最长连续切换的长度

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摘　要：作为一类简单数学模型,均匀硬币的独立抛掷模型在诸多领域具有广泛的应用,关于其最长连续头部长度的研究长期吸引着学者们,时至今日仍有大量研究致力于这个问题的推广及应用.本文研究最长连续切换的长度,给出了几个极限定理.

关键词：最长连续头部；最长连续切换；极限定理；Borel-Cantelli 引理
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On the length of the longest consecutive switches

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Abstract: As a class of simple mathematical models, tossing an unbiased coin independently has extensive applications in many fields. The length of the longest head-run has been long explored by many scholars. Up to now, there is still a lot of results on the extension of this problem and their applications. In this paper, we study the length of the longest consecutive switches and present several limit theorems.

Keywords: Longest head-run; Longest consecutive switches; Limit theorem; Borel-Cantelli lemma
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1　Introduction

An unbiased coin with two sides named by "head" and "tail" respectively, is tossed n times independently and sequentially. We use 0 to denote "tail" and 1 to denote "head". For simplicity, we assume that all the random variables in the following are defined in a probability space (Ω, F, P). Let $\{X_i, i \geqslant 1\}$ be a sequence of independent and identically distributed random variables with $P\{X_1=0\}=P\{X_1=1\}=\frac{1}{2}$. Let $S_0=0, S_n=X_1+X_2+\cdots+X_n, n=1,2,\cdots$, and

$$I(N,K)=\max_{0\leqslant n\leqslant N-K}(S_{n+K}-S_n),\quad N\geqslant K, N, K\in\mathbf{N} \tag{1}$$

Denote by Z_N the largest integer for which $I(N, Z_N)=Z_N$. Then Z_N is the length of the longest head-run of pure heads in N Bernoulli trials.

The statistic Z_N has been long studied because it has extensive applications in reliability theory, biology, quality control, pattern recognition, finance, etc. Erdös and Rényi[1] proved the following result:

Theorem 1.1　Let $0<C_1<1<C_2<\infty$. Then for almost all $\omega\in\Omega$, there exists a finite $N_0=N_0(\omega, C_1, C_2)$ such that $[C_1\log N]\leqslant Z_N\leqslant$

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$[C_2 \log N]$ if $N \geqslant N_0$.

Hereafter, we denote by "log" the logarithm with base 2, and by $[x]$ the largest integer which is no more than x. Theorem 1.1 was extended by Komlós and Tusnády[2]. Erdös and Rényi[3] presented several sharper bounds of Z_N including the following four theorems among others.

Theorem 1.2 Let ε be any positive number. Then for almost all $\omega \in \Omega$, there exists a finite $N_0 = N_0(\omega, \varepsilon)$ such that if $N \geqslant N_0$, then $Z_N \geqslant [\log N - \log\log\log N + \log\log \mathrm{e} - 2 - \varepsilon]$.

Theorem 1.3 Let ε be any positive number. Then for almost all $\omega \in \Omega$, there exists an infinite sequence $N_i = N_i(\omega, \varepsilon)(i = 1, 2, \cdots)$ of integers such that $Z_{N_i} < [\log N_i - \log\log\log N_i + \log\log \mathrm{e} - 1 + \varepsilon]$.

Theorem 1.4 Let $\{\gamma_n\}$ be a sequence of positive numbers for which $\sum_{n=1}^{\infty} 2^{-\gamma_n} = \infty$. Then for almost all $\omega \in \Omega$, there exists an infinite sequence $N_i = N_i(\omega, \{\gamma_n\})(i = 1, 2, \cdots)$ of integers such that $Z_{N_i} \geqslant \gamma_{N_i}$.

Theorem 1.5 Let σ_n be a sequence of positive numbers for which $\sum_{n=1}^{\infty} 2^{-\sigma_n} < \infty$. Then for almost all $\omega \in \Omega$, there exists a positive integer $N_0 = N_0(\omega, \{\delta_n\})$ such that $Z_N < \delta_N$ if $N \geqslant N_0$.

These limit theorems have been extended by many authors. We refer to Guibas and Odlyzko[4], Samarova[5], Révész[6], Nemetz and Kusolitsch[7], Grill[8] and Vaggelatou[9].

The distribution function of Z_N and some related problems have been studied by Goncharov[10], Földes[11], Arratia *et al.*[12], Novak[13-15], Schilling[16], Binswanger and Embrechts[17], Muselli[18], Vaggelatou[9], Túri[19], Novak[20]. Mao *et al.*[21] studied the large deviation behavior for the length of the longest head run. For more recent related references, we refer to Asmussen *et al.*[22], Chen and Yu[23], Li and Yang[24], Pawelec and Urbański[25], and Mezhennaya[26].

In 2012, Anush posed the definition of "switch", and considered the bounds for the number of coin tossing switches. In 2013, Li[27] considered the number of switches in unbiased coin-tossing, and established the central limit theorem and the large deviation principle for the total number of switches. According to Li[27], a "head" switch is the tail followed by a head and a "tail" switch is the head followed by a tail.

Motivated by the study of the longest head-run and the work of Li[27], we will study the length of the longest consecutive switches in this paper. At first, we introduce some notations. For m, $n \in \mathbf{N}$, define

$$S_n^{(m)}(H) = \sum_{i=m+1}^{n+m-1} (1 - X_{i-1}) X_i,$$

$$S_n^{(m)}(T) = \sum_{i=m+1}^{n+m-1} X_{i-1} (1 - X_i).$$

Then $S_n^{(m)}(H)$ (resp. $S_n^{(m)}(T)$) denotes the number of "head" switches (resp. "tail" switches) in the trials $\{X_m, X_{m+1}, \cdots, X_{m+n-1}\}$. Set

$$S_n^{(m)} = S_n^{(m)}(H) + S_n^{(m)}(T) \tag{2}$$

Then $S_n^{(m)}$ denotes the total number of switches in the sequence $\{X_m, X_{m+1}, \cdots, X_{m+n-1}\}$.

For i, $N \in \mathbf{N}$, define

$$H_{n,i}^{(N)} = \bigcup_{i \leqslant m \leqslant i+N-n+1} \{S_n^{(m)} = n-1\}, n = 1, \cdots, N.$$

Then $\omega \in H_{n,i}^{(N)}$ implies that there exists at least one sequence of consecutive switches of length $n-1$ in the sequence $\{X_i(\omega), X_{i+1}(\omega), \cdots, X_{i+N+1}(\omega)\}$. Define

$$M_N^{(i)} = \max_{1 \leqslant n \leqslant N} \{n - 1 \mid H_{n,i}^{(N)} \neq \varnothing\} \tag{3}$$

which stands for *the number of switches in the longest consecutive switches* in the sequence $\{X_i, X_{i+1}, \cdots, X_{i+N+1}\}$. When $i = 1$, we denote M_N instead of $M_N^{(1)}$.

***Remark* 1** Note that by (3), the length of the longest consecutive switches in the sequence $\{X_i, X_{i+1}, \cdots, X_{i+N+1}\}$ is $M_N^{(i)} + 1$.

We use $A_1 + A_2 + \cdots + A_n$ instead of $A_1 \cup A_2 \cdots \cup A_n$ when the sets A_i, $i = 1, \cdots, n$ are disjoint. The rest of this paper is organized as follows. In Section 2, we present main results. The proofs will be given in Section 3. In Section 4, we give some final remarks.

2 The main results

In this section, we present several limit results on M_N. Corresponding to Theorems 1.1~1.5, we have the following five theorems.

Theorem 2.1 We have

$$\lim_{N\to\infty}\frac{M_N}{\log N}=1 \quad \text{a. s.} \tag{4}$$

Theorem 2.2 Let ε be any positive number. Then for almost all $\omega\in\Omega$, there exists a finite $N_0=N_0(\omega,\varepsilon)$ such that if $N>N_0$,

$$M_N\geqslant[\log N-\log\log\log N+\log\log e-2-\varepsilon]:=\alpha_1(N) \tag{5}$$

Theorem 2.3 Let ε be any positive number. Then for almost all $\omega\in\Omega$, there exists an infinite sequence $N_i=N_i(\omega,\varepsilon)(i=1,2,\cdots)$ of integers such that

$$M_{N_i}<[\log N_i-\log\log\log N_i+\log\log e-1+\varepsilon]:=\alpha_2(N) \tag{6}$$

Theorem 2.4 Let $\{\gamma_n\}$ be a sequence of positive numbers for which $\sum_{n=1}^{\infty}2^{-\gamma_n}=\infty$. Then for almost all $\omega\in\Omega$, there exists an infinite sequence $N_i=N_i(\omega,\{\gamma_n\})(i=1,2,\cdots)$ of integers such that $M_{N_i}\geqslant\gamma_{N_i}-1$.

Theorem 2.5 Let σ_n be a sequence of positive numbers for which $\sum_{n=1}^{\infty}2^{-\sigma_n}<\infty$. Then for almost all $\omega\in\Omega$, there exists a positive integer $N_0=N_0(\omega,\{\delta_n\})$ such that $M_N<\delta_N-1$ if $N\geqslant N_0$.

The last two theorems can be reformulated as follows.

Theorem 2.4* Let $\{\gamma_n\}$ be a sequence of positive numbers for which $\sum_{n=1}^{\infty}2^{-\gamma_n}=\infty$. Then for almost all $\omega\in\Omega$, there exists an infinite sequence $N_i=N_i(\omega,\{\gamma_n\})(i=1,2,\cdots)$ of integers such that $S_{\gamma_{N_i}}^{(N_i-\gamma_{N_i})}\geqslant\gamma_{N_i}-1$.

Theorem 2.5* Let σ_n be a sequence of positive numbers for which $\sum_{n=1}^{\infty}2^{-\sigma_n}<\infty$. Then for almost all $\omega\in\Omega$, there exists a positive integer $N_0=N_0(\omega,\{\delta_n\})$ such that $S_{\delta_i}^{(N_i-\delta_{N_i})}<\delta_{N_i}-1$ if $N\geqslant N_0$.

Remark 2 The closely related result with respect to Theorems 2.4 and 2.5 is Guibas and Odlyzko[4] (Theorem 1).

3 The proofs of the main results

Proof of Theorem 2.1 Step 1. We prove $\liminf_{N\to\infty}\frac{M_N}{\log N}\geqslant 1$ a. s. For any $0<\varepsilon<1$, $N\in\mathbf{N}$ and $N\geqslant 2$, we introduce the following notations:

$$t=[(1-\varepsilon)\log N]+1, \bar{N}=[N/t]-1,$$
$$U_k=S_t^{(tk+1)}, k=0,1,\cdots,\bar{N},$$

where $S_t^{(tk+1)}$ is defined by (2). Then the sequence $\{U_k, 0\leqslant k\leqslant\bar{N}\}$ of random variables are independent and identically distributed with $U_k\leqslant t-1$ and $P\{U_k=t-1\}=2\cdot\frac{1}{2^t}=\frac{1}{2^{t-1}}$. It follows that

$$P\{U_0<t-1,U_1<t-1,\cdots,U_{\bar{N}}<t-1\}=\left(1-\frac{1}{2^{t-1}}\right)^{\bar{N}+1}.$$

By a simple calculation, we get that $\sum_{N=1}^{\infty}\left(1-\frac{1}{2^{t-1}}\right)^{\bar{N}+1}<\infty$. Then by the Borel-Cantelli lemma, we get that $\liminf_{N\to\infty}\frac{M_N}{\log N}\geqslant 1-\varepsilon$. By the arbitrariness of ε, we obtain that $\liminf_{N\to\infty}\frac{M_N}{\log N}\geqslant 1$ a. s.

Step 2. We prove $\limsup_{N\to\infty}\frac{M_N}{\log N}\leqslant 1$ a. s. For any $\varepsilon>0$ and $N\in\mathbf{N}$, we introduce the following notations:

$$u=[(1+\varepsilon)\log N]+1, A_N=\bigcup_{k=1}^{N-u+1}\{S_u^{(k)}=u-1\}.$$

We have $P\{S_u^{(k)}=u-1\}=\frac{1}{2^{u-1}}$ and thus $P(A_N)\leqslant\frac{N}{2^{[(1+\varepsilon)\log N]}}$. For any $T\in\mathbf{N}$ with $T\varepsilon>1$, $k\in\mathbf{N}$, it holds that

$$\sum_{k=1}^{\infty}P(A_{k^T})\leqslant 2\sum_{k=1}^{\infty}\frac{k^T}{k^{T(1+\varepsilon)}}=2\sum_{k=1}^{\infty}\frac{1}{k^{T\varepsilon}}<\infty,$$

which together with the Borel-Cantelli lemma implies that

$$P(A_{k^T}\ \text{i. o.})=P(\limsup_{k\to\infty}\bigcup_{i=0}^{k^T-u+1}\{S_u^{(i)}=u-1\})=0$$

It follows that

$$\limsup_{N\to\infty}\frac{M_N}{\log N}\leqslant 1 \text{ a. s.} \tag{7}$$

Let $k^T<n<(k+1)^T$. By (7), we have

$$M_n \leqslant M_{(k+1)^T} \leqslant (1+\varepsilon)\log (k+1)^T \leqslant (1+2\varepsilon)\log k^T \leqslant (1+2\varepsilon)\log n$$

with probability 1 for all but finitely many n. Hence we get that $\limsup_{N\to\infty}\frac{M_N}{\log N}\leqslant 1$ a. s.

Now we turn to the proofs for Theorems 2. 2, 2. 3, 2. 4* and 2. 5*. The basic idea of these comes from Ref. [7]. For the reader's convenience, we spell out the details. At first, as in Ref. [7, Theorem 5], we give an estimate for the length of consecutive switches, which is very useful in our proofs.

Theorem 3. 1 Let $N,K\in\mathbf{N}$ and let M_N be defined in (3) with $i=1$. Then, if $N\geqslant 2K$, then

$$\left(1-\frac{K+2}{2^K}\right)^{\left[\frac{N}{K}\right]-1} \leqslant P(M_N<K-1) \leqslant \left(1-\frac{K+2}{2^K}\right)^{\left[\frac{1}{2}\left[\frac{N}{K}\right]\right]} \tag{8}$$

To prove Theorem 3. 1, we need the following lemma.

Lemma 3. 2 Let $N,m\in\mathbf{N}$ and let $M_N^{(m)}$ be defined in (3). Then

$$P(M_{2N}^{(m)}\geqslant N-1)=\frac{N+2}{2^N} \tag{9}$$

Proof Since $\{M_{2N}^{(i)},i\in\mathbf{N}\}$ are identically distributed, we only consider the case that $i=1$ in the following. Let

$$A=\{M_{2N}\geqslant N-1\},$$
$$A_k=\{M_N^{(k+1)}=N-1\},k=0,1,\cdots,N.$$

Then we have

$$A=A_0+\overline{A}_0A_1+\overline{A}_0\overline{A}_1\overline{A}_2+\cdots+\bar{A}_0\bar{A}_1\cdots\bar{A}_{N-1}A_N \tag{10}$$

and

$$P(A_0)=\frac{1}{2^{N-1}},P(\bar{A}_0\bar{A}_1\cdots\bar{A}_{k-1}A_k)=\frac{1}{2^N},k=1,\cdots,N.$$

Hence

$$P(A)=P(A_0)+P(\overline{A}_0A_1)+P(\overline{A}_0\overline{A}_1\overline{A}_2)+\cdots+P(\bar{A}_0\bar{A}_1\cdots\bar{A}_{N-1}A_N)=\frac{1}{2^{N-1}}+\frac{N}{2^N}=\frac{N+2}{2^N}.$$

Proof of Theorem 3. 1 Let $N,K\in\mathbf{N}$ with $N\geqslant 2K$. Denote

$$B_j=\{M_K^{(j+1)}\geqslant K-1\},j=0,1,\cdots,N-K,$$

$$C_l=\bigcup_{j=lk}^{(l+1)k}B_j,l=0,1,\cdots,\left[\frac{N-2K}{K}\right].$$

Then for any $l=0,1,\cdots,\left[\frac{N-2K}{K}\right]$, we have $C_l=\{M_{2K}^{(lK+1)}\geqslant K-1\}$, and for any $l=0,1,\cdots,\left[\frac{N-2K}{K}\right]-2$, we have $C_l\cap C_{l+2}=\varnothing$. By Lemma 3. 2, we know that for any $l=0,1,\cdots,\left[\frac{N-2K}{K}\right]$, $P(C_l)=\frac{K+2}{2^K}$.

Let

$$D_0=C_0+C_2+\cdots+C_{2\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]\right]},$$
$$D_1=C_1+C_3+\cdots+C_{2\left[\frac{1}{2}\left(\left[\frac{N-2K}{K}\right]-1\right)\right]+1} \tag{11}$$

By the independence of $\{C_0, C_2, \cdots, C_{2\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]\right]}\}$, we have

$$P(\bar{D}_0)=P(\bar{C}_0)P(\bar{C}_2)\cdots P(\bar{C}_{2\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]\right]})=\left(1-\frac{K+2}{2^K}\right)^{\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]\right]+1} \tag{12}$$

Similarly we have

$$P(\bar{D}_1)=\left(1-\frac{K+2}{2^K}\right)^{\left[\frac{1}{2}\left(\left[\frac{N-2K}{K}\right]-1\right)\right]+1} \tag{13}$$

By the obvious fact that $D_0\subset\{M_N\geqslant K-1\}$, we get that

$$P(M_N<K-1)\leqslant P(\bar{D}_0)=\left(1-\frac{K+2}{2^K}\right)^{\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]\right]+1} \tag{14}$$

In the following, we prove that $P(D_1D_0)\geqslant P(D_1)P(D_0)$. To this end, by (11), it is enough to prove that for any $i=2l$, $l=0,1,\cdots,\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]\right]$, $P(D_1C_i)\geqslant P(D_1)P(C_i)$. Below we give the proof for $l=0$ and the proofs for $l=1,\cdots,\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]\right]$ are similar. We omit them.

For $i=1,\cdots,K+1$, denote $F_i=\{(X_i,\cdots,X_{i+K-1})$ is the first section of consecutive switches of length $K-1$ in the sequence $(X_i,\cdots,X_{2K})\}$. Then we have

$$F_i\cap F_j=\varnothing,\forall i\neq j,C_0=\bigcup_{i=1}^{K+1}F_i,$$

and

$$P(F_1)=P\{(X_1,\cdots,X_K)\text{ has consecutive switches}\}=\frac{1}{2^{K-1}},$$

$$P(F_i)=P\{(X_j,\cdots,X_{j+K-1})\text{ has consecutive switches},X_{j-1}=X_j\}=\frac{1}{2^K},i=2,\cdots,K.$$

By the independence of $\{X_j, j=1,2,\cdots,N\}$, we have

$$P(D_1F_1)=P(D_1)P(F_1) \quad (15)$$

$P(D_1F_2)=$
$P(D_1\cap\{(X_2,\cdots,X_{K+1})\}$ has consecutive switches$,X_1=X_2\})=$
$P(D_1\cap\{(X_2,\cdots,X_{K+1})\}$ has consecutive switches$,X_1=X_2,X_{K+1}=1\})+$
$P(D_1\cap\{(X_2,\cdots,X_{K+1})\}$ has consecutive switches$,X_1=X_2,X_{K+1}=0\})=$
$2P(D_1\cap\{(X_2,\cdots,X_{K+1})\}$ has consecutive switches$,X_1=X_2,X_{K+1}=1\})=$
$2P(D_1\cap\{X_{K+1}=1\})P\{(X_2,\cdots,X_K)$ has consecutive switches$,X_K=0,X_1=X_2\})=$
$\frac{1}{2^K}P(D_1)=P(D_1)P(F_2) \quad (16)$

$P(D_1F_3)=$(suppose that $K\geqslant 3$)
$P(D_1\cap\{(X_3,\cdots,X_{K+2})\}$ has consecutive switches$,X_2=X_3\})=$
$P(D_1\cap\{(X_3,\cdots,X_{K+2})\}$ has consecutive switches$,X_2=X_3,(X_{K+1},X_{K+2})=(0,1)\})+$
$P(D_1\cap\{(X_3,\cdots,X_{K+2})\}$ has consecutive switches$,X_2=X_3,(X_{K+1},X_{K+2})=(1,0)\})=$
$2P(D_1\cap\{(X_3,\cdots,X_{K+2})\}$ has consecutive switches$,X_2=X_3,(X_{K+1},X_{K+2})=(0,1)\})=$
$2P(\{(X_3,\cdots,X_K)$ has consecutive switches, $X_K=1,X_2=X_3\})\times P(D_1\cap\{(X_{K+1},X_{K+2})=(0,1)\})=\frac{1}{2^{K-2}}P(D_1\cap\{(X_{K+1},X_{K+2})=(0,1)\})=4P(D_1\cap\{(X_{K+1},X_{K+2})=(0,1)\})P(F_3) \quad (17)$

By the definition of D_1, we know that

$P(D_1\cap\{(X_{K+1},X_{K+2})=(0,1)\})\geqslant P(D_1\cap\{(X_{K+1},X_{K+2})=(0,0)\})$,
$P(D_1\cap\{(X_{K+1},X_{K+2})=(0,1)\})\geqslant P(D_1\cap\{(X_{K+1},X_{K+2})=(1,1)\})$,
$P(D_1\cap\{(X_{K+1},X_{K+2})=(0,1)\})=P(D_1\cap\{(X_{K+1},X_{K+2})=(1,0)\})$,

which together with (17) implies that

$$P(D_1F_3)-P(D_1)P(F_3)=\{P(\{(X_{K+1},X_{K+2})=(0,1)\}\cap D_1)-P(\{(X_{K+1},X_{K+2})=(0,0)\}\cap D_1)+P(\{(X_{K+1},X_{K+2})=(0,1)\}\cap D_1)-P(\{(X_{K+1},X_{K+2})=(1,1)\}\cap D_1)\}P(F_3)\geqslant 0 \quad (18)$$

Similarly, if $K\geqslant 4$, we have that

$$P(D_1F_i)\geqslant P(D_1)P(F_i),\ \forall i=4,\cdots,K \quad (19)$$

Finally, by the definitions of D_1 and F_{K+1}, we know that $D_1\cap F_{K+1}=F_{K+1}$. Hence we have

$$P(D_1F_{K+1})=P(F_{K+1})\geqslant P(D_1)P(F_{K+1}) \quad (20)$$

By (15), (16), (18), (19) and (20), we obtain

$$P(D_1\cap C_0)=\sum_{i=1}^{K+1}P(D_1\cap F_i)\geqslant \sum_{i=1}^{K+1}P(D_1)P(F_i)=P(D_1)P(C_0).$$

It is easy to check that

$$P(D_1D_0)\geqslant P(D_1)P(D_0)\Leftrightarrow P(\bar{D}_1\bar{D}_0)\geqslant P(\bar{D}_1)P(\bar{D}_0).$$

Then by (12) and (13), we get

$$P(\bar{D}_1\bar{D}_0)\geqslant \left(1-\frac{K+2}{2^K}\right)^{\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]\right]+1+\left[\frac{1}{2}\left(\left[\frac{N-2K}{K}\right]-1\right)\right]+1} \quad (21)$$

As to the right-hand side of (21), we have

(i) when $\left[\frac{N-2K}{K}\right]$ is even, the exponential part on the right-hand side is equal to

$$\frac{1}{2}\left[\frac{N-2K}{K}\right]+\frac{1}{2}\left[\frac{N-2K}{K}\right]-1+2=\left[\frac{N-2K}{K}\right]+1;$$

(ii) when $\left[\frac{N-2K}{K}\right]$ is odd, the exponential part on the right-hand is equal to

$$\frac{1}{2}\left[\frac{N-2K}{K}\right]-\frac{1}{2}+\left[\frac{1}{2}\left[\frac{N-2K}{K}\right]-\frac{1}{2}\right]+2=\left[\frac{N-2K}{K}\right]+1.$$

Hence

$$P(M_N<K-1)=P(\bar{D}_0\bar{D}_1)\geqslant \left(1-\frac{K+2}{2^K}\right)^{\left[\frac{N-2K}{K}\right]+1} \quad (22)$$

By (14) and (22), we complete the proof.

To prove Theorem 2.2, we need the following lemma.

Lemma 3.3 Let $\{\alpha_j, j\geqslant 1\}$ be a sequence of positive numbers. Suppose that $\lim_{j\to\infty}\alpha_j=a>0$. Then we have $\sum_{j=1}^{\infty}\frac{1}{\alpha_j^{\log j}}<+\infty$ if $a>2$, and $\sum_{j=1}^{\infty}\frac{1}{\alpha_j^{\log j}}=+\infty$ if $a<2$.

Proof We have

$$\sum_{j=1}^{\infty}\frac{1}{\alpha_j^{\log j}}=\sum_{j=1}^{\infty}\frac{1}{\alpha_j^{\frac{\log_{\alpha_j} j}{\log_{\alpha_j} 2}}}=\sum_{j=1}^{\infty}j^{-\frac{1}{\log_{\alpha_j} 2}}=\sum_{j=1}^{\infty}j^{-\log\alpha_j}.$$

If $a>2$, then $p=\log a>1$ and thus

$$\sum_{j=1}^{\infty}j^{-\log\alpha_j}=\sum_{j=1}^{\infty}j^{-\log a}\frac{j^{-\log\alpha_j}}{j^{-\log a}}=$$
$$\sum_{j=1}^{\infty}j^{-p}\cdot j^{-\log\frac{a}{\alpha_j}}<\infty.$$

If $a<2$, then $p=\log a<1$ and thus

$$\sum_{j=1}^{\infty}j^{-\log\alpha_j}=\sum_{j=1}^{\infty}j^{-\log a}\frac{j^{-\log\alpha_j}}{j^{-\log a}}=$$
$$\sum_{j=1}^{\infty}j^{-p}\cdot j^{-\log\frac{a}{\alpha_j}}=\infty.$$

Proof of Theorem 2.2　Let N_j be the smallest integer with $\alpha_1(N_j)+1=j$. Since $\lim\limits_{j\to\infty}\left(1-\frac{j+2}{2^j}\right)^{-\frac{3}{2}}=1$, there exists $M\in\mathbf{N}$ such that $M>2$ and

$$\left(1-\frac{j+2}{2^j}\right)^{-\frac{3}{2}}\leqslant 2,\ \forall j>M.$$

Then by Theorem 3.1, we have

$$\sum_{j=1}^{\infty}P\{M_{N_j}<\alpha_1(N_j)\}=\sum_{j=1}^{\infty}P\{M_{N_j}<j-1\}\leqslant$$
$$\sum_{j=1}^{\infty}\left(1-\frac{j+2}{2^j}\right)^{\left[\frac{1}{2}\left[\frac{N_j}{j}\right]\right]}=$$
$$\sum_{j=1}^{M}\left(1-\frac{j+2}{2^j}\right)^{\left[\frac{1}{2}\left[\frac{N_j}{j}\right]\right]}+$$
$$\sum_{j=M+1}^{\infty}\left(1-\frac{j+2}{2^j}\right)^{\left[\frac{1}{2}\left[\frac{N_j}{j}\right]\right]}:=$$
$$\beta+\sum_{j=M+1}^{\infty}\left(1-\frac{j+2}{2^j}\right)^{\left[\frac{1}{2}\left[\frac{N_j}{j}\right]\right]}\leqslant$$
$$\beta+\sum_{j=M+1}^{\infty}\left(1-\frac{j+2}{2^j}\right)^{\frac{N_j}{2j}-\frac{3}{2}}\leqslant$$
$$\beta+2\sum_{j=M+1}^{\infty}\left(1-\frac{j+2}{2^j}\right)^{\frac{N_j}{2j}}=$$
$$\beta+2\sum_{j=M+1}^{\infty}(\sqrt{e_j})^{-\frac{N_j}{2j}\cdot\frac{j+2}{j}},$$

where $e_j:=\left(1+\frac{j+2}{2^j}\right)^{-\frac{2^j}{j+2}}$. By $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\mathrm{e}$, we have

$$\lim_{j\to\infty}e_j=\mathrm{e}\tag{23}$$

Without loss of generality, we assume that $e_j\geqslant 2$ for any $j>M$. By $\alpha_1(N_j)+1=j$, we have

$$j\leqslant\log N_j-\log\log\log N_j+\log\log e-1-\varepsilon$$

and thus

$$\frac{N_j}{2^j}\geqslant\frac{2^{1+\varepsilon}\cdot\log\log N_j}{\log e},\ \log j<\log\log N_j.$$

Thus

$$\sum_{j=M+1}^{\infty}(\sqrt{e_j})^{-\frac{N_j}{2j}\cdot\frac{j+2}{j}}=\sum_{j=M+1}^{\infty}\left(\frac{1}{e_j}\right)^{\frac{1}{2}\cdot\frac{N_j}{2j}\cdot\frac{j+2}{j}}\leqslant$$
$$\sum_{j=M+1}^{\infty}\left(\frac{1}{e_j}\right)^{\frac{1}{2}\cdot\frac{2^{1+\varepsilon}\cdot\log\log N_j}{\log e}\cdot\frac{j+2}{j}}=$$
$$\sum_{j=M+1}^{\infty}\left(\frac{1}{\mathrm{e}^{\frac{2^{\varepsilon}}{\log e}}}\right)^{\frac{j+2}{j}\cdot\log\log N_j}\leqslant\sum_{j=M+1}^{\infty}\left(\frac{1}{\mathrm{e}^{\frac{2^{\varepsilon}}{\log e}}}\right)^{\frac{j+2}{j}\cdot\log j}.$$

For any $\varepsilon>0$, by (23), we have $\lim\limits_{j\to\infty}e_j^{\frac{2^{\varepsilon}}{\log e}}=\mathrm{e}^{\frac{2^{\varepsilon}}{\log e}}=(\mathrm{e}^{\ln 2})^{2^{\varepsilon}}=2^{2^{\varepsilon}}>2$. Then by Lemma 3.3 and the Borel-Cantelli lemma, we complete the proof.

To prove Theorem 2.3, we need the following version of Borel-Cantelli lemma.

Lemma 3.4(Kochen and Stone[28])　Let A_1, A_2, … be arbitrary events, fulfilling the conditions $\sum\limits_{n=1}^{\infty}P(A_n)=\infty$ and

$$\liminf_{n\to\infty}\frac{\sum\limits_{1\leqslant k\leqslant l\leqslant n}P(A_kA_l)}{\sum\limits_{1\leqslant k\leqslant l\leqslant n}P(A_k)P(A_l)}=1\tag{24}$$

Then $P(\limsup\limits_{n\to\infty}A_n)=1$.

Proof of Theorem 2.3　Let $\delta>0$. Let $N_j=N_j(\delta)$ be the smallest integer for which $\alpha_2(N_j)=[j^{1+\delta}]$ with $\alpha_2(N_j)$ given by (6). Define

$$A_j:=\{M_{N_j}<\alpha_2(N_j)\}=$$
$$\{M_{N_j}<([j^{1+\delta}]+1)-1\},\ j\geqslant 1\tag{25}$$

By Theorem 3.1, we have

$$\sum_{j=1}^{\infty}P(A_j)\geqslant\sum_{j=1}^{\infty}\left(1-\frac{[j^{1+\delta}]+3}{2^{[j^{1+\delta}]+1}}\right)^{\left[\frac{N_j}{[j^{1+\delta}]+1}\right]-1}\geqslant$$
$$\sum_{j=1}^{\infty}\left(1-\frac{[j^{1+\delta}]+3}{2^{[j^{1+\delta}]+1}}\right)^{-\frac{2^{[j^{1+\delta}]+1}}{[j^{1+\delta}]+3}\cdot\left(-\frac{[j^{1+\delta}]+3}{2^{[j^{1+\delta}]+1}}-\frac{N_j}{[j^{1+\delta}]+1}\right)}:=$$
$$\sum_{j=1}^{\infty}f_j^{-\frac{[j^{1+\delta}]+3}{[j^{1+\delta}]+1}-\frac{N_j}{2^{[j^{1+\delta}]+1}}}\tag{26}$$

where $f_j:=\left(1+\frac{[j^{1+\delta}]+3}{2^{[j^{1+\delta}]+1}}\right)^{-\frac{2^{[j^{1+\delta}]+1}}{[j^{1+\delta}]+3}}$. As in (23), we have

$$\lim_{j\to\infty}f_j=\mathrm{e}\tag{27}$$

Then there exists $M\in\mathbf{N}$ such that $M>2$ and

$$f_j\geqslant 2,\ \forall j>M\tag{28}$$

By (6) we have

$$\log N_j-\log\log\log N_j+\log\log e+\varepsilon\leqslant[j^{1+\delta}]+1,$$

which implies that

$$\frac{N_j}{2^{[j^{1+\delta}]+1}} \leqslant \frac{\log\log N_j}{2^{\varepsilon}\log e}.$$

Then by (26) and (28), we get

$$\sum_{j=1}^{\infty} P(A_j) \geqslant \sum_{j=1}^{M} P(A_j) + \sum_{j=M+1}^{\infty} f_j^{-\frac{[j^{1+\delta}]+3}{[j^{1+\delta}]+1}\frac{\log\log N_j}{2^{\varepsilon}\log e}} \tag{29}$$

Let $0<\varepsilon<1$ satisfy

$$[j^{1+\delta}] = [\log N_j - \log\log\log N_j + \log\log e + \varepsilon] > \varepsilon_0 \log N_j, \forall j = 1, 2, \cdots.$$

Then $\log N_j < \frac{1}{\varepsilon_0} \cdot j^{1+\delta}$, which implies that

$$\log\log N_j < \log\frac{1}{\varepsilon_0} + (1+\delta)\log j \tag{30}$$

Hence we have

$$\sum_{j=M+1}^{\infty} f_j^{-\frac{[j^{1+\delta}]+3}{[j^{1+\delta}]+1}\frac{\log\log N_j}{2^{\varepsilon}\log e}} = \sum_{j=M+1}^{\infty} f_j^{-\frac{[j^{1+\delta}]+3}{[j^{1+\delta}]+1}\cdot 2^{-\log\log e-\varepsilon\cdot(-\log\log N_j)}} \geqslant \sum_{j=M+1}^{\infty} f_j^{-\frac{[j^{1+\delta}]+3}{[j^{1+\delta}]+1}\cdot 2^{-\log\log e-\varepsilon\cdot(-\log\frac{1}{\varepsilon_0}-(1+\delta)\log j)}} \tag{31}$$

For any given positive number ε, by (27), we have

$$\lim_{j\to\infty} f_j^{-\frac{[j^{1+\delta}]+3}{[j^{1+\delta}]+1}\cdot\frac{1+\delta}{2^{\varepsilon}\log e}} = 2^{\frac{1+\delta}{2^{\varepsilon}}} < 2 \tag{32}$$

when δ is small enough. By Lemma 3.3, (29), (31) and (32), we get that $\sum_{n=1}^{\infty} P(A_n) = \infty$.

Recall that $\{A_j, j\geqslant 1\}$ is defined in (25). For $i<j$, we define

$$B_{i,j} = \begin{cases} \{M_{N_i} < \alpha_2(N_j)\}, & N_i \geqslant \alpha_2(N_j), \\ \Omega, & \text{otherwise}, \end{cases}$$

$$C_{i,j} = \{M_{N_j-N_i}^{(N_i+1)} < \alpha_2(N_j)\}.$$

We claim that

$$P(A_j) = P(B_{i,j})P(C_{i,j})(1+o(1)), \quad i<j\to\infty \tag{33}$$

In fact, by the definitions of A_j, $B_{i,j}$ and $C_{i,j}$, we know that

$$A_j = B_{i,j} \cap C_{i,j} \cap \{M_{2(\alpha_2(N_j)+1)}^{((N_i-\alpha_2(N_j))\vee 0+1)} < \alpha_2(N_j)\}.$$

Then (33) is equivalent to

$$\lim_{j\to\infty}\frac{P(B_{i,j}\cap C_{i,j}\cap\{M_{2(\alpha_2(N_j)+1)}^{((N_i-\alpha_2(N_j))\vee 0+1)} < \alpha_2(N_j)\})}{P(B_{i,j}\cap C_{i,j})} = 1$$
$$\Leftrightarrow \lim_{j\to\infty} P(\{M_{2(\alpha_2(N_j)+1)}^{((N_i-\alpha_2(N_j))\vee 0+1)} < \alpha_2(N_j)\} \mid B_{i,j}\cap C_{i,j}) = 1$$
$$\Leftrightarrow \lim_{j\to\infty} P(\{M_{2(\alpha_2(N_j)+1)}^{((N_i-\alpha_2(N_j))\vee 0+1)} \geqslant \alpha_2(N_j)\} \mid B_{i,j}\cap C_{i,j}) = 0 \tag{34}$$

By the definition of $C_{i,j}$, we get that

$$\begin{aligned} & P(\{M_{2(\alpha_2(N_j)+1)}^{((N_i-\alpha_2(N_j))\vee 0+1)} \geqslant \alpha_2(N_j)\} \mid B_{i,j}\cap C_{i,j}) = \\ & P\Big(\bigcup_{k=(N_i-\alpha_2(N_j))\vee 0+1}^{N_i-1} [M_{\alpha_2(N_j)+1}^{(k)} = \alpha_2(N_j)] \mid B_{i,j}\cap C_{i,j}\Big) \leqslant \\ & \sum_{k=(N_i-\alpha_2(N_j))\vee 0+1}^{N_i-1} P([M_{\alpha_2(N_j)+1}^{(k)} = \alpha_2(N_j)] \mid B_{i,j}\cap C_{i,j}) \leqslant \\ & \sum_{k=(N_i-\alpha_2(N_j))\vee 0+1}^{N_i-1} \frac{P(M_{\alpha_2(N_j)+1}^{(k)} = \alpha_2(N_j))}{P(B_{i,j})P(C_{i,j})} = \\ & \frac{N_i-1}{2^{\alpha_2(N_j)}P(B_{i,j})P(C_{i,j})} \leqslant \\ & \frac{N_j-1}{2^{\alpha_2(N_j)}} \cdot \frac{1}{P(B_{i,j})P(C_{i,j})} \end{aligned} \tag{35}$$

By Theorem 3.1, we have that when j is large enough,

$$\begin{aligned} P(C_{i,j}) &= P(M_{N_j-N_i}^{(N_i+1)} < \alpha_2(N_j)) = P(M_{N_j-N_i}^{(N_i+1)} < (\alpha_2(N_j)+1)-1) \geqslant \\ & \left(1-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)^{\left[\frac{N_j-N_i}{\alpha_2(N_j)+1}\right]-1} = \left(1-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)^{-\frac{2^{\alpha_2(N_j)+1}}{\alpha_2(N_j)+3}\cdot\left(\left[\frac{N_j-N_i}{\alpha_2(N_j)+1}\right]-1\right)\cdot\left(-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)} \geqslant \\ & \left(1-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)^{-\frac{2^{\alpha_2(N_j)+1}}{\alpha_2(N_j)+3}\cdot\left(\left[\frac{N_j}{\alpha_2(N_j)+1}\right]-1\right)\cdot\left(-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)} \to \frac{1}{e} \end{aligned} \tag{36}$$

as $j\to\infty$. When $N_i \geqslant \alpha_2(N_j)$ and $i<j$, by Theorem 3.1, we have

$$\begin{aligned} P(B_{i,j}) &= P(M_{N_j-N_i} < \alpha_2(N_j)) \geqslant P(M_{N_{j-1}} < (\alpha_2(N_j)+1)-1) \geqslant \left(1-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)^{\left[\frac{N_{j-1}}{\alpha_2(N_j)+1}\right]-1} = \\ & \left(1-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)^{-\frac{2^{\alpha_2(N_j)+1}}{\alpha_2(N_j)+3}\cdot\left(\left[\frac{N_{j-1}}{\alpha_2(N_j)+1}\right]-1\right)\cdot\left(-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)} \geqslant \left(1-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)^{-\frac{2^{\alpha_2(N_j)+1}}{\alpha_2(N_j)+3}\cdot\left(\frac{N_{j-1}}{\alpha_2(N_j)+1}\right)\cdot\left(-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)} = \end{aligned}$$

$$\left(1-\frac{\alpha_2(N_j)+3}{2^{\alpha_2(N_j)+1}}\right)^{-\frac{2^{\alpha_2(N_j)+1}}{\alpha_2(N_j)+3}\cdot\left(\frac{N_{j-1}}{2^{\alpha_2(N_j)+1}}\right)\cdot\left(-\frac{\alpha_2(N_j)+3}{\alpha_2(N_j)+1}\right)} \to 1 \tag{37}$$

as $j\to\infty$, where we used $\lim_{j\to\infty}\frac{N_{j-1}}{2^{\alpha_2(N_j)+1}}=0$, which can be deduced by the definitions of N_j and N_{j-1}, and the mean-value theorem for the function $x^{1+\delta}$. By (35)~(37), we obtain (34). Hence (33) holds.

Similar to (33), we have

$$P(A_iA_j)=P(A_i)P(C_{i,j})(1+o(1)),\quad i<j\to\infty \tag{38}$$

By (33) and (38), we get

$$P(A_iA_j)=\frac{P(A_i)P(A_j)}{P(B_{i,j})}(1+o(1)),\quad i<j\to\infty,$$

which together with (37) implies that

$$\frac{P(A_iA_j)}{P(A_i)P(A_j)}=\frac{1+o(1)}{P(B_{i,j})}=1+o(1),\quad i<j\to\infty \tag{39}$$

By the fact that $\sum_{n=1}^{\infty}P(A_n)=\infty$ and (39), we know that (34) holds. By Lemma 3.4, we complete the proof.

Proof of Theorem 2.4* Let $A_n=\{S_{\gamma_n}^{(n-\gamma_n)}\geqslant\gamma_n-1\}$. Then we have $P(A_n)=2\cdot 2^{-\gamma_n}$, which together with the assumption implies that $\sum_{n=1}^{\infty}P(A_n)=\infty$. By following the method in the proof of Theorem 2.3, we have

$$\frac{P(A_iA_j)}{P(A_i)P(A_j)}=1+o(1),\quad j\to\infty.$$

Then we get that (34) holds and thus by Lemma 3.4 we complete the proof.

Proof of Theorem 2.5* Let $B_n=\{S_{\delta_n}^{(n-\delta_n)}\geqslant\delta_n-1\}$. Then we have $\sum_{n=1}^{\infty}P(B_n)=2\cdot 2^{-\delta_n}$, which together with the assumption implies that $\sum_{n=1}^{\infty}P(B_n)<\infty$. By the Borel-Cantelli lemma, we get the result.

4 Final remark

After the first version of our paper was uploaded to arXiv, Professor Laurent Tournier sent two emails to us and gave some helpful comments. In particular, he told us one way to reduce consecutive switches to pure heads or pure tails by doing the following: introduce a sequence (Y_n) such that $Y_{2n}=X_{2n}, Y_{2n+1}=1-X_{2n+1}$. Then (Y_n) is again a sequence of independent and unbiased coin tosses. And a sequence of consecutive switches for X is equivalent to a sequence of pure heads or pure tails for Y. Then Theorems 2.1 and 2.5 can be deduced easily from Theorems 1.1 and 1.5, respectively.

We spell out all the proofs with two reasons. One is for the reader's convenience. The other is that as to biased coin tosses, it seems that Theorems 2.1 and 2.5 can not be deduced directly from Theorems 1.1 and 1.5, respectively, and our proof may be moved to that case. We will consider the biased coin tosses in a forthcoming paper.

References:

[1] Erdös P, Rényi A. On a new law of large numbers [J]. J Anal Math, 1970, 22: 103.

[2] Komlós J, Tusnády G. On sequences of " pure heads" [J]. Ann Probab, 1975, 3: 608.

[3] Erdös P, Révész P. On the length of the longest head-run [J]. Coll Math Soc J Bolyai, 1976, 16: 219.

[4] Guibas L J, Odlyzko A M. Long repetitive patterns in random sequences [J]. Z Wahrsch Verw Geb, 1980, 53: 241.

[5] Samarova S S. On the length of the longest heads-run for a Markov chain with two states [J]. Theor Probab Appl, 1981, 26: 498.

[6] Révész P. Limit theorems in probability and statistics [M]. Amsterdam: North-Holland Publishing Company, 1984.

[7] Nemetz T, Kusolitsch N. On the longest run of coincidences [J]. Z Wahrsch Verw Geb, 1982, 61: 59.

[8] Grill K. Erdös-Révész type bounds for the length of the longest run from a stationary mixing sequence [J]. Probab Theor Related Fields, 1987, 75: 77.

[9] Vaggelatou E. On the length of the longest run in a multi-state Markov chain [J]. Stat Probab Lett, 2003, 62: 211.

[10] Goncharov V L. On the field of combinatory analysis [J]. Amer Math Soc Trans, 1943, 19: 1.

[11] Földes A. The limit distribution of the length of the longest head-run [J]. Period Math Hungar, 1979, 10: 301.

[12] Arratia R, Goldstein L, Gordon L. Two moments suffice for Poisson approximation [J]. Ann Probab, 1989, 17: 9.

[13] Novak S Y. Asymptotic expansions in the problem of the length of the longest head-run for Markov chain with two states (in Russian) [J]. Trudy Inst Math, 1989, 13: 136.

[14] Novak S Y. Rate of convergence in the problem of the longest head run [J]. Siberian Math J, 1991, 32: 444.

[15] Novak S Y. Longest runs in a sequence of m-dependent random variables [J]. Probab Theor Related Fields, 1992, 91: 269.

[16] Schilling M F. The longest run of heads [J]. Coll Math J, 1990, 21: 196.

[17] Binwanger K, Embrechts P. Longest runs in coin tossing [J]. Insurance Math and Econom, 1994, 15: 139.

[18] Muselli M. Useful inequalities for the longest run distribution [J]. Stat Probab Lett, 2000, 46: 239.

[19] Túri J. Limit theorems for the longest run [J]. Ann Math Inform, 2009, 36: 133.

[20] Novak S Y. On the length of the longest head run [J]. Stat Probab Lett, 2017, 130: 111.

[21] Mao Y H, Wang F, Wu X Y. Large deviation behavior for the longest head run in an IID Bernoulli sequence [J]. J Theor Probab, 2015, 28: 259.

[22] Asmussen S, Ivanous J, Nielsen A R. Time inhomogeneity in longest gap and longest problems [J]. Stoch Process Appl, 2017, 127: 574.

[23] Chen H, Yu M. A generalization of the Erdös and Rényi limit theorem and the corresponding multifractal analysis [J]. J Number Theory, 2018, 192: 307.

[24] Li Z X, Yang X F. On the longest runs in Markov chain [J]. Probab Math Stat, 2018, 38: 407

[25] Pawelec L, Urbanski M. Long hitting times for expanding systems [EB/OL]. [2020-02-21]. arXiv, 2018, 1812.09231v3.

[26] Mezhennaya N. On the number of event appearances in a Markov chain [J]. Inter J Appl Math, 2019, 32: 537.

[27] Li W. On the number of switches in unbiased coin-tossing [J]. Stat Probab Lett, 2013, 83: 1613.

[28] Kochen S, Stone C. A note on the Borel-Cantelli lemma [J]. Illinois J Math, 1964, 8: 248.

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