(\cdot)|_{L_F^\infty(0,T;L^p(\Omega;H))=0} \qquad (47)$$

Step 2. Since $(\bar{x}(\cdot),\bar{u}(\cdot))$ is the optimal pair of Problem (OP), we find that

$$0\le \lim_{\varepsilon\to 0^+}\frac{J(u^\varepsilon(\cdot))-J(\bar{u}(\cdot))}{\varepsilon}= E\int_0^T (\langle g_1(t,\bar{x}(t),\bar{u}(t)),x_2(t)\rangle_H + \langle g_2(t,\bar{x}(t),\bar{u}(t)),\delta u(t)\rangle_{H_1})dt + E\langle h_x(\bar{x}(T)),x_2(T)\rangle_H \qquad (48)$$

Since $(y(\cdot),Y(\cdot))$ is a transposition solution to (7), for any $u(\cdot)\in U[0,T]$ satisfying (40), we deduce that

$$E\int_0^T \langle a_2^*(t)y(t)+b_2^*(t)Y(t)-g_2(t,\bar{x}(t),\bar{u}(t)),u(t)-\bar{u}(t)\rangle_{H_1} dt\le 0 \qquad (49)$$

Hence

$$\langle a_2^*(t)y(t)+b_2^*(t)Y(t)-g_2(t,\bar{x}(t),\bar{u}(t)), u-\bar{u}(t)\rangle_{H_1}\le 0, (t,\omega)\in[0,T]\times\Omega,\forall u\in U \qquad (50)$$

The proof is complete.

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