关于 Jordan 函数的 gcd 和函数的渐近估计

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摘　要：将整数 k 和 j 的最大公约数记为 $\gcd(k,j)$. 设 k 为正整数，f 为任意算术函数，r 是任意固定整数，n 为任意正整数. 对实数 $x\geqslant 2$，定义与 f 关联的 gcd 和函数 $M_r(x;f)$ 为 $M_r(x;f)\coloneqq\sum_{k\leqslant x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r f(\gcd(k,j))$. 本文利用 Kiuchi 在 2017 年得到的关于 $M_r(x;f)$ 的一个恒等式及初等和解析方法给出了 $M_r(x;J_k)$ 的渐近公式，其中的若当函数 J_k 定义为 $J_k(n)\coloneqq n^k\prod_{p|n}(1-\frac{1}{p^k})$. 本文的结果加强了 Kiuchi 和 Saadeddin 在 2018 年得到的结果.

关键词：GCD 和函数；若当函数；均值；部分和；黎曼 Zeta 函数
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An asymptotic formula for the gcd-sum function of Jordan's totient function

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Abstract: Let $\gcd(k,j)$ denote the greatest common divisor of the positive integers k and j, r be any fixed positive integer. Let $M_r(x;f)\coloneqq\sum_{k\leqslant x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r f(\gcd(k,j))$, where $x\geqslant 2$ is any large real number and f is any arithmetical function. Let J_k denote Jordan's totient function defined for any integer $n\geqslant 1$ by $J_k(n)\coloneqq n^k\prod_{p|n}(1-\frac{1}{p^k})$. In this paper, by using the identity of Kiuchi on $M_r(x;f)$ together with some analytic techniques, we present an asymptotic formula of $M_r(x;J_k)$. These complement and strengthen the corresponding results obtained by Kiuchi and Saadeddin in 2018.

Keywords: Gcd-sum function; Jordan's totient function; Mean value; Partial summation; Riemann Zeta-function
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1　Introduction

Let $\gcd(k,j)$ be the greatest common divisor of the integers k and j. The gcd-sum function, which is also known as Pillai's arithmetical function, is defined by

$$P(n)=\sum_{k=1}^{n}\gcd(k,n).$$

This function has been studied by many authors such as Broughan[1], Bordelles[2], Tanigawa and

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Zhai[3], Toth[4], and others. Analytic properties for partial sums of the gcd-sum function $f(\gcd(j,k))$ were recently studied by Inoue and Kiuchi [5]. We recall that the symbol $*$ denotes the Dirichlet convolution of two arithmetical functions f and g which is defined by

$$f*g(n)=\sum_{d|n}f(d)g(n/d)$$

for every positive integer n. For any arithmetical function f, Kiuchi[6] showed that for any fixed positive integer r and any large positive number $x\geqslant 2$,

$$M_r(x;f):=\sum_{k\leqslant x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r f(\gcd(k,j))=\frac{1}{2}\sum_{n\leqslant x}\frac{f(n)}{n}+\frac{1}{r+1}\sum_{dl\leqslant x}\frac{\mu*f(d)}{d}+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{dl\leqslant x}\frac{\mu*f(d)}{d}\frac{1}{l^{2m}} \quad (1)$$

Here, as usual, the function μ denotes the Mobius function,

$$\mu(x)=\begin{cases}1, \text{ if } x=1,\\ (-1)^k, \text{ if } x \text{ is the product of } k \text{ distinct primes}, x=p_1p_2\cdots p_k,\\ 0, \text{ if } x \text{ is divisible by the square of a prime},\end{cases}$$

$B_m=B_m(0)$ is a Bernoulli number, where $B_m(x)$ is a Bernoulli polynomial defined by the generating function

$$\frac{z\,e^{xz}}{e^z-1}=\sum_{m=0}^{\infty}B_m(x)\frac{z^m}{m!}$$

with $|z|<2$. Many applications of (1) have been given in Refs. [6-8]. Some related results are refereed to Refs. [9-11].

Let k be a positive integer. In number theory, Jordan's totient function J_k is defined by

$$J_k(n)=n^k\prod_{p|n}(1-\frac{1}{p^k}),$$

where p ranges through the prime divisors of n. Then J_k is multiplicative, $\sum_{d|n}J_k(d)=n^k$, $J_k*1=id_k$ and $J_k=\mu*id_k$, where the function id_k is defined by $id_k(x)=x^k$. Let $id:=id_1$, $id(x)=x$. Then $J_1=\varphi$, $\varphi=id*\mu$ and $\varphi*1=id$. We can state the main result of this paper as follows.

Theorem 1.1 Let a be a positive integer >1. Then for any fixed positive integer r and any large positive number $x>5$, we have

$$M_r(x;J_{a+1})=\left(\frac{1}{(a+1)\zeta(a+2)}+\frac{1}{r+1}\frac{a+1}{\zeta(a+1)\,\zeta^2(a+2)}+\frac{1}{(r+1)(a+1)\,\zeta^2(a+2)}\sum_{t=1}^{[r/2]}\binom{r+1}{2t}B_{2t}\zeta(a+2t+1)\right)x^{a+1}+O(x^a).$$

2 Preliminaries

Lemma 2.1 (i) Let $\tau=1*1$ be the divisor function. Then for any large positive number x, we have

$$\sum_{n\leqslant x}\tau(n)=x(\log x+2\gamma-1)+\Delta(x) \quad (2)$$

where γ is the Euler constant and error term $\Delta(x)$ can be estimated by $\Delta(x)=O(x^{\theta+\epsilon})$, it is known that one can take $\frac{1}{4}\leqslant\theta\leqslant\frac{1}{3}$;

(ii) Let $\sigma_u=id_u*1$ be the generalized divisor function for any real number u and let $a\geqslant 1$ be an integer. Then for any large positive number $x\geqslant 2$, we have

$$\sum_{n\leqslant x}\sigma_a(n)=\frac{\zeta(a+1)}{a+1}x^{a+1}+O(x^a) \quad (3)$$

where ζ denotes the Riemann Zeta-function, that is $\zeta(a)=\sum_{n=1}^{\infty}\frac{1}{n^a}$;

(iii) For any large positivenumber $x>5$, we have

$$\sum_{n\leqslant x}\frac{\mu(n)}{n}=O(\delta(x)) \quad (4)$$

$$\sum_{n\leqslant x}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}+O(\frac{\delta(x)}{x}) \quad (5)$$

$$\sum_{n\leqslant x}\frac{\mu(n)}{n^k}=\frac{1}{\zeta(k)}+O(\frac{\delta(x)}{x^{k-1}}) \quad (6)$$

where $\delta(x) := \exp\left[-C\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right]$.

Proof (2) is Theorem 7.3 of Ref. [12], (3)~(6) follow from Ref. [13].

Lemma 2.2 Let k be an integer with $k>1$. Then for any large positive number $x>5$, we have

$$\sum_{n\leqslant x}\frac{(\mu*\mu)(n)}{n}=O(\delta(x)) \tag{7}$$

$$\sum_{n\leqslant x}\frac{(\mu*\mu)(n)}{n^2}=\frac{1}{\zeta^2(2)}+O(\frac{\delta(x)}{x}) \tag{8}$$

$$\sum_{n\leqslant x}\frac{(\mu*\mu)(n)}{n^k}=\frac{1}{\zeta^2(k)}+O(\frac{\delta(x)}{x^{k-1}}) \tag{9}$$

Proof (7) and (8) are shown in Ref. [5]. Now let us show (9). Since

$$\sum_{n=1}^{\infty}\frac{(\mu*\mu)(n)}{n^k}=(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^k})^2=(\frac{1}{\zeta(k)})^2,$$

it follows that

$$\sum_{n\leqslant x}\frac{(\mu*\mu)(n)}{n^k}=\sum_{n=1}^{\infty}\frac{(\mu*\mu)(n)}{n^k}-\sum_{n>x}\frac{(\mu*\mu)(n)}{n^k}=\frac{1}{\zeta^2(k)}-\sum_{n>x}\frac{(\mu*\mu)(n)}{n^k}.$$

Putting $g(n)=\frac{1}{n^{k-1}}$, it can be easily shown that

$$g(n+1)-g(n)=O(\frac{1}{n^k}).$$

Therefore by partial summation and (7), one has

$$\sum_{n>x}\frac{(\mu*\mu)(n)}{n^k}=\sum_{n>x}\frac{(\mu*\mu)(n)}{n}g(n)=-\sum_{n\leqslant x}\frac{(\mu*\mu)(n)}{n}g([x]+1)-\sum_{n>x}\sum_{l<n}\frac{(\mu*\mu)(l)}{n}(g(n+1)-g(n))=O(\frac{\delta(x)}{x^{k-1}})+O(\sum_{n>x}\frac{\delta(n)}{n^k}).$$

In what follows, we claim that $\sum_{n>x}\frac{1}{n^k}=O(\frac{1}{x^{k-1}})$. To prove the claim, we use the partial summation formula with $f(n)=1$ and $g(n)=\frac{1}{n^k}$ to get $F(t)=[t]$ and $g'(t)=-kt^{-k-1}$. We have

$$\sum_{n>x}\frac{1}{n^k}=-[x]\frac{1}{x^k}-\int_x^{\infty}[t](-kt^{-k-1})\mathrm{d}t=-\frac{1}{x^{k-1}}+\{x\}\frac{1}{x^k}-(-k\int_x^{\infty}t^{-k}\mathrm{d}t+k\int_x^{\infty}\{t\}t^{-k-1}\mathrm{d}t)=-\frac{1}{x^{k-1}}-\frac{k}{1-k}\frac{1}{x^{k-1}}+O(\frac{1}{x^k})=\frac{1}{k-1}\frac{1}{x^{k-1}}+O(\frac{1}{x^k})=O(\frac{1}{x^{k-1}}),$$

as claimed. This completes the proof of the claim.

Since $\delta(x)$ is monotonic decreasing, it follows from the claim that

$$\sum_{n>x}\frac{(\mu*\mu)(n)}{n^k}=O(\frac{\delta(x)}{x^{k-1}})+O(\delta(x)\sum_{n>x}\frac{1}{n^k})=O(\frac{\delta(x)}{x^{k-1}}).$$

Finally, one deduces that

$$\sum_{n\leqslant x}\frac{(\mu*\mu)(n)}{n^k}=\frac{1}{\zeta^2(k)}+O(\frac{\delta(x)}{x^{k-1}}).$$

Thus (9) is true. Lemma 2.2 is proved.

Lemma 2.3 Let a be a fixed number with $1<a$. Then for any real number $x>1$, we have

$$\sum_{n\leqslant x}n^a=\frac{x^{a+1}}{a+1}+O(x^a) \tag{10}$$

Proof We use partial summation with $f(n)=1$ and $g(t)=x^a$. Then $F(t)=[t]$ and $g'(t)=at^{a-1}$. Thus

$$\sum_{n\leqslant x}n^a=[x]x^a-\int_1^x[t]at^{a-1}\mathrm{d}t=x^{a+1}-\{x\}x^a-(\int_1^x at^a\mathrm{d}t-\int_1^x\{t\}at^{a-1}\mathrm{d}t)=x^{a+1}-\frac{a}{a+1}x^{a+1}+O(x^a)=\frac{x^{a+1}}{a+1}+O(x^a),$$

as required. This completes the proof of Lemma 2.3.

Lemma 2.4[14] Let m,t be a fixed real number with $m>1$ and $t>1$. Then for any sufficiently large number $x>2$, we have

$$\sum_{n\leqslant x}\frac{\sigma_m(n)}{n^t}=\frac{\zeta(m+1)}{m-t+1}x^{m-t+1}+O(x^{m-t}) \tag{11}$$

Lemma 2.5 Let $a>1$ be a positive integer. For any large positive number $x>5$, we have

$$\sum_{n\leqslant x}\frac{J_{1+a}(n)}{n}=\frac{1}{(1+a)\zeta(2+a)}x^{1+a}+O(x^a).$$

Proof Since $J_k=\mu*id_k$, we have

$$\frac{J_k}{id}=\frac{id_k*\mu}{id}=\frac{id_k}{id}*\frac{\mu}{id}=id_{k-1}*\frac{\mu}{id}=\frac{\mu}{id}*id_{k-1}.$$

Hence

$$\sum_{n\leqslant x}\frac{J_{1+a}(n)}{n}=\sum_{n\leqslant x}(\frac{J_{1+a}}{id})(n)=$$

$$\sum_{n\leqslant x}(\frac{\mu}{id}*id_a)(n)=\sum_{n\leqslant x}\sum_{d\mid n}\frac{\mu(d)}{d}id_a(\frac{n}{d}).$$

Let $l=\frac{n}{d}$. Then $dl=n\leqslant x$, $l\leqslant\frac{x}{d}$. Thus

$$\sum_{n\leqslant x}\sum_{d\mid n}\frac{\mu(d)}{d}id_a(\frac{n}{d})=\sum_{d\leqslant x}\frac{\mu(d)}{d}\sum_{l\leqslant\frac{x}{d}}id_a(l).$$

By (10), one gets that

$$\sum_{d\leqslant x}\frac{\mu(d)}{d}\sum_{l\leqslant\frac{x}{d}}id_a(l)=\sum_{d\leqslant x}\frac{\mu(d)}{d}(\frac{(\frac{x}{d})^{a+1}}{a+1}+$$

$$O((\frac{x}{d})^a))=\frac{x^{1+a}}{1+a}\sum_{d\leqslant x}\frac{\mu(d)}{d^{2+a}}+\sum_{d\leqslant x}\frac{\mu(d)}{d^{1+a}}O(x^a).$$

By (6), we arrive at

$$\sum_{n\leqslant x}\frac{J_{1+a}(n)}{n}=\sum_{d\leqslant x}\frac{\mu(d)}{d}\sum_{l\leqslant\frac{x}{d}}id_a(l)=$$

$$\frac{1}{1+a}x^{1+a}(\frac{1}{\zeta(2+a)}+O(\frac{\delta(x)}{x^{1+a}}))+$$

$$(\frac{1}{\zeta(1+a)}+O(\frac{\delta(x)}{x^a}))O(x^a)=$$

$$\frac{1}{(1+a)\zeta(2+a)}x^{1+a}+O(x^a),$$

as desired. This finishes the proof of Lemma 2.5.

Lemma 2.6 Let $a>1$ be a positive integer. For any large positive number $x>5$, we have

$$\sum_{dl\leqslant x}\frac{(\mu*J_{1+a})(d)}{d}=\sum_{n\leqslant x}\sum_{d\mid n}\frac{(\mu*J_{1+a})(d)}{d}=$$

$$\frac{a+1}{\zeta(a+1)\zeta^2(a+2)}x^{a+1}+O(x^a).$$

Proof Since $J_k=\mu*id_k$, one has

$$\frac{(\mu*J_k)}{id}*1=\frac{id_k*\mu*\mu}{id}*1=\frac{id_k}{id}*\frac{\mu*\mu}{id}*1=$$

$$id_{k-1}*\frac{\mu*\mu}{id}*1=\frac{\mu*\mu}{id}*id_{k-1}*1=$$

$$\frac{\mu*\mu}{id}*\sigma_{k-1}.$$

By the identity $\frac{(\mu*J_k)}{id}*1=\frac{\mu*\mu}{id}*\sigma_{k-1}$, we get

$$\sum_{n\leqslant x}\sum_{d\mid n}\frac{(\mu*J_{1+a})(d)}{d}=\sum_{n\leqslant x}(\frac{(\mu*J_{1+a})}{id}*1)(n)=$$

$$\sum_{n\leqslant x}(\frac{(\mu*\mu)}{id}*\sigma_a)(n)=$$

$$\sum_{n\leqslant x}\sum_{d\mid n}\frac{(\mu*\mu)(d)}{d}\sigma_a(\frac{n}{d}).$$

Let $l=\frac{n}{d}$. Then $dl=n\leqslant x$, $l\leqslant\frac{x}{d}$. Hence

$$\sum_{n\leqslant x}\sum_{d\mid n}\frac{(\mu*\mu)(d)}{d}\sigma_a(\frac{n}{d})=$$

$$\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d}\sum_{l\leqslant\frac{x}{d}}\sigma_a(l).$$

By (3), we derive that

$$\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d}\sum_{l\leqslant\frac{x}{d}}\sigma_a(l)=$$

$$\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d}(\frac{a+1}{\zeta(a+1)}(\frac{x}{d})^{a+1}+$$

$$O((\frac{x}{d})^a))=\frac{a+1}{\zeta(a+1)}\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d^{a+2}}$$

$$x^{a+1}+O(\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d^{a+1}}x^a).$$

By (9), we derive that

$$\sum_{n\leqslant x}\sum_{d\mid n}\frac{(\mu*J_{1+a})(d)}{d}=\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d}$$

$$\sum_{l\leqslant\frac{x}{d}}\sigma_a(l)=\frac{a+1}{\zeta(a+1)}\Big(\frac{1}{\zeta^2(a+2)}+$$

$$O\Big(\frac{\delta(x)}{x^{a+1}}\Big)\Big)x^{a+1}+\Big(\frac{1}{\zeta^2(a+1)}+O\Big(\frac{\delta(x)}{x^a}\Big)\Big)$$

$$O(x^a)=\frac{a+1}{\zeta(a+1)\zeta^2(a+2)}x^{a+1}+O(x^a),$$

as required. The proof of Lemma 2.6 is complete.

Lemma 2.7 Let t be an any positive integer. For any large positive number $x>5$, one has

$$\sum_{dl\leqslant x}\frac{(\mu*J_{1+a})(d)}{d}\frac{1}{l^{2t}}=$$

$$\sum_{n\leqslant x}\sum_{d\mid n}\frac{\mu*J_{1+a}(d)}{d}(\frac{d}{n})^{2t}=$$

$$\frac{\zeta(a+2t+1)}{(a+1)\zeta^2(a+2)}x^{a+1}+O(x^a).$$

Proof Since $J_{1+a}=\mu*id_{1+a}$, one has

$$id_{-2m}*id_{-2t}=\sum_{d\mid n}d^{-2m}(\frac{n}{d})^{-2t}=$$

$$\sum_{d\mid n}\frac{d^{2t-2m}}{n^{2t}}\frac{1}{n^{2t}}\sum_{d\mid n}d^{2t-2m}=\frac{\sigma_{2t-2m}}{id_{2t}}.$$

It follows that

$$\frac{\mu*J_{1+a}}{id}*id_{-2t}=\frac{\mu*id_{1+a}*\mu}{id}*id_{-2t}=$$

$$\frac{id_{1+a}}{id}*\frac{\mu*\mu}{id}*id_{-2t}=id_a*\frac{\mu*\mu}{id}*id_{-2t}=$$

$$\frac{\mu*\mu}{id}*id_a*id_{-2t}=\frac{\mu*\mu}{id}*\frac{\sigma_{2t+a}}{id_{2t}}.$$

Then we can deduce that

$$\sum_{n\leqslant x}\sum_{d\mid n}\frac{(\mu*J_{1+a})(d)}{d}\frac{1}{l^{2t}}=\sum_{n\leqslant x}\frac{(\mu*J_{1+a})}{id}*$$

$$id_{-2t}(n)=\sum_{n\leqslant x}\left(\frac{(\mu*\mu)}{id}*\frac{\sigma_{2t+a}}{id_{2t}}\right)(n)=\sum_{n\leqslant x}\sum_{d\mid n}\frac{(\mu*\mu)(d)}{d}\frac{\sigma_{2t+a}}{id_{2t}}\left(\frac{n}{d}\right):=\Delta_1.$$

Let $l=\frac{n}{d}$. Then $dl=n\leqslant x, l\leqslant\frac{x}{d}$. Hence

$$\Delta_1=\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d}\sum_{l\leqslant\frac{x}{d}}\frac{\sigma_{2t+a}}{id_{2t}}(l).$$

Then, by (11), we obtain that

$$\Delta_1=\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d}\left(\frac{\zeta(a+2t+1)}{a+1}\left(\frac{x}{d}\right)^{a+1}+O\left(\frac{x}{d}\right)^{a}\right)=\frac{\zeta(a+2t+1)}{a+1}\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d^{a+2}}x^{a+1}+\sum_{d\leqslant x}\frac{(\mu*\mu)(d)}{d^{a+1}}O(x^a).$$

By (9), we get that

$$\Delta_1=\frac{\zeta(a+2t+1)}{(a+1)}x^{a+1}\left(\frac{1}{\zeta^2(a+2)}+O\left(\frac{\delta(x)}{x^{a+1}}\right)\right)+O(x^a)\left(\frac{1}{\zeta^2(a+1)}+O\left(\frac{\delta(x)}{x^a}\right)\right)=\frac{\zeta(a+2t+1)}{(a+1)\zeta^2(a+2)}x^{a+1}+O(x^a).$$

This concludes the proof of Lemma 2.7.

3 The proof of Theorem 1.1

Letting $f=J_{1+a}$ in (1) gives us that

$$M_r(x;J_{1+a}):=\sum_{k\leqslant x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r J_{1+a}(\gcd(k,j))=\frac{1}{2}\sum_{n\leqslant x}\frac{J_{1+a}(n)}{n}+\frac{1}{r+1}\sum_{dl\leqslant x}\frac{(\mu*J_{1+a})(d)}{d}+\frac{1}{r+1}\sum_{t=1}^{[r/2]}\binom{r+1}{2t}B_{2t}\sum_{dl\leqslant x}\frac{(\mu*J_{1+a})(d)}{d}\frac{1}{l^{2t}}=J_1+J_2+J_3 \quad (12)$$

where

$$J_1=\frac{1}{2}\sum_{n\leqslant x}\frac{J_{1+a}(n)}{n},$$

$$J_2=\frac{1}{r+1}\sum_{dl\leqslant x}\frac{(\mu*J_{1+a})(d)}{d},$$

$$J_3=\frac{1}{r+1}\sum_{t=1}^{[r/2]}\binom{r+1}{2t}B_{2t}\sum_{dl\leqslant x}\frac{(\mu*J_{1+a})(d)}{d}\frac{1}{l^{2t}}.$$

From Lemmas 2.5 to 2.7 one has

$$J_1=\frac{1}{(1+a)\zeta(2+a)}x^{a+1}+O(x^a) \quad (13)$$

$$J_2=\frac{1}{r+1}\left(\frac{a+1}{\zeta(a+1)\zeta^2(a+2)}x^{a+1}+O(x^a)\right) \quad (14)$$

$$J_3=\frac{1}{r+1}\sum_{t=1}^{[r/2]}\binom{r+1}{2t}B_{2t}\left(\frac{\zeta(a+2t+1)}{(a+1)\zeta^2(a+2)}x^{a+1}+O(x^a)\right)=\frac{x^{a+1}}{(r+1)(a+1)\zeta^2(a+2)}\sum_{t=1}^{[r/2]}\binom{r+1}{2t}B_{2t}\times\zeta(a+2t+1)+O(x^a) \quad (15)$$

Putting (13) to (15) into (12), the desired result follows immediately. This finishes the proof of Theorem 1.1.

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