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带弱奇异项的二阶微分方程正周期解的存在性

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摘　要：本文运用 Schauder 不动点定理获得了一类二阶非线性微分方程 $u''+a(t)u=f(t,u)+c(t)$ 正周期解的存在性，其中 $a\in L^1(\mathbf{R}/T\mathbf{Z};\mathbf{R}_+)$，$c\in L^1(\mathbf{R}/T\mathbf{Z};\mathbf{R})$，$f$ 为 Carathéodory 函数. 本文的主要结果推广了一些已有结果.

关键词：正周期解；Schauder 不动点定理；弱奇异

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Existence of positive periodic solutions of second-order differential equation with weak singularity

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Abstract: The existence of positive periodic solutions of the following second-order differential equation $u''+a(t)u=f(t,u)+c(t)$ is considered via Schauder's fixed point theorem, where $a\in L^1(\mathbf{R}/T\mathbf{Z};\mathbf{R}_+)$, $c\in L^1(\mathbf{R}/T\mathbf{Z};\mathbf{R})$, f is a Carathéodory function. Our main results generalize some known results.

Keywords: Positive periodic solution; Schauder's fixed point theorem; Weak singularity
(2010 MSC 39A10; 39A12)

1　Introduction

In this paper, we are concerned with the existence of positive periodic solutions of the second-order differential equation

$$u''+a(t)u=f(t,u)+c(t) \tag{1}$$

under the following assumption:

(C0) $a\in L^1(\mathbf{R}/T\mathbf{Z};\mathbf{R}_+)$, $c\in L^1(\mathbf{R}/T\mathbf{Z};\mathbf{R})$, $f\in Car(\mathbf{R}/T\mathbf{Z}\times(0,\infty)$, which means $a|_{[0,T]}:[0,T]\to\mathbf{R}^+$, and $C|_{[0,T]}:[0,T]\to\mathbf{R}^+$ are L^1 fanctions with period T, $\mathbf{R}$), which means $f|_{[0,T]}:[0,T]\times(0,\infty)\to\mathbf{R}$ is a L^1-Carathéodory function, and f is singular at $u=0$.

In the case that $a(t)\equiv 0$ and $f(t,u)=\frac{1}{u^\lambda}$, (1) reduces to the special equation

$$u''=\frac{1}{u^\lambda}+c(t) \tag{2}$$

which was initially studied by Lazer and Solimini [1]. There they proved that for $\lambda\geqslant 1$ (called strong force condition in a terminology first introduced by Gordon [2,3]), a necessary and sufficient condition for the existence of a positive periodic solution of (2) is that the mean value of c is negative,

$$\bar{c}:=\frac{1}{T}\int_0^T c(t)dt<0.$$

Moreover, if $0<\lambda<1$ (weak force condition),

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they found examples of functions c with negative mean values and such that periodic solutions do not exist.

If compared with the literature available for strong singularities, see Refs. [4-10] and the references therein, the study of the existence of periodic solutions under the presence of a weak singularity is much more recent and the number of references is considerably smaller. The likely reason may be that with a weak singularity, the energy near the origin becomes finite, and this fact is not helpful for obtaining a priori bound needed for a classical application of the degree theory, and also is not helpful for the fast rotation needed in recent versions of the Poincaré-Birkhoff theorem. The first existence result with weak force condition appears in Ref. [11]. Since then, the equation (1) with f has weak singularities has been studied by several authors, we refer the reader to Refs. [12-18] and the references therein.

Recently, Torres[13] showed how a weak singularity can play an important role if Schauder's fixed point theorem is chosen in the proof of the existence of positive periodic solution for (1). From now on, for a given function $\xi\in L^\infty[0,T]$, we denote the essential supremum and infimum of ξ by ξ^* and ξ_*, respectively. We write $\xi\succ 0$ if $\xi\geqslant 0$ for a. e. $t\in[0,T]$ and it is positive in a set of positive measure. Under the assumption

(H0) The linear equation $u''+a(t)u=0$ is nonresonant and the corresponding Green's function

$$G(t,s)\geqslant 0,\ (t,s)\in[0,T]\times[0,T],$$

Torres showed the following three results:

Theorem 1.1[13] Let (C0), (H0) hold and define

$$\gamma(t)=\int_0^T G(t,s)c(s)\mathrm{d}s \tag{3}$$

Assume that

(H1) there exist $b\in L^1(0,T)$ with $b\succ 0$ and $\lambda>0$ such that $0\leqslant f(t,u)\leqslant\frac{b(t)}{u^\lambda}$, for all $u>0$, a. e. $t\in[0,T]$.

If $\gamma_*>0$, then there exists a positive T-periodic solution of (1).

Theorem 1.2[13] Let (C0), (H0) hold. Assume that

(H2) there exist two functions $b,\hat{b}\in L^1(0,T)$ with $b,\hat{b}\succ 0$ and a constant $\lambda\in(0,1)$ such that

$$0\leqslant\frac{\hat{b}(t)}{u^\lambda}\leqslant f(t,u)\leqslant\frac{b(t)}{u^\lambda},\ u\in(0,\infty),$$

a. e. $t\in[0,T]$.

If $\gamma_*=0$, then (1) has a positive T-periodic solution.

Theorem 1.3[13] Let (C0), (H0) and (H2) hold. Let

$$\hat{\beta}_*=\min_{t\in[0,T]}\left(\int_0^T G(t,s)\hat{b}(s)\mathrm{d}s\right),$$

$$\beta^*=\min_{t\in[0,T]}\left(\int_0^T G(t,s)b(s)\mathrm{d}s\right).$$

If $\gamma_*\leqslant 0$ and

$$\gamma_*\geqslant\left(\frac{\hat{\beta}_*}{(\beta^*)^\lambda}\lambda^2\right)\frac{1}{1-\lambda^2}\left(1-\frac{1}{\lambda^2}\right),$$

then (1) has a positive T-periodic solution.

From the proof of Theorem 1.1～1.3, it is easy to see that (H1) and (H2), in which f is bounded by functions of form $\frac{1}{u^\lambda}$, play a key role in the using of Schauder's fixed point theorem.

Obviously, (H1) and (H2) are too restrictive so that the above mentioned results are only applicable to (1) with nonlinearity which is bounded at origin and infinity by a function of the form $\frac{1}{u^\lambda}$. Very recently, Ma *et al*. [14] generalized Theorems 1.1～1.3 under some conditions which allow the nonlinearity f to be bounded by two different functions $\frac{1}{u^\alpha}$ and $\frac{1}{u^\beta}$ with $0<\alpha,\beta<1$. It is easy to check that for fixed $\alpha\in(0,1)$,

$$\lim_{u\to 0^+}\frac{\ln(1+u)}{u^\alpha}=0,\ \lim_{u\to+\infty}\frac{\ln(1+u)}{u^\alpha}=0,$$

and there exists a constant $C(\alpha)$ such that

$$\frac{C(\alpha)}{u^\alpha}<\frac{1}{\ln(1+u)},u\in(0,\infty).$$

Of course, a natural question is what would happen if (H1) and (H2) are replaced by the following weaker conditions (A1) and (A2), respectively:

(A1) there exist $b\in L^1(0,T)$ with $b\succ 0$ and α

>0 such that

$$0\leqslant f(t,u)\leqslant\frac{b(t)}{\ln(1+u^{2\alpha})},\ u\in(0,\infty),$$

a. e. $t\in[0,T]$;

(A2) there exist two functions $b,\hat{b}\in L^1(0,T)$ with $b,\hat{b}>0$, $\alpha\in(0,1)$, such that

$$0\leqslant\frac{\hat{b}(t)}{u^{\alpha}}\leqslant f(t,u)\leqslant\frac{b(t)}{\ln(u+1)},\ u\in(0,\infty),$$

a. e. $t\in[0,T]$.

Let

$$B(t):=\int_0^T G(t,s)b(s)\mathrm{d}s.$$

$$\hat{B}(T):=\int_0^T G(t,s)\,\hat{b}(s)\mathrm{d}s.$$

In this paper, the following three theorems are obtained.

Theorem 1.4 Let (C0), (H0) and (A1) hold. If $\gamma_*>0$, then (1) has a positive T-periodic solution.

Theorem 1.5 Let (C0), (H0) and (A2) hold. If $\gamma_*=0$, then (1) has a positive T-periodic solution.

Theorem 1.6 Let (C0), (H0) and (A2) hold. Assume that there exists r_0 such that

$$\frac{\ln(r_0+1)}{B^*}\geqslant\frac{(r_0-\gamma_*)^{1/\alpha}}{\hat{b}_*^{1/\alpha}} \tag{4}$$

and

$$r_0\leqslant\frac{\hat{b}_*^{1/\alpha}}{(r_0-\gamma_*)^{1/\alpha}} \tag{5}$$

If $\gamma^*\leqslant 0$, then (1) has a positive T-periodic solution.

***Remark* 1** Theorem 1.4 generalizes Theorem 1.1. Theorem 1.5 generalizes Theorem 1.2. Theorem 1.6 deals with a case which can not be covered by Theorem 1.3, see Example 1.

2 Main results

Proof of Theorem 1.4 We denote the set of continuous T-periodic functions as C_T. Notice that a T-periodic solution of (1) is just a fixed point of the completely continuous map $A:C_T\to C_T$ defined as

$$(Au)(t):=\int_0^T G(t,s)(f(s,u(s))+c(s))\mathrm{d}s=\int_0^T G(t,s)f(s,u(s))\mathrm{d}s+\gamma(t),$$

where $\gamma(t)$ is defined as in (3). By Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set defined as

$$K=\{u\in C_T:r\leqslant u(t)\leqslant R,\text{ for all }t\in[0,T]\}$$

into itself, where $R>r>0$ are positive constants to be fixed properly.

For given $u\in K$, by the nonnegative sign of G and f, we have

$$(Au)(t)=\int_0^T G(t,s)f(s,u(s))\mathrm{d}s+\gamma(t)\geqslant\gamma(t)\geqslant\gamma_*=:r.$$

For every $u\in K$,

$$(Au)(t)=\int_0^T G(t,s)f(s,u(s))\mathrm{d}s+\gamma(t)\leqslant\int_0^T G(t,s)\frac{b(s)}{\ln(1+u^{2\alpha})}\mathrm{d}s+\gamma^*\leqslant\frac{B^*}{\ln(1+r^{2\alpha})}+\gamma^*=:R.$$

Therefore, $A(K)\subset K$ if $r=\gamma^*$ and $R=\frac{B^*}{\ln(1+\gamma_*^{2\alpha})}+\gamma^*$. Clearly, $R>r>0$ so the proof is finished.

Proof of Theorem 1.5 We follow the same strategy and notations as in the proof of Theorem 1.4. Define a closed convex set

$$K=\{u\in C_T:r\leqslant u(t)\leqslant R,\text{ for all }t\in[0,T]\}.$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set K into itself, where R and r are positive constants to be fixed properly and they should satisfy $R>r>0$.

Then, for given $u\in K$, by the nonnegative sign of G and f, it follows that

$$(Au)(t)=\int_0^T G(t,s)f(s,u(s))\mathrm{d}s+\gamma(t)\leqslant\int_0^T G(t,s)\frac{b(s)}{\ln(u+1)}\mathrm{d}s+\gamma^*\leqslant\frac{B^*}{\ln(r+1)}+\gamma^*.$$

On the other hand, for every $u\in K$,

$$(Au)(t)=\int_0^T G(t,s)f(s,u(s))\mathrm{d}s+\gamma(t)\geqslant\int_0^T G(t,s)\frac{\hat{b}(s)}{u^{\alpha}}\mathrm{d}s+\gamma_*\geqslant$$

$$\int_0^T G(t,s)\,\frac{\hat{b}(s)}{R^\alpha}\mathrm{d}s \geqslant \frac{\hat{B}_*}{R^\alpha}.$$

Thus $Au\in K$ if r,R are chosen so that

$$\frac{\hat{B}_*}{R^\alpha}\geqslant r,\quad \frac{B^*}{\ln(r+1)}+\gamma^*\leqslant R.$$

Note that $\hat{B}_*, B^*>0$ and if we fix $\frac{\hat{B}_*}{R^\alpha}=r$, then

$$R=(\frac{\hat{B}_*}{r})^{1/\alpha} \tag{6}$$

the second inequality holds if r verifies

$$\frac{B^*}{\ln(r+1)}+\gamma^*\leqslant(\frac{\hat{B}_*}{r})^{1/\alpha} \tag{7}$$

Since $\lim\limits_{r\to 0}\frac{\ln(1+r)}{r^{\frac{1}{\alpha}}}=\infty$, inequality (7) holds if we choose r small enough. Therefore (6) is satisfied and $R>r>0$.

Remark 2 It is worth remarking that Theorem 1.5 is also valid for the special case that $c(t)\equiv 0$, which implies that $\gamma_*=0$.

Proof of Theorem 1.6 Define a closed convex set

$$K=\{u\in C_T: r\leqslant u(t)\leqslant R,\text{ for all } t\in[0,T]\}.$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set K into itself, where R and r are positive constants to be fixed properly and they should satisfy $R>r>0$.

For given $u\in K$,

$$(Au)(t)=\int_0^T G(t,s)f(s,u(s))\mathrm{d}s+\gamma(t)\leqslant \int_0^T G(t,s)\,\frac{b(s)}{\ln(u+1)}ds+\gamma^*\leqslant\frac{B^*}{\ln(r+1)}.$$

On the other hand, for every $u\in K$,

$$(Au)(t)=\int_0^T G(t,s)f(s,u(s))ds+\gamma(t)\geqslant \int_0^T G(t,s)\,\frac{\hat{b}(s)}{u^\alpha}ds+\gamma_*\geqslant\frac{\hat{B}_*}{R^\alpha}+\gamma_* \tag{8}$$

In this case, to prove that $A(K)\subset K$, it is sufficient to find $0<r<R$ such that

$$\frac{\hat{B}_*}{R^\alpha}+\gamma_*\geqslant r,\quad \frac{B^*}{\ln(r+1)}\leqslant R \tag{9}$$

Now, let us take

$$r=r_0,\quad R=\frac{\hat{B}^{1/\alpha}{}_*}{(r_0-\gamma_*)^{1/\alpha}}.$$

Then (5) yields $r<R$. Moreover, (4) and (8) imply that (9) is true. The proof is end.

3 An example

Let us consider the following boundary value problem

$$u''+u=\frac{1}{\ln(u+1)}-\varepsilon,\quad 0<t<1 \tag{10}$$

$$u(0)=u(1),\quad u'(0)=u'(1) \tag{11}$$

It is easy to check that the function $\frac{1}{\ln(u+1)}$ satisfies (A2) with

$$\alpha=\frac{1}{2},\quad \hat{b}(t)=1.2,\quad b(t)=1.$$

This impies that

$$\frac{1.2}{\sqrt{u}}\leqslant\frac{1}{\ln(u+1)}-\varepsilon\leqslant\frac{1}{\ln(u+1)},\quad u\in(0,\infty).$$

Since the Green function of the linear problem

$$\begin{cases}u''+u=0,0<t<1,\\ u(0)=u(1),u'(0)=u'(1)\end{cases}$$

can be explicitly given by

$$G(t,s)=\frac{1}{2(1-\cos 1)}.$$

$$\begin{cases}\sin(t-s)+\sin(1-t+s), & 0\leqslant s\leqslant t\leqslant 1,\\ \sin(s-t)+\sin(1-s+t), & 0\leqslant t\leqslant s\leqslant 1,\end{cases}$$

it follows that

$$\gamma(t)=\int_0^1 G(t,s)(-\varepsilon)\mathrm{d}s=-\varepsilon,$$

$$\hat{B}(t)=\int_0^1 G(t,s)\,\hat{b}(s)\mathrm{d}s=1.2,$$

$$B(t)=\int_0^1 G(t,s)b(s)\mathrm{d}s=1,$$

and subsequently

$$\gamma_*=-\varepsilon,\quad \hat{B}_*=1.2,\quad B^*=1.$$

Now take $\gamma_0=0.2$ and let $\varepsilon\in(0,0.1]$ be a constant. Then the conditions (4) and (5) are satisfied. In fact,

$$0.1823\approx\frac{\ln(1.2)}{1}\geqslant\frac{(0.2+0.1)^2}{1.2^2}=\frac{0.09}{1.44}= 0.0625\geqslant\frac{(0.2-(-\varepsilon))^2}{1.2^2},$$

$$0.2<\frac{1.2^2}{(0.2+0.1)^2}=16\leqslant\frac{1.2^2}{(0.2-(-\varepsilon))^2}.$$

Therefore, we have from Theorem 1.6 that (10), (11) has a positive 1-periodic solution for each $\varepsilon\in(0,0.1]$.

However, it is easy to see that wecan not apply the results of Ref. [18] to guarantee the existence of positive 1-periodic solutions of (10), (11) since $\frac{1}{\ln(u+1)}$ can not be bounded by a function of form $\frac{C}{u^{\alpha}}$, where $\alpha \in (0,1)$ and C is a constant.

References:

[1] Lazer A C, Solimini S. On periodic solutions of nonlinear differential equations with singularities [J]. P Am Math Soc, 1987, 99: 109.

[2] Gordon W B. Conservative dynamical systems involving strong forces [J]. T Am Math Soc, 1975, 204: 113.

[3] Gordon W B. A minimizing property of Keplerian orbits [J]. Am J Math, 1977, 99: 961.

[4] Bonheure D, Fabry C, Smets D. Periodic solutions of forced isochronous oscillators at resonance [J]. Discrete Cont Dyn-A, 2002, 8: 907.

[5] Fonda A. Periodic solutions of scalar second order differential equations with a singularity [J]. Mém Cl Sci Acad R B, 1993, 8: 68.

[6] Fonda A, Manásevich R, Zanolin F. Subharmonics solutions for some second order differential equations with singularities [J]. SIAM J Math Anal, 1993, 24: 1294.

[7] Habets P, Sanchez L. Periodic solutions of some Liénard equations with singularities [J]. P Am Math Soc, 1990, 109: 1135.

[8] Jiang D, Chu J, Zhang M. Multiplicity of positive periodic solutions to superlinear repulsive singular equations [J]. J Differ Equations, 2005, 211: 282.

[9] Del Pino M, Manásevich R. Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity [J]. J Differ Equations, 1993, 103: 260.

[10] Del Pino M, Manásevich R, Montero A. T-periodic solutions for some second order differential equations with singularities [J]. P Roy Soc Edinb A, 1992, 120: 231.

[11] Rachunková, Rachunková I, Tvrdý M, Vrkoc I. Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems [J]. J Differ Equations, 2001, 176: 445.

[12] Torres P J. Existence of one-signed periodic solutions of some second order differential equations via a Krasnoselskii fixed point theorem [J]. J Differ Equations, 2003, 190: 643.

[13] Torres P J. Weak singularities may help periodic solutions to exist [J]. J Differ Equations, 2007, 232: 277.

[14] Ma R, Chen R, He Z. Positive periodic solutions of second-order differential equations with weak singularities [J]. Appl Math Comput, 2014, 232: 97.

[15] Chu J, Torres P J, Wang F. Twist periodic solutions for differential equations with a combined attractive-repulsive singularity [J]. J Math Anal Appl, 2016, 437: 1070.

[16] Hakl R, Zamora M. Periodic solutions to second-order indefinite singular equations [J]. J Differ Equations, 2017, 263: 451.

[17] Lomtatidze A, Sremr J. On periodic solutions to second-order Duffing type equations [J]. Nonlinear Anal-Real, 2018, 40: 215.

[18] Ma M T. Existence of positive solution for a class of nonlinear second order periodic problems [J]. J Sichua Univ: Nat Sci Ed(四川大学学报：自然科学版), 2018, 55: 693.

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