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二阶非线性积分边值问题正解的存在唯一性

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摘　要：本文运用双度量空间中的广义 Krasnoselskii's 压缩不动点定理研究了二阶非线性积分边值问题 $u''+a(t)f(t,u(t),u'(t))=0$，$t\in(0,1)$，$u(0)=0$，$u(1)=\alpha\int_0^\eta u(s)\mathrm{d}s$ 正解的存在唯一性，其中 $0<\eta<1,0<\alpha<\frac{2}{\eta^2},a\in C([0,1],[0,\infty))$，$f:[0,1]\times[0,\infty)\times\mathbf{R}\to[0,\infty)$ 连续，且当 $t_0\in[\eta,1]$ 时 $a(t_0)>0$.

关键词：正解；存在唯一性；积分边值问题；广义压缩不动点定理
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Existence and uniqueness of positive solutions for second-order nonlinear integral boundary value problem

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Abstract: In this paper, by using the method of generalized Krasnoselskii's contractive fixed point theorem in bimetric spaces, we study the existence and uniqueness of the positive solutions for the following-second-order nonlinear integral boundary value problem $u''+a(t)f(t,u(t),u'(t))=0$, $t\in(0,1)$, $u(0)=0$, $u(1)=\alpha\int_0^\eta u(s)\mathrm{d}s$ where $0<\eta<1,0<\alpha<\frac{2}{\eta^2},a\in C([0,1],[0,\infty))$, $f:[0,1]\times[0,+\infty)\times\mathbf{R}\to[0,+\infty)$ is continuous and $a(t_0)>0$ when $t_0\in[\eta,1]$.

Keywords: Positive solution; Existence and uniqueness; Integral boundary value problem; Generalized contractive fixed point theorem
(2010, MSC 26A33)

1　引　言

许多实际问题可归结为常微分方程边值问题. 近年来,很多学者对带有一般边值条件[1-6]和积分边值条件[7,8]的常微分方程边值问题进行了研究.

2008 年,文献[9]获得了二阶三点边值问题

$$x''(t)+f(t,x)=0,\ t\in(0,1),$$
$$x'(0)=0,\ x(1)=\delta x(\eta)$$

解的存在唯一性,其中 $f\in C(I\times\mathbf{R},\mathbf{R})$, $I=[0,1]$, $0<\eta<1,\delta>0$. 2008 年,王和刘[10]讨论了二阶三点边值问题

$$-u''=b(t)f(u(t)),\ t\in(0,1),$$
$$u'(0)=0,\ u(1)=\alpha u(\eta)$$

正解的存在性与多重性,其中常数 $\alpha,\eta\in(0,1)$, $f\in C((0,\infty),(0,\infty))$, $b\in C([0,1],[0,\infty))$,且存在

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$t_0\in[0,1]$使得 $b(t_0)>0$. 2010 年,姚[11]研究了二阶非线性积分边值问题

$$u''+a(t)f(t,u(t))=0,\ t\in(0,1),$$
$$u(0)=0,\ u(1)=\alpha\int_0^\eta u(s)\mathrm{d}s$$

正解的存在性,其中 $0<\eta<1$,$0<\alpha<\frac{2}{\eta^2}$,$\alpha\in C([0,1],[0,\infty))$,且存在 $t_0\in[\eta,1]$使得 $a(t_0)>0$. 2014 年, Berzig[12]介绍了广义 Krasnoselskii's 压缩不动点定理并证明了两点边值问题

$$u''=f(t,u(t)),\ t\in I,$$
$$u(a)=\alpha,\ u(b)=\beta$$

解的存在性,这里 $I=[a,b]$,$0<a<b<\infty$,且 $f:I\times\mathbf{R}\to\mathbf{R}$.

本文主要运用双度量空间中广义 Krasnoselskii's 压缩不动点定理研究二阶非线性积分边值问题

$$\begin{cases}u''+a(t)f(t,u(t),u'(t))=0,\ t\in(0,1) & (1)\\ u(0)=0,\ u(1)=\alpha\int_0^\eta u(s)ds & (2)\end{cases}$$

正解的存在唯一性. 与文献[11]相比,本文方法不同且非线性项 f 中含一阶导数项 u'.

2 预备知识

为证明问题(1)-(2)正解的存在唯一性,本文假设:

(H1) f 是连续的;

(H2) $|f(t,u,u')-f(t,v,v')|\le\frac{2(2-\alpha\eta^2)^2}{\pi(2\alpha\eta^2+1)}\arctan[(|u-v|)+(|u'-v'|)]$, $\forall u,v,u',v'\in C^1[0,1]$.

记

$$d(u,v)=\max(|u(t)-v(t)|+|u'(t)-v'(t)|),t\in[0,1],$$
$$\delta(u,v)=\frac{1}{2}d(u,v),u,v,u',v'\in C^1[0,1],$$

θ 是 $\theta:[0,\infty)\to[0,\infty)$的辅助函数族,对每个 $t>0$ 都有$\lim_{n\to\infty}\theta^n(t)=0$. 这里 $\theta^n(t)$是 θ 的 n 次迭代,且对每个 $\theta\in\theta$,当 $t>0$ 时都有 $0<\theta(t)<t$.

定义 2.1[12] 设(X,d)是一个度量空间,且 $T:X\to X$ 是一个映射. 如果任意 $0<a<b<\infty$,存在 $\theta_{(a,b)}\in\theta$,有

$$\forall x,y\in X,a\le d(x,y)\le b\Rightarrow$$
$$d(Tx,Ty)\le\theta_{a,b}(d(x,y)),$$

则 T 是一个广义 Krasnoselskii's 压缩映射.

定理 2.2[12] 设(X,δ,d)是一个双度量空间,定义算子 $T:X\to X$,当满足下列条件:

(i) X 中的每个 d-Cauchy 列也是 δ-Cauchy 列;

(ii) X 关于 δ 是完备的;

(iii) T 是广义的 Krasnoselskii's 压缩映射,

则 T 在 X 中存在唯一的不动点.

引理 2.3[11] 设 $\alpha\eta^2\ne2$,$p(t)\in C[0,1]$,则问题

$$\begin{cases}u''+p(t)=0,\ t\in(0,1) & (3)\\ u(0)=0,\ u(1)=\alpha\int_0^\eta u(s)\mathrm{d}s & (4)\end{cases}$$

有解

$$u(t)=-\int_0^t(t-s)p(s)\mathrm{d}s-\frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta(\eta-s)^2p(s)\mathrm{d}s+\frac{2t}{2-\alpha\eta^2}\int_0^1(1-s)p(s)\mathrm{d}s.$$

由引理 2.3 知,问题(1)-(2)有解

$$u(t)=-\int_0^t(t-s)a(s)f(s,u(s),u'(s))\mathrm{d}s-\frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta(\eta-s)^2a(s)f(s,u(s),u'(s))\mathrm{d}s+\frac{2t}{2-\alpha\eta^2}\int_0^1(1-s)a(s)f(s,u(s),u'(s))\mathrm{d}s. \quad (5)$$

定义算子 $T:C^1[0,1]\to C^1[0,1]$,

$$(Tu)(t)=-\int_0^t(t-s)a(s)f(s,u(s),u'(s))\mathrm{d}s-\frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta(\eta-s)^2a(s)f(s,u(s),u'(s))\mathrm{d}s+\frac{2t}{2-\alpha\eta^2}\int_0^1(1-s)a(s)f(s,u(s),u'(s))\mathrm{d}s.$$

引理 2.4 T 是广义 Krasnoselskii's 压缩算子.

证明 对任意的 $u,v,u',v,\in C^1[0,1]$,考虑

$$|(Tu)(t)-(Tv)(t)|=$$
$$\left|\int_0^1G(t,s)[f(s,u(s),u'(s))-f(s,v(s),v'(s))]\mathrm{d}s\right|\le$$
$$\int_0^1G(t,s)|[f(s,u(s),u'(s))-f(s,v(s),v'(s))]|\mathrm{d}s\le$$
$$\int_0^1G(t,s)\mathrm{d}s\int_0\frac{2(2-\alpha\eta^2)^2}{\pi(2\alpha\eta^2+1)}\arctan(|u(s)-v(s)|+|u'(s)-v'(s)|)\mathrm{d}s\le$$

$$\frac{2(2-\alpha\eta^2)^2}{\pi(2\alpha\eta^2+1)}\frac{2\alpha\eta^2+1}{2(2-\alpha\eta^2)}\arctan(d(u,v))\leq$$
$$\frac{(2-\alpha\eta^2)}{\pi}\arctan(d(u,v)),$$

其中,很容易得到,对 $\forall t\in(0,1)$,有

$$\int_0^1 G(t,s)\mathrm{d}s=\max\left\{\frac{1}{2(2-\alpha\eta^2)},\frac{2\alpha\eta^2+1}{2(2-\alpha\eta^2)}\right\}=$$
$$\frac{2\alpha\eta^2+1}{2(2-\alpha\eta^2)}.$$
$$\max|(Tu)(t)-(Tv)(t)|\leq$$
$$\frac{2(2-\alpha\eta^2)^2}{\pi(2\alpha\eta^2+1)}\arctan(d(u,v)),$$

其中

$$\theta_{0,1}(d(u,v))=\frac{2(2-\alpha\eta^2)^2}{\pi(2\alpha\eta^2+1)}\arctan(d(u,v)).$$

因此

$$d(Tu,Tv)\leq\theta_{0,1}(d(u,v)),$$
$$|(Tu)'(t)-(Tv)'(t)|=$$
$$\left|\int_0^1 G_t(t,s)[f(s,u(s),u'(s))-f(s,v(s),v'(s))]\mathrm{d}s\right|\leq$$
$$\int_0^1 G_t(t,s)\mathrm{d}s\int_0^1\left|[f(s,u(s),u'(s))-f(s,v(s),v'(s))]\right|\mathrm{d}s\leq$$
$$\frac{2(2-\alpha\eta^2)}{\pi(2\alpha\eta^2+1)}\arctan(d(u,v))\cdot\max\int_0^1 G_t(t,s)\mathrm{d}s\leq$$
$$\frac{2(2-\alpha\eta^2)}{\pi(2\alpha\eta^2+1)}\frac{2}{2-\alpha\eta^2}\arctan(d(u,v))\leq$$
$$\frac{4(2-\alpha\eta^2)}{\pi(2\alpha\eta^2+1)}\arctan(d(u,v)),$$

其中

$$\theta_{0,1}(d(u,v))=\frac{4(2-\alpha\eta^2)}{\pi(2\alpha\eta^2+1)}\arctan(d(u,v)).$$

因此

$$d((Tu)'-(Tv)')\leq\theta_{0,1}(d(u,v)).$$

由上得 T 是 $C^1[0,1]$上的广义 Krasnoselskii's 压缩算子.

3 主要结果

定理 3.1 若(H1)-(H2)成立,则(1)-(2)存在唯一的不动点 $x^*\in C^1[0,1]$.

证明 首先构造序列$\{f(x_n)\}$. 设 $f(x)\in C^1[0,1]$,$f(x_1)=T(f(x_0))$. 如果 $f(x_1)=f(x_0)$,结论得证. 如果 $f(x_1)\neq f(x_0)$,则对充分大的整数 n_0,当$\frac{1}{n_0}\leq d(f(x_0),f(x_1))\leq n_0$ 时,由定义2.1有

$$d(f(x_1),f(x_0))=$$
$$d(T(f(x_0)),T(f(x_1)))\leq$$
$$\theta_{\frac{1}{n_0},n_0}(d(f(x_0),f(x_1)))<$$
$$d(f(x_0),f(x_1)).$$

设 $f(x_2)=T(f(x_1))$. 如果 $f(x_2)=f(x_1)$,则 $T(f(x_1))=f(x_1)$,结论得证. 如果 $f(x_2)\neq f(x_1)$,则对充分大的整数 n_1,当$\frac{1}{n_1}\leq d(f(x_1),f(x_2))\leq n_1$ 时,由定义 2.1 有

$$d(f(x_2),f(x_3))=$$
$$d(T(f(x_1)),T(f(x_2)))\leq$$
$$\theta_{\frac{1}{n_1},n_1}(d(f(x_1),f(x_2)))<$$
$$d(f(x_1),f(x_2)).$$

继续上述过程,假设对所有的 $n\geq 0$,当 $f(x_{n+1})\neq f(x_n)$时可得序列$\{f(x_n)\}$. 则对所有的 $k>0$ 及充分大的整数 n_k,当$\frac{1}{n_k}\leq d(f(x_{k+1}),f(x_k))\leq n_k$ 时,由定义 2.1 有

$$d(f(x_{k+1}),f(x_k))=$$
$$d(T(f(x_k)),T(f(x_{k-1})))\leq$$
$$\theta_{\frac{1}{n_k},n_k}(d(f(x_k),f(x_{k-1})))<$$
$$d(f(x_k),f(x_{k-1})).$$

因此可知非负序列$\{d(f(x_n),f(x_{n+1}))\}$是非减的. 所以

$$\lim_{n\to\infty}d(f(x_n),f(x_{n+1}))=\varepsilon\geq 0 \tag{6}$$

下证$(C^1[0,1],\delta)$是完备的. 需证$\{f(x_n)\}$关于 δ 是 Cauchy 列. 我们先证$\{f(x_n)\}$关于 d 是 Cauchy 列. 要证式(6)中 $\varepsilon=0$ 反设 $\varepsilon>0$. 则对充分大的整数 N,有

$$\varepsilon\leq d(f(x_{N+k-1}),f(x_{N+k}))\leq$$
$$\varepsilon+1,k=1,2,\cdots.$$

由定义 2.1 有

$$d(f(x_{N+k}),f(x_{N+k+1}))=$$
$$d(T(f(x_{N+k-1})),T(f(x_{N+k})))\leq$$
$$\theta_{\varepsilon,\varepsilon+1}(d(f(x_{N+k-1}),f(x_{N+k}))),k=1,2,\cdots,$$
$$d(f(x_{N+k-1}),f(x_{N+k}))=$$
$$d(T(f(x_{N+k-2})),T(f(x_{N+k-1})))\leq$$
$$\theta_{\varepsilon,\varepsilon+1}(d(f(x_{N+k-2}),f(x_{N+k-1}))),k=1,2,\cdots.$$

所以

$$d(f(x_{N+k}),f(x_{N+k+1}))\leq$$
$$\theta^2_{\varepsilon,\varepsilon+1}(d(f(x_{N+k-2}),f(x_{N+k-1}))).$$

继续上述过程得

$$d(f(x_{N+k}),f(x_{N+k+1}))\leq$$
$$\theta^k_{\varepsilon,\varepsilon+1}(d(f(x_N),f(x_{N+1})))\leq$$

$\theta^k_{\varepsilon,\varepsilon+1}(\varepsilon+1), k=1,2,\cdots$.

又由于$\lim\limits_{k\to\infty}\theta^k_{\varepsilon,\varepsilon+1}(\varepsilon+1)=0$,所以$\lim\limits_{k\to\infty}(d(f(x),f(x_{N+1})))=0$. 矛盾!

设对任意的$\varepsilon>0$,存在$N>0$,使得

$$d(f(x_N),f(x_{N+1}))\le\frac{\varepsilon}{2}-\frac{1}{2}\theta_{\frac{\varepsilon}{2},\varepsilon}(\varepsilon) \tag{7}$$

令$B=\{f(x)\in C^1[0,1],d(f(x),f(x_N))\le\varepsilon\}$. 下证$T(B)\subseteq B$. 分两种情况进行讨论:

(i) 若$d(f(x),f(x_N))\le\frac{\varepsilon}{2}$,则对充分大的整数$L$有$\frac{1}{L}d(f(x),f(x_N))\le\varepsilon$. 则

$$\begin{aligned}&d(T(f(x),f(x_N)))\le\\&\quad d(T(f(x),f(x_{N-1})))+\\&\quad d(T(f(x_{N-1}),f(x_N)))\le\\&\quad \theta_{\frac{1}{L},\frac{\varepsilon}{2}}(d(f(x),f(x_N))+\\&\quad d(f(x_N),f(x_{N+1})))\le\\&\quad \theta_{\frac{1}{L},\frac{\varepsilon}{2}}(\frac{\varepsilon}{2})+\frac{\varepsilon}{2}+\frac{1}{2}\theta_{\frac{\varepsilon}{2},\varepsilon}(\varepsilon)\le\\&\quad \frac{\varepsilon}{2}+\frac{\varepsilon}{2}-\frac{1}{2}\theta_{\frac{\varepsilon}{2},\varepsilon}(\varepsilon)\le\varepsilon.\end{aligned}$$

(ii) 若$\frac{\varepsilon}{2}\le d(f(x),f(x_N))\le\varepsilon$,有

$$\begin{aligned}&d(T(f(x),f(x_N)))\le\\&\quad d(T(f(x),f(x_{N-1})))+\\&\quad d(T(f(x_{N-1}),f(x_N)))\le\\&\quad \theta_{\frac{\varepsilon}{2},\varepsilon}(d(f(x),f(x_{N-1}))+\\&\quad d(f(x_N),f(x_{N+1})))\le\\&\quad \theta_{\frac{\varepsilon}{2},\varepsilon}(\varepsilon)+\frac{\varepsilon}{2}-\frac{1}{2}\theta_{\frac{\varepsilon}{2},\varepsilon}(\varepsilon)\le\\&\quad \frac{\varepsilon}{2}+\frac{\varepsilon}{2}-\frac{1}{2}\theta_{\frac{\varepsilon}{2},\varepsilon}(\varepsilon)\le\varepsilon.\end{aligned}$$

所以$T(B)\subseteq B$. 由式(7)可得$f(x_{N+1})\in B$, $f(x_{N+2})\in B,\cdots$,即当$n>N$时,$f(x_n)\in B$. 从而证得$\{f(x_n)\}$是d-Cauchy 列. 进而$(C^1[0,1],d)$是完备度量空间,且$(C^1[0,1],\delta)$也是完备度量空间.

由式(6)可得问题(1)-(2)的 Green 函数

$$G(t,s)=\begin{cases}\dfrac{2t(1-s)}{2-\alpha\eta^2},\ 0\le t\le\eta\le s\le1,\\ \dfrac{2t(1-s)}{2-\alpha\eta^2}-\dfrac{\alpha t(\eta-s)^2}{2-\alpha\eta^2}-(t-s),\ 0\le s\le t,\eta\le1,\\ \dfrac{2t(1-s)}{2-\alpha\eta^2}-\dfrac{\alpha t(\eta-s)^2}{2-\alpha\eta^2},\ 0\le t\le s\le\eta\le1.\end{cases}$$

由引理 2.4 和定理 2.2,T在$C^1[0,1]$上存在不动点,即存在$x^*\in C^1[0,1]$,使得$x^*=Tx^*$.

最后,证T是唯一的不动点. 假设存在$y^*\in C^1[0,1]$, $y^*\ne x^*$, $Tx^*=x^*$, $Ty^*=y^*$. 则存在充分大的n,使得当$\frac{1}{n}\le d(x^*,y^*)\le n$时,

$$\begin{aligned}&d(x^*,y^*)=d(Tx^*,Ty^*)\le\\&\quad \theta_{\frac{1}{n},n}(d(x^*,y^*))<d(x^*,y^*).\end{aligned}$$

矛盾. 因此$d(x^*,y^*)=0$,即$x^*=y^*$. 证毕.

4 例 子

考虑二阶非线性积分边值问题

$$u''+\left[\frac{1}{80}+\frac{t^{\frac{1}{2}}(\frac{4}{3}-t)\frac{1}{2}}{20}\right][\ln(u+1)+u'']=0,$$

$$t\in(0,1), t\in(0,1), u(0)=0, u(1)=6\int_0^{\frac{1}{2}}u(s)\mathrm{d}s.$$

令

$$f(t,u,u')=\left[\frac{1}{80}+\frac{t^{\frac{1}{2}}(\frac{4}{3}-t)\frac{1}{2}}{20}\right][\ln(u+1)+u''].$$

容易证得f连续且满足

$$|f(t,u,u')-f(t,v,v')|\le\frac{1}{8\pi}\arctan(|u-v|+|u'-v'|).$$

则(H1)-(H2)成立. 由定理 3.1,边值问题存在唯一的不动点.

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