doi：10.3969/j.issn.0490-6756.2019.05.006

各向异性线弹性问题的鲁棒 V-循环多重网格法

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摘　要：本文对各向异性线弹性方程的双线性有限元法离散系统构造一种“鲁棒”的 V-循环多重网格法. 通过 Xu-Zikatanov (XZ)等式，本文得到了所构造多重网格算法的不依赖于各向异性参数 ε，而弱依赖于 h 的拟最优收敛性. 由于分析中未用到线弹性方程的“正则性”假设，该收敛性结果可以推广到一般的可剖分成矩形网格的区域上. 数值实验验证了理论结果.

关键词：线弹性；各向异性；双线性元；多重网格法
中图分类号：O241.82　　**文献标识码**：A　　**文章编号**：0490-6756(2019)05-0819-08

Robust V-cycle multigrid method for anisotropic linear elasticity problems

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Abstract: A robust V-cycle multigrid method is constructed for the linear systems arising from the bilinear finite element discretization of anisotropic linear elasticity equations. By using the Xu-Zikatanov (XZ) identity, quasi-optimal convergence of the method is established in the sense that the multigrid method is independent of the parameter ε and weakly dependent on h. Since the "regularity assumption" is not used in the analysis, the results can be extended to domains consisting of rectangles. Numerical experiments confirm the theoretical results.

Keywords: Linear elasticity; Anisotropy; Bilinear element; Multigrid method
(2010 MSC 65M60)

1　Introduction

As one of the most efficient methods for approximations to solutions of partial differential equations, multigrid methods have been used extensively (see Refs. [1-14] and the references therein). This paper will construct multigrid methods for anisotropic linear elasticity equations and present convergence analysis of the constructed multigrid method without the "regularity assumption". There are so many papers focus on the construction and analysis of multigrid meth-

收稿日期：2019-04-26
基金项目：国家自然科学基金(11701481)
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ods for isotropic linear elasticity equations (for example Refs. [15-16]), but no such results for anisotropic linear elasticity equations.

The existing multigrid theories for anisotropic problems are focused on the second order anisotropic elliptic equation. The theories can be classified into three categories. One follows the standard multigrid framework proposed by Hackbusch[14] and later extended by Bramble and Pasciak[7], imposes the "regularity and approximation" assumption[4, 19-20]. One follows the framework of multigrid proposed by Bramble, Pasciak, Wang and Xu[5, 8] (see also Xu[25] and Yserentant[26]), see Refs. [13, 17]. The last one follows the multigrid framework developed by Xu and Zikatanov[23], which do not need any "regularity" of partial differential equations[22, 27]. In all of these works, only scalar equations were considered and do not consider any coupling of variables.

This paper will construct multigrid methods for anisotropic linear elasticity equations and analyze the convergence of the methods following the framework developed by Xu and Zikatanov[23]. Because of using the framework[23], we do not need any "regularity" assumptions in the analysis. The main difficulties of the analysis are how to choose a proper space decomposition (or proper smoothers) and how to construct stable quasi-interpolation operators. To overcome these difficulties we use line smoothers in x-direction to the first variable of the displacement field and line smoothers in the y-direction for the second variable of displacement field, and we use the quasi-interpolation operators constructed in Ref. [22]. The main difference of the analysis of multigrid methods for anisotropic linear elasticity equations between second order elliptic equation is that we need to consider the coupling of variable, and this cause some difficulties in the proof of stability decomposition of spaces and the stable of quasi-interpolation operators. For simplicity of exposition, we present our analysis in the unit square domain. The analysis, however, can be easily generalized to domains for which the full regularity does not hold following the existing work[22].

In this paper, we use notation $a\lesssim b$ (or $a\gtrsim b$) to represent that there exists a constant C independent of mesh size h and the Lame constant λ such that $a\leqslant Cb$ (or $a\geqslant Cb$), and use $a\simeq b$ to denote $a\lesssim b\lesssim a$.

The rest of this paper is organized as follows. In section 2, we describe the model problem, and review the successive subspace correction algorithm. Section 3 constructs the multigrid methods and obtains the error operator. Section 4 gives the property of the space decomposition of $M_k\times M_k$. Section 5 presents the convergence of multigrid methods. In the final section, we give some numerical results.

2 Preliminaries

In this section, we present a model problem and review the successive subspace correction algorithm.

2.1 Model problem

Let $\Omega=(0,1)^2\subset\mathbf{R}^2$ be the unit square. We consider the 2 dimensional anisotropic linear elasticity problem

$$\begin{cases}-\mathrm{div}\ \boldsymbol{\varepsilon}(\boldsymbol{u})=\boldsymbol{f} & \text{in } \Omega,\\ \boldsymbol{u}=0 & \text{on } \partial\Omega\end{cases}\tag{1}$$

where $\boldsymbol{u}=(\boldsymbol{u},\boldsymbol{v})^T\in\mathbf{R}^2$ is the displacement field, and $\epsilon(\boldsymbol{u})=\frac{1}{2}(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T)$ the strain tensor, $\boldsymbol{f}\in\mathbf{R}^2$ the body loading density, and $\mathbf{C}$ the symmetric positive definite elasticity module tensor with

$$\boldsymbol{C}\epsilon(\boldsymbol{u})=\begin{pmatrix}a\,\partial_x u+b\,\partial_y v & \varepsilon(\partial_y u+\partial_x v)\\ \varepsilon(\partial_y u+\partial_x v) & b\,\partial_x u+a\,\partial_y v\end{pmatrix}\tag{2}$$

where a,b,ε are positive constants, a,b are given by $a=\frac{E}{1-\nu^2}$, $b=\frac{E\nu}{1-\nu^2}$ for plane stress problems, and

$$a=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)},$$

$$b=\frac{E\nu}{(1+\nu)(1-2\nu)}$$

for plane strain problems, with $0<\nu<0.5$ the

Poisson ratio and E the Young's modulus. We are interested in the case that $a\gg\varepsilon$ and do not consider the case $\nu\to 0.5$, so we always have $a>b$ and $a-b\simeq b\simeq a\gg\varepsilon$. Such kinds of problems main arise from the anisotropic orthotropy materials.

The weak form of equation(1) is: find $\boldsymbol{u}\in(H_0^1(\Omega))^2$ such that

$$a(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad \text{for all } \boldsymbol{v}\in(H_0^1(\Omega))^2 \tag{3}$$

where

$$a(\boldsymbol{u},\boldsymbol{v})=\int_\Omega \boldsymbol{C}\epsilon(\boldsymbol{u}):\epsilon(\boldsymbol{v})\mathrm{d}x\mathrm{d}y,$$

$$(\boldsymbol{f},\boldsymbol{v})=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\mathrm{d}y$$

defined two inner products on $(H_0^1(\Omega))^2$. We define $\|\cdot\|_A^2=a(\cdot,\cdot)$ to be the energy norm on $(H_0^1(\Omega))^2$.

Assume that $T_0\subset T_1\subset\cdots\subset T_J$ is a sequence of nested square partitions of Ω. The finest mesh T_J is obtained by divided Ω into $2^{J+1}\times 2^{J+1}$ small squares of equal size, and T_k for $0\leqslant k\leqslant J-1$ is obtained by uniformly coarsening of T_{k+1}. Let h_k $(0\leqslant k\leqslant J)$ denotes the mesh size on the kth level mesh. For $1\leqslant k\leqslant J$, let N_k be the integer such that T_k partitions Ω into $(N_k+1)\times(N_k+1)$ small squares. Define

$$\Omega_{k,j}^x=\{(x,y)\in\Omega:(j-1)h_k<y<(j+1)h_k\},\quad 2\leqslant j\leqslant N_k$$

and

$$\Omega_{k,j}^y=\{(x,y)\in\Omega:(j-1)h_k<x<(j+1)h_k\},\quad 2\leqslant j\leqslant N_k.$$

Namely, $\Omega_{k,j}^x$ is a horizontal strip of width $2h_k$ in the x-direction and $\Omega_{k,j}^y$ is a vertical strip of width $2h_k$ in the y-direction.

Let M_k be the bilinear finite element space[21] of $H_0^1(\Omega)$ associate to T_k. We obtain a sequence of nested spaces

$$M_0\subset M_1\subset\cdots\subset M_J.$$

We shall develop multigrid algorithms[23, 25] for solving the problem on the finest grid: given $\boldsymbol{f}\in M_J\times M_J$, find $\boldsymbol{u}\in M_J\times M_J$ satisfying

$$a(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v})\quad \text{for all } \boldsymbol{v}\in M_J\times M_J \tag{4}$$

2.2 Successive subspace correction

In this subsection, we will review successive subspace correction (SSC) algorithm[23, 25]. Let V be a Hilbert space. Assume that $V_i\subset V$ $(i=1,2,\cdots,J)$ is a subspace of V, and satisfy $V=\sum_{i=1}^J V_i$. Let $A:V\mapsto V$ be a symmetric and positive definite operator, and the norm induced by A is denoted as

$$\|\cdot\|_A^2=(A\cdot,\cdot)=a(\cdot,\cdot).$$

Define

$A_i: V_i\mapsto V_i$ be the restriction of A to V_i, i. e. A_i satisfying $(A_iv_i,w_i)=a(v_i,w_i)$ for all $v_i,w_i\in V_i$;

$Q_i: V\mapsto V_i$ be the L^2 projection, i. e. Q_i satisfying $(Q_iv,w_i)=(w,v_i)$ for all $w\in V$ and $w_i\in V_i$;

$P_i:V\mapsto V_i$ be the projection operator in the inner product induce by A, i. e. P_i satisfying $a(P_iv,w_i)=a(v,w_i)$ for all $w\in V$ and $w_i\in V_i$.

The one iterate step of SSC algorithm for solving equation: find $u\in V$, such that

$$a(u,v)=(f,v)\quad \text{for all } v\in V$$

can be reads as:

(**Algorithm SSC**) Give $u^k\in V$ to obtain $u^{k+1}\in V$.

(1) Let $v=u^k$;

(2) For $i=1:J$, define $v=v+A_i^{-1}Q_i(f-Av)$;

(3) Let $u^{k+1}=v$.

The error operator of SSC can be written as[23]

$$E_r=(I-P_1)(I-P_2)\cdots(I-P_J).$$

The following fundamental identity developed by Ref. [23] for the multiplication of operators (see also Ref. [12] for alternative proofs).

Theorem 2.1 (XZ identity) Assume that V is a Hilbert space with the $a(\cdot,\cdot)$-inner product and $V_i\subset V$ $(i=1,2,\cdots,J)$ are closed subspaces satisfying $V=\sum_{i=1}^J V_i$. Let $P_i: V\mapsto V_i$ be the orthogonal projection in the $a(\cdot,\cdot)$-inner product. Then the following identity holds:

$$\|(I-P_J)(I-P_{J-1})\cdots(I-P_1)\|_A^2=1-\frac{1}{1+c_0},$$

where

$$c_0=\sup_{\|v\|_A=1}\inf_{\sum_{i=1}^J v_i=v}\sum_{i=1}^J\left\|P_i\sum_{j=i+1}^J v_j\right\|_A^2.$$

3 Multigrid algorithms

3.1 Algorithm

To describe the multigrid algorithm we introduce the following auxiliary operators. For $0 \leqslant k \leqslant J$ define the operator $A_k : M_k \times M_k \mapsto M_k \times M_k$ by

$$(A_k \boldsymbol{w}, \boldsymbol{\varphi}) = a(\boldsymbol{w}, \boldsymbol{\varphi}) \text{ for all } \boldsymbol{w}, \boldsymbol{\varphi} \in M_k \times M_k.$$

The operator A_k is symmetric and positive definite with respect to the L^2 inner product. We define the projection operator $P_k : M_J \times M_J \mapsto M_k \times M_k$ in the $a(\cdot,\cdot)$-inner product as

$$a(P_k \boldsymbol{w}, \boldsymbol{\varphi}) = a(\boldsymbol{w}, \boldsymbol{\varphi}) \text{ for all } \boldsymbol{\varphi} \in M_k \times M_k,$$

and the L^2 projection $Q_k : L^2(\Omega) \times L^2(\Omega) \mapsto M_k \times M_k$ as

$$(Q_k \boldsymbol{w}, \boldsymbol{\varphi}) = (\boldsymbol{w}, \boldsymbol{\varphi}) \text{ for all } \boldsymbol{\varphi} \in M_k \times M_k.$$

For $1 \leqslant k \leqslant J$, we define

$$M_{k,j}^x = \{ v \in M_k : v = 0 \text{ in } \Omega \backslash \Omega_{k,j}^x \}, \quad j = 1, 2, \cdots, N_k,$$

$$M_{k,j}^y = \{ v \in M_k : v = 0 \text{ in } \Omega \backslash \Omega_{k,j}^y \}, \quad j = 1, 2, \cdots, N_k.$$

Then M_k can be decomposed as

$$M_k = \sum_{j=2}^{N_k} M_{k,j}^x = \sum_{j=2}^{N_k} M_{k,j}^y.$$

Similarly, for $k=0$, let $N_0 = 1$ and $M_{0,1}^x = M_{0,1}^y = M_0$.

Let $P_{k,j}^x : H_0^1(\Omega) \times 0 \mapsto M_{k,j}^x \times 0$, $P_{k,j}^y : 0 \times H_0^1(\Omega) \mapsto 0 \times M_{k,j}^y$ be the projections with respect to the inner product $a(\cdot,\cdot)$, and $Q_{k,j}^x : L^2(\Omega) \times 0 \mapsto M_{k,j}^x \times 0$, $Q_{k,j}^y : 0 \times L^2(\Omega) \mapsto 0 \times M_{k,j}^y$ be the projections with respect to the L^2 inner product $(\cdot,\cdot)$. Let $A_{k,j}^x : M_{k,j}^x \times 0 \mapsto M_{k,j}^x \times 0$ and $A_{k,j}^y : 0 \times M_{k,j}^y \mapsto 0 \times M_{k,j}^y$ be the operators satisfying

$$a(w, v) = (A_{k,j}^x w, v) \text{ for all } w, v \in M_{k,j}^x \times 0,$$

$$a(w, v) = (A_{k,j}^y w, v) \text{ for all } w, v \in 0 \times M_{k,j}^y.$$

It is easy to verify the following relations:

$$A_k P_k = Q_k A_k, \quad A_{k,j}^x P_{k,j}^x = Q_{k,j}^x A_k, \quad A_{k,j}^y P_{k,j}^y = Q_{k,j}^y A_k \tag{5}$$

We decompose $M_J \times M_J$ as

$$M_J \times M_J = \sum_{k=J}^{1} M_k \times M_k + M_0 \times M_0 + \sum_{k=1}^{J} M_k \times M_k = \sum_{k=J}^{1} \Big(\sum_{j=2}^{N_k} (M_{k,j}^x \times 0) + \sum_{j=2}^{N_k} (0 \times M_{k,j}^y) \Big) + M_0 \times M_0 + \sum_{k=1}^{J} \Big(\sum_{j=2}^{N_k} (M_{k,j}^x \times 0) + \sum_{j=2}^{N_k} (0 \times M_{k,j}^y) \Big).$$

Applying successive subspace correction (SSC) to the above space decomposition and choose exact subspace solvers P_0 on $M_0 \times M_0$, $P_{k,j}^x$ on $M_{k,j}^x \times 0$ and $P_{k,j}^y$ on $0 \times M_{k,j}^y$, we obtain the V-cycle multigrid method with only one pre-smoothing and post-smoothing step in each V-cycle iteration and the smoother is x-line smoother for the first variable of $\boldsymbol{u}$ and y-line smoother for the second variable of $\boldsymbol{u}$.

3.2 Error operator

Use the body capital letter E_r to denote the error operator, then by the theory of SSC E_r can be written as

$$E_r = \Big(\prod_{k=0}^{J} \Big(\prod\nolimits_{j=2}^{N_k} (I - P_{k,j}^x) \prod\nolimits_{j=2}^{N_k} (I - P_{k,j}^y) \Big) \Big)^* \Big(\prod_{k=0}^{J} \Big(\prod\nolimits_{j=2}^{N_k} (I - P_{k,j}^x) \prod\nolimits_{j=2}^{N_k} (I - P_{k,j}^y) \Big) \Big).$$

where $(\cdot)^*$ is the adjoint in the inner product $a(\cdot,\cdot)$, $P_{0,1}^x = P_{0,1}^y = P_0$. Let $P_{k,j} = P_{k,j}^x$, $j = 2, \cdots, N_k$ and $P_{k,j+N_k} = P_{k,j+1}^y$, $j = 1, \cdots, N_k - 1$. Then

$$\| E_r \|_A = \Big\| \prod_{k=0}^{J} \Big(\prod\nolimits_{j=2}^{2N_k - 1} (I - P_{k,j}) \Big) \Big\|_A^2 \tag{6}$$

In the following sections, we will estimate the A-norm of E_r. Our analysis relies on Theorem 2.1. We shall show that the constant c_0 in Theorem 2.1 associated with $\| E_r \|_A$ is uniformly bounded with respect to ε and depends on h in a very weakly way.

4 The decomposition of space M_k

In this section we give the property of the space decomposition of $M_k \times M_k$. By the stable decomposition of the L^2-norm, we immediately have

Lemma 4.1 For any $\boldsymbol{v}_k \in M_k \times M_k$, let $\boldsymbol{v}_k = \sum_{i=2}^{N_k} \boldsymbol{w}_{k,i} + \sum_{i=2}^{N_k} \boldsymbol{\psi}_{k,i}$, $\boldsymbol{w}_{k,i} \in M_{k,i}^x \times 0$, $\boldsymbol{\psi}_{k,i} \in 0 \times$

$M^y_{k,i}$, $i=2,3,\cdots,N_k$. Then we have

$$\sum_{i=2}^{N_k}\|\boldsymbol{w}_{k,i}\|_A^2+\sum_{i=2}^{N_k}\|\boldsymbol{\psi}_{k,i}\|_A^2\lesssim \|\boldsymbol{v}_k\|_A^2+\frac{\varepsilon}{h_k^2}\|\boldsymbol{v}_k\|^2 \quad (7)$$

Proof Note that, we can write $\boldsymbol{w}_{k,i}$ and $\boldsymbol{\psi}_{k,i}$ as $\boldsymbol{w}_{k,i}=(w_{k,i}\ 0)^T$ and $\boldsymbol{\psi}_{k,i}=(0\ \psi_{k,i})^T$,

$$\|\boldsymbol{w}_{k,i}\|_A^2=\int_\Omega (a(\partial_x w_{k,i})^2+\varepsilon(\partial_y w_{k,i})^2)\mathrm{d}x\mathrm{d}y,$$

$$\|\boldsymbol{\psi}_{k,i}\|_A^2=\int_\Omega (a(\partial_y \psi_{k,i})^2+\varepsilon(\partial_x \psi_{k,i})^2)\mathrm{d}x\mathrm{d}y.$$

Assume that $\boldsymbol{v}_k=(u_k\ v_k)^T$. Then $u_k=\sum_{i=2}^{N_k} w_{k,i}$ and $v_k=\sum_{i=2}^{N_k}\psi_{k,i}$. u is only decomposition in the y-direction and v is only decomposition in the x-direction. Therefore, by the stability of the decomposition in the L^2-norm, we have

$$\sum_{i=2}^{N_k}\|\partial_x w_{k,i}\|^2\lesssim\left\|\sum_{i=2}^{N_k}\partial_x w_{k,i}\right\|^2=\|\partial_x u_k\|^2,$$

$$\sum_{i=2}^{N_k}\|\partial_y \psi_{k,i}\|^2\lesssim\left\|\sum_{i=2}^{N_k}\partial_y \psi_{k,i}\right\|^2=\|\partial_y v_k\|^2.$$

On the other direction, we use the inverse inequality to get

$$\sum_{i=2}^{N_k}\|\partial_y w_{k,i}\|^2\lesssim h_k^{-2}\sum_{i=2}^{N_k}\|w_{k,i}\|^2\lesssim h_k^{-2}\|u_k\|^2,$$

$$\sum_{i=2}^{N_k}\|\partial_x \psi_{k,i}\|^2\lesssim h_k^{-2}\sum_{i=2}^{N_k}\|\psi_{k,i}\|^2\lesssim h_k^{-2}\|v_k\|^2.$$

A linear combination of the above inequalities leads to

$$\sum_{i=2}^{N_k}\|w_{k,i}\|_A^2+\sum_{i=2}^{N_k}\|\psi_{k,i}\|_A^2\lesssim a(\|\partial_x u_k\|^2+\|\partial_y v_k\|^2)+\frac{\varepsilon}{h_k^2}\|v_k\|^2 \quad (8)$$

Note that

$$\|\boldsymbol{v}_k\|_A^2=a(\|\partial_x u_k\|^2+\|\partial_y v_k\|^2)+\varepsilon\|\partial_y u_k+\partial_x v_k\|^2+2b\int_\Omega \partial_x u_k\partial_y v_k\mathrm{d}x\mathrm{d}y\geqslant (a-b)(\|\partial_x u_k\|^2+\|\partial_y v_k\|^2)+\varepsilon\|\partial_y u_k+\partial_x v_k\|^2.$$

By inequality (8) and the assumptions on the constants a,b, the desired result follows.

5 Convergence analysis without "regularity"

In this section, we will analyze the convergence of V-cycle multigrid method constructed in this paper without using any "regularity" assumption of the anisotropic linear elasticity equation. We will introduce a stable quasi interpolation operator[18, 22] and then give the convergence results by using XZ identity[23].

5.1 Stable quasi-interpolation operators

In this subsection, we will introduce stable quasi-interpolation operators[18, 22]. Here we briefly review the definition and the properties of the operators. Let $\varphi_i^k\in P_1(x_i^k,x_{i+1}^k)$ (or $\in P_1(y_i^k,y_{i+1}^k)$) be the one dimensional linear nodal base at the point x_i^k or y_i^k. On the edge (x_i^k,x_{i+1}^k), we choose

$$\theta_i^k=h_k^{-1}(4\varphi_i^k-2\varphi_{i+1}^k)\in P_1(x_i^k,x_{i+1}^k).$$

Direct computation shows

$$\int_{x_i^k}^{x_{i+1}^k}\theta_i^k\varphi_i^k=1,\quad \int_{x_i^k}^{x_{i+1}^k}\theta_i^k\varphi_{i+1}^k=0,$$

where $P_1(x_i^k,x_{i+1}^k)$ is the space of polynomial of degree less than or equal to 1 on the edge (x_i^k,x_{i+1}^k). Similar definition applies to $\theta_j^k(y)$ for the edge (y_j^k,y_{j+1}^k).

For a function $v\in H^1(\Omega)$, we define I_k^x and I_k^y as follows.

$$(I_k^x v)(x,y)=\sum_{i=2}^{N_k} v_i(y)\varphi_i^k(x),$$

$$(I_k^y v)(x,y)=\sum_{j=2}^{N_k} v_j(x)\varphi_j^k(y),$$

where

$$v_i(y)=\int_{x_i^k}^{x_{i+1}^k}\theta_i^k(x)v(x,y)\mathrm{d}x,$$

$$v_j(x)=\int_{y_j^k}^{y_{j+1}^k}\theta_j^k(y)v(x,y)\mathrm{d}y.$$

We then introduce a quasi-interpolation $I_k: H_0^1(\Omega)\to M_k$ by

$$I_k v=\sum_{i=2}^{N_k}\sum_{j=2}^{N_k} v_{i,j}\varphi_i^k(x)\varphi_j^k(y),$$

where

$$v_{i,j}=\int_{x_i^k}^{x_{i+1}^k}\int_{y_j^k}^{y_{j+1}^k}\theta_i^k(x)\theta_j^k(y)v(x,y)\,\mathrm{d}x\mathrm{d}y.$$

In this definition, since the boundary nodes are not included, we can easily to see that $I_k v|_{\partial\Omega}=0$.

Lemma 5.1[18, 22] The following properties hold for the interpolation I_k, I_k^x, and I_k^y:

(i) Preservation of bilinear finite element functions: $I_k v_k = v_k$ for $v_k \in M_k$;

(2) Approximation property:

$\|v - I_k v\| \lesssim h_k |v|_1$ for $v \in H^1(\Omega)$;

(3) Operators I_k^x and I_k^y are interchangeable and $I_k = I_k^x I_k^y = I_k^y I_k^x$;

(4) I_k^x and I_k^y are stable in both L^2-norm and corresponding one dimensional H^1-norm. Namely, for all $v \in H^1$, we have

$$\|I_k^x v\| \lesssim \|v\|, \|I_k^y v\| \lesssim \|v\|,$$

and

$$\|\partial_x I_k^x v\| \lesssim \|\partial_x v\|, \|\partial_y I_k^y v\| \lesssim \|\partial_y v\|;$$

(5) For $v \in M_J$, we have

$$\|\partial_x I_k^y v\| \lesssim \|\partial_x v\|, \|\partial_y I_k^x v\| \lesssim \|\partial_y v\|.$$

For any $\boldsymbol{u} = (u, v)^T \in M_J \times M_J$, define $\Pi_k \boldsymbol{u} = (I_k u, I_k v)^T$. Using the properties in Lemma 5.1, we prove Π_k is stable in the energy norm.

Lemma 5.2 For any $\boldsymbol{u} = (u, v)^T \in M_J \times M_J$, it holds

$$\|\Pi_k \boldsymbol{u}\|_A \lesssim \|\boldsymbol{u}\|_A.$$

Proof For any $\boldsymbol{u} \in M_J \times M_J$, it holds that

$$\|\Pi_k \boldsymbol{u}\|_A^2 = a(\|\partial_x I_k u\|^2 + \|\partial_y I_k v\|^2) + \varepsilon\|\partial_y I_k u + \partial_x I_k v\|^2 + 2b\int_\Omega \partial_x I_k u\, \partial_y I_k v \,\mathrm{d}x\mathrm{d}y \lesssim (a+b)(\|\partial_x u\|^2 + \|\partial_y v\|^2) + \varepsilon\|\partial_y u\|^2 + \varepsilon\|\partial_x v\|^2 \lesssim \|\boldsymbol{u}\|_A^2.$$

Then the desired result follows.

5.2 Convergence

In this subsection, we will use XZ identity to prove the convergence of V-Cycle multigrid method for anisotropic 2D linear elasticity equations.

Theorem 5.3 There exists a positive constant C independent of ε and h, such that

$$\|E_r\|_A \leqslant 1 - \frac{1}{1 + C|\log h|},$$

i. e, the V-cycle multigird method is convergent with rate $1 - \frac{1}{1 + C|\log h|}$.

Proof For any $\boldsymbol{v} \in M_J \times M_J$, we define

$$\boldsymbol{v}_k = (\Pi_k - \Pi_{k-1})\boldsymbol{v} \in M_k \times M_k, k = 0, 1, \cdots, J,$$

where $\Pi_{-1} = 0$. It is easy to check that $\boldsymbol{v} = \sum_{k=0}^{J} \boldsymbol{v}_k$. Assume that $\boldsymbol{v}_k = (u_k, v_k)^T$. We can decompose u_k and v_k as $u_k = \sum_{i=2}^{N_k} u_{k,i}$, $v_k = \sum_{i=2}^{N_k} v_{k,i}$, where for $i = 2, \cdots, N_k$, $u_{k,i} \in M_{k,i}^x$ and $v_{k,i} \in M_{k,i}^y$. Define

$$\boldsymbol{u}_{k,j} = (u_{k,j}, 0)^T, j = 2, 3, \cdots, N_k$$

and

$$\boldsymbol{u}_{k,j+N_k} = (0, v_{k,j+1})^T, j = 1, 2, \cdots, N_k - 1,$$

thus $\boldsymbol{v} = \sum_{k=0}^{J}\sum_{j=2}^{2N_k-1} \boldsymbol{u}_{k,j}$. By Theorem 2.1, we have

$$c_0 = \sup_{\|\boldsymbol{v}\|_A = 1} \inf_{\sum_{k=0}^{J}\sum_{i=2}^{2N_k-1} \boldsymbol{u}_{k,i} = \boldsymbol{v}} \sum_{k=0}^{J}\sum_{i=2}^{2N_k-1} \left\| P_{k,i} \sum_{(l,j)>(k,i)} \boldsymbol{u}_{l,j} \right\|_A^2,$$

where the ordering $(l,j) > (k,i)$ is defined by

$$(l,j) > (k,i) \text{ if } \begin{cases} l = k \text{ but } j > i, \\ l > k. \end{cases}$$

By the decomposition of $\boldsymbol{v}$, it is easy to check that

$$\sum_{(l,j)>(k,i)} \boldsymbol{u}_{l,j} = \sum_{l=k+1}^{J}\sum_{j=2}^{2N_k-1} \boldsymbol{u}_{l,j} + \sum_{j=i+1}^{2N_k-1} \boldsymbol{u}_{k,j} = \sum_{l=k+1}^{J} \boldsymbol{v}_l + \sum_{j=i+1}^{2N_k-1} \boldsymbol{u}_{k,j} = \boldsymbol{v} - \Pi_k \boldsymbol{v} + \sum_{j=i+1}^{2N_k-1} \boldsymbol{u}_{k,j}.$$

Immediately, we get

$$\sum_{i=2}^{2N_k-1} \left\| P_{k,i} \sum_{(l,j)>(k,i)} \boldsymbol{u}_{l,j} \right\|_A^2 \lesssim \|\boldsymbol{v} - \Pi_k \boldsymbol{v}\|_A^2 + \sum_{i=2}^{2N_k-1} \|\boldsymbol{u}_{k,i}\|_A^2.$$

By Lemma 5.2, we have $\|\boldsymbol{v} - \Pi_k \boldsymbol{v}\|_A \lesssim \|\boldsymbol{v}\|_A$.

For $1 \leqslant k \leqslant J$, using Lemmas 4.1 and 5.2, it holds that

$$\sum_{i=2}^{2N_k-1} \|\boldsymbol{u}_{k,i}\|_A^2 \lesssim \|\boldsymbol{v}_k\|_A^2 + \frac{\varepsilon}{h_k^2}\|\boldsymbol{v}_k\|^2 \lesssim \|\boldsymbol{v}\|_A^2 + \frac{\varepsilon}{h_k^2}(\|\boldsymbol{v} - \Pi_k \boldsymbol{v}\|^2 + \|\boldsymbol{v} - \Pi_{k-1}\boldsymbol{v}\|^2) \lesssim \|\boldsymbol{v}\|_A^2 + \varepsilon\|\boldsymbol{v}\|_1^2 \lesssim \|\boldsymbol{v}\|_A^2.$$

For $k = 0$, it holds that

$$\|\boldsymbol{v}_{0,i}\|_A^2 = \|\boldsymbol{v}_0\|_A^2 = \|\Pi_0 \boldsymbol{v}\|_A^2 \lesssim \|\boldsymbol{v}\|_A^2.$$

Thus we have

$$c_0 \leqslant C(J+1) \leqslant C|\log h|.$$

The desired result follows.

6 Numerical experiments

In this section, we use two examples to verify theoretical results in the above sections. We choose parameters in equation (1) as $a=1.5, b=1$ and let $\varepsilon \to 0$. The domain for one example is $\Omega=(0,1)^2$ and another one is $\Omega=(0,1)^2 \backslash (0.5,1)^2$. The body force for both examples is $\boldsymbol{f}=1$. When the related error $\frac{\|\boldsymbol{f}-A_J\boldsymbol{u}\|}{\|\boldsymbol{f}\|}<10^{-8}$ or the iteration steps is larger than 2000, we stop the iteration.

We compute the multigrid method with line smoother in this paper (LMG) and the classical multigrid with one point pre-smoothing and post-smoothing in each iteration step (CMG), and listed the results of different ε and h in Tables 1 and 2. Through the numerical results, we can see that the multigrid method analyzed in this paper is robust with respect to both ε and h, while the classical (CMG) one is dependent on both ε and h.

Tab. 1　The results of LMG and CMG for different ε and h on square domain

Method	h^{-1}	ε				
		1	0.1	0.01	0.001	0.0001
LMG	8	20	13	16	17	18
	16	24	14	15	16	17
	32	30	14	15	16	17
	64	32	14	14	15	16
	128	32	14	14	14	15
CMG	8	19	36	110	157	166
	16	24	46	234	490	576
	32	29	48	322	1249	1895

Tab. 2　The results of LMG and CMG for different ε and h on L-shape domain

Method	h^{-1}	ε				
		1	0.1	0.01	0.001	0.0001
LMG	8	19	13	16	17	17
	16	28	14	16	17	17
	32	35	14	15	16	17
	64	39	15	15	15	16
	128	40	15	14	15	15
CMG	8	19	39	106	154	162
	16	29	48	235	485	598
	32	35	51	335	1287	1971

参考文献：

[1]　Bank R E, Douglas C C. Sharp estimates for multigrid rates of convergence with general smoothing and acceleration [J]. SIAM J Numer Anal, 1985, 22: 617.

[2]　Bank R E, Dupont T. An optimal order process for solving finite element equations [J]. Math Comput, 1981, 36: 35.

[3]　Braess D, Hackbusch W. A new convergence proof for the multigrid method including the V-cycle [J]. SIAM J Numer Anal, 1983, 20: 967.

[4]　Bramble J H, Zhang X. Uniform convergence of the multigrid V-cycle for an anisotropic problem [J]. Math Comput, 1998, 70: 453.

[5]　Bramble J H, Pasciak J E, Wang J, *et al*. Convergence estimates for multigrid algorithms without regularity assumptions [J]. Math Comput, 1991, 57: 23.

[6]　Bramble J H, Pasciak J E, Wang J, *et al*. Convergence estimates for product iterative methods with applications to domain decomposition [J]. Math Comput, 1991, 195: 1.

[7]　Bramble J H, Pasciak J E. New convergence estimates for multigrid algorithms [J]. Math Comput, 1987, 49: 311.

[8]　Bramble J H, Pasciak J E. New estimates for multilevel algorithms including the V-cycle [J]. Math Comput, 1993, 60: 447.

[9]　Bramble J H, Pasciak J E, Xu J. The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms [J]. Math Comput, 1991, 56: 1.

[10]　Bramble J H, Xu J. Some estimates for a weighted L^2 projection [J]. Math Comput, 1991, 56: 463.

[11]　Brandt A. Multi-level adaptive solutions to boundary-value problems [J]. Math Comput, 1977, 31: 333.

[12]　Cho D, Xu J, Zikatanov L. New estimates for the rate of convergence of the method of subspace corrections [J]. Numer Math-Theory Me, 2008, 1: 44.

[13]　Griebel M, Oswald P. Tensor-product-type subspace splittings and multilevel methods for anisotropic problems [J]. Adv Comput Math, 1995, 4: 171.

[14]　Hackbusch W. Multi-grid methods and applications [M]. Berlin: Spring, 1985.

[15]　Lee C O. Multigrid methods for the pure traction problem of linear elasticity: mixed formulation [J]. SIAM J Numer Anal, 1998, 35: 121.

[16]　Lee Y J, Wu J, Chen J. Robust multigrid method for the planar linear elasticity problems [J]. Numer

Math, 2009, 113: 473.
[17] Neuss N. V-cycle convergence with unsymmetric smoothers and application to an anisotropic model problem [J]. SIAM Numer Anal, 1998, 35: 1201.
[18] Scott L R, Zhang S. Finite element interpolation of nonsmoothing functions satisfying boundary conditions [J]. Math Comput, 1990, 54: 483.
[19] Stevenson R. Robustness of multi-grid applied to anisotropic equations on convex domains and on domains with reentrant corners [J]. Numer Math, 1993, 66: 373.
[20] Stevenson R. New estimates of the contraction number of V-cycle multi-grid with applications to anisotropic equations [M]. Braunschweig: Vieweg, 1993.
[21] Wang L, Luo K, Zhang S Q. Weak Galerkin finite element method for fourth order singular perturbation problems [J]. J Sichuan Univ: Nat Sci Ed(四川大学学报:自然科学版), 2018, 55: 1141.
[22] Wu Y, Chen L, Xie X, *et al*. Convergence analysis of V-Cycle multigrid methods for anisotropic elliptic equations [J]. IMA Numeri Anal, 2012, 32: 1329.
[23] Xu J, Zikatanov L. The method of alternating projections and the method of subspace corrections in Hilbert space [J]. J Am Math Soc, 2002, 15: 573.
[24] Xu J. Theory of multilevel methods [D]. Ithace: Cornell University, 1989.
[25] Xu J. Iterative methods by space decomposition and subspace correction [J]. SIAM Rev, 1992, 34: 581.
[26] Yserentant H. Old and new convergence proofs for multigrid methods [J]. Acta Numer, 1993, 2: 285.
[27] Yu G, Xu J, Zikatanov L. Analysis of a two-level method for anisotropic diffusion equations on aligned and nonaligned grids [J]. Numer Linear Algebr Appl, 2013, 20: 832.

引用本文格式:
中　文:白艳红,吴永科,覃艳梅. 各向异性线弹性问题的鲁棒V-循环多重网格法[J]. 四川大学学报:自然科学版, 2019, 56: 819.
英　文:BAI Y H, WU Y K, QIN Y M. Robust V-cycle multigrid method for anisotropic linear elasticity problems [J]. J Sichuan Univ: Nat Sci Ed, 2019, 56: 819.