具有对称实焦点的分段线性类-Liénard 系统的穿越周期轨

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摘　要：本文研究了具有对称实焦点的分段线性类-Liénard 系统的穿越周期轨. 通过把系统约化成一个有更少参数的正规形并构造左右子系统的 Poincaré 映射，本文证明了系统至少存在一个穿越周期轨. 此外，本文还给出了一个系统不存在穿越周期轨的充分条件并在一定条件下对穿越周期轨的个数给出了一个上界.

关键词：焦点；正规形；周期轨；分段线性系统
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On the crossing periodic orbits of a piecewise linear Liénard-like system with symmetric admissible foci

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Abstract: We investigate the crossing periodic orbits of a piecewise linear Liénard-like system with symmetric admissible foci. By reducing this system to a normal form, which has less parameters, and constructing the Poincaré maps of the left and right subsystems, we prove the existence of at least one crossing periodic orbit, give a sufficient condition for the non-existence of crossing periodic orbits, and provide an upper bound for the number of crossing periodic orbits under some conditions.

Keywords: Focus; Normal form; Periodic orbit; Piecewise linear system
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1　Introduction

Planar piecewise smooth differential systems are used as mathematical models in many fields such as power electronics[1] and feedback systems in control systems[2]. Because of the non-smoothness of vector fields, the qualitative analysis is much more difficult than smooth systems even for piecewise linear systems with one switching line

$$\dot{X}=\begin{cases}\boldsymbol{F}^-(X)=(F_1^-(X),F_2^-(X))^T=\\ \quad A^-X+\boldsymbol{b}^-,\ \text{if } x<0,\\ \boldsymbol{F}^+(X)=(F_1^+(X),F_2^+(X))^T=\\ \quad A^+X+\boldsymbol{b}^+,\ \text{if } x>0,\end{cases} \tag{1}$$

where $X=(x,y)^T\in\mathbf{R}^2$, $A^\pm=(a_{ij}^\pm)$ are 2×2 constant matrices and $\boldsymbol{b}^\pm=(b_1^\pm,b_2^\pm)^T$ are constant vectors in $\mathbf{R}^2$.

As indicated in Ref. [3], System (1) has no crossing periodic orbits when $a_{12}^+a_{12}^-\leqslant 0$. Thus, we assume that $a_{12}^+a_{12}^->0$ in this paper. By Ref. [3],

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System (1) can be transformed into

$$\dot{X}=\begin{cases}\begin{pmatrix}T^- & -1\\ D^- & 0\end{pmatrix}X-\begin{pmatrix}0\\ a^-\end{pmatrix}, & \text{if } x<0,\\ \begin{pmatrix}g_{11}^+ & -1\\ g_{21}^+ & g_{22}^+\end{pmatrix}X-\begin{pmatrix}-b\\ a^+\end{pmatrix}, & \text{if } x>0\end{cases} \tag{2}$$

by

$$X\to\begin{pmatrix}1 & 0\\ \dfrac{a_{22}^-}{a_{12}^-} & -\dfrac{1}{a_{12}^-}\end{pmatrix}X+\begin{pmatrix}0\\ -\dfrac{b_1^-}{a_{12}^-}\end{pmatrix} \tag{3}$$

where $(T^-, D^-, a^+, a^-, b, g_{11}^+, g_{21}^+, g_{22}^+)\in\mathbf{R}^8$. Clearly, the transformation (3) does not change the discontinuity line. When $g_{22}^+=0$, we say that System (2) is of Liénard-like canonical form as defined in Ref. [4], *i.e.*,

$$\dot{X}=\begin{cases}\begin{pmatrix}T^- & -1\\ D^- & 0\end{pmatrix}X-\begin{pmatrix}0\\ a^-\end{pmatrix}, & \text{if } x<0,\\ \begin{pmatrix}g_{11}^+ & -1\\ g_{21}^+ & 0\end{pmatrix}X-\begin{pmatrix}-b\\ a^+\end{pmatrix}, & \text{if } x>0\end{cases} \tag{4}$$

It is not hard to check that both equilibria of subsystems in System (4) are foci if and only if

$$T^-\neq 0,\ T^{-2}-4D^-<0,\ g_{11}^+\neq 0,\ g_{11}^{+2}-4g_{21}^+<0 \tag{5}$$

The focus of the left subsystem is admissible (resp. virtual) if additionally $a^-<0$ (resp. $a^->0$). The focus of the right subsystem is admissible (resp. virtual) if additionally $a^+>0$ (resp. $a^+<0$).

For System (4), the existence of two crossing periodic orbits is proved in Ref. [3] when the two foci are virtual and the existence of three crossing periodic orbits is proved in Ref. [5] when one of these two foci is admissible and the other is virtual.

In this paper, we study the number of crossing periodic orbits for System (4) having two admissible foci. We reduce System (4) to a normal form and state our main result in Section 2 and give the proofs in Section 3.

2 Main results

System (4) has 7 parameters. In order to simplify it, we firstly find a normal form with less parameters. For convenience, we write T^-, D^- as $2\alpha^-$, $(\alpha^-)^2+(\omega^-)^2$ with $\omega^->0$, respectively, and denote the eigenvalue by $\lambda^-:=\alpha^-\pm i\omega^-$ for the left subsystem. Note that (5) holds because the equilibria of subsystems in (4) are both foci.

Theorem 2.1 System (4) having two foci is equivalently transformed into

$$\dot{X}=\begin{cases}\begin{pmatrix}2\delta & -1\\ 1+\delta^2 & 0\end{pmatrix}X-\begin{pmatrix}0\\ a\end{pmatrix}, & \text{if } x<0,\\ \begin{pmatrix}\alpha & -1\\ \beta & 0\end{pmatrix}X-\begin{pmatrix}-b\\ \rho\end{pmatrix}, & \text{if } x>0\end{cases} \tag{6}$$

by $(x,y,t)\to(x/\omega^-, y, t/\omega^-)$, where

$$\alpha:=\frac{g_{11}^+}{\omega^-},\ \beta:=\frac{g_{21}^+}{\omega^{-2}},\ \rho:=\frac{a^+}{\omega^-},$$
$$\delta:=\frac{\alpha^-}{\omega^-},\ a:=\frac{a^-}{\omega^-} \tag{7}$$

satisfying

$$\alpha\delta a\neq 0,\ \alpha^2-4\beta<0 \tag{8}$$

When there are two admissible foci, for System (4) we get an equivalent System (6), which has 6 parameters $\alpha,\beta,\rho,\delta,a,b$. Thus System (6) is regarded as a normal form of System (4). Observing that System (6) is invariant under the change

$$(x,y,t,\alpha,\beta,\rho,\delta,a,b)\to(x,-y,-t,-\alpha,\beta,\rho,-\delta,a,-b),$$

we only consider $b\geq 0$ in System (6). In the following, we consider the existence of crossing periodic orbits of System (6).

Theorem 2.2 Assume that the two foci of System (6) are admissible and symmetric with respect to y-axis. The following statements hold:

(a) For the case $\delta\alpha<0$, System (6) has at least one crossing periodic orbits if $\delta^2+1-\beta=b=0$;

(b) For the case $\delta\alpha>0$, System (6) has at most two crossing periodic orbits and there exists $M>0$ such that for all $|\delta|>M$ System (6) has no crossing periodic orbits.

In Theorem 2.2, for the case $\delta\alpha<0$, we give a sufficient condition for the existence of crossing periodic orbits. However, it is hard to get a condition for the non-existence of crossing periodic orbits as we provide for the case $\delta\alpha>0$. Thus in the end of this paper we give two examples for the case $\delta\alpha<0$ to exemplify the possibility of non-existence of crossing periodic orbits. On the other hand, for the case $\delta\alpha>0$ we provide an upper bound for the number of crossing periodic orbits, but we are not able to judge if it is reachable.

3 The proofs of the main results

Proof of Theorem 2.1 By transformation $(x,y)\to(x/\omega^-,y)$, System (2) having two foci is transformed into

$$\dot{X}=\begin{cases}\begin{pmatrix}\omega^- & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}T^- & -1\\ D^- & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\omega^-} & 0\\ 0 & 1\end{pmatrix}X-\begin{pmatrix}\omega^- & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ a^-\end{pmatrix}, & \text{if } x<0,\\ \begin{pmatrix}\omega^- & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}g_{11}^+ & -1\\ g_{21}^+ & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\omega^-} & 0\\ 0 & 1\end{pmatrix}X-\begin{pmatrix}\omega^- & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-b\\ a^+\end{pmatrix}, & \text{if } x>0\end{cases}=$$

$$\begin{cases}\begin{pmatrix}2\alpha^- & -\omega^-\\ \omega^-+\frac{\alpha^{-2}}{\omega^-} & 0\end{pmatrix}X-\begin{pmatrix}0\\ a^-\end{pmatrix}, & \text{if } x<0,\\ \begin{pmatrix}g_{11}^+ & -\omega^-\\ \frac{g_{21}^+}{\omega^-} & 0\end{pmatrix}X-\begin{pmatrix}-b\omega^-\\ a^+\end{pmatrix}, & \text{if } x>0\end{cases} \tag{9}$$

Further, by a time rescaling $t\to t/\omega^-$ System (9) is changed into System (6) with new parameters defined in (7). By (5), new parameters in system (6) satisfy (8).

In order to prove Theorem 2.2, we need some preliminaries. For the left subsystem in (6), the solution satisfies $(x(0),y(0))=(0,y)$ is of the form

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=e^{\delta t}\begin{pmatrix}\cos t+\delta\sin t & -\sin t\\ (1+\delta^2)\sin t & \cos t-\delta\sin t\end{pmatrix}\begin{pmatrix}-x_L\\ y-y_L\end{pmatrix}+\begin{pmatrix}x_L\\ y_L\end{pmatrix} \tag{10}$$

where

$$(x_L,y_L):=\left(\frac{a}{1+\delta^2},\frac{2\delta a}{1+\delta^2}\right)$$

is the coordinate of the unique equilibrium of the left subsystem. For the right subsystem in (6), the solution satisfying $(x(0),y(0))=(0,y_0)$ is of the form

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=e^{At}\begin{pmatrix}\cos Bt+(\alpha-A)\frac{\sin Bt}{B} & -\frac{\sin Bt}{B}\\ (B^2+(\alpha-A)^2)\frac{\sin Bt}{B} & \cos Bt-(\alpha-A)\frac{\sin Bt}{B}\end{pmatrix}\begin{pmatrix}-x_R\\ y_0-y_R\end{pmatrix}+\begin{pmatrix}x_R\\ y_R\end{pmatrix} \tag{11}$$

where $A:=\alpha/2, B:=\sqrt{|\alpha^2-4\beta|}/2$ and

$$(x_R,y_R):=\left(\frac{\rho}{\beta},\frac{\alpha\rho+b\beta}{\beta}\right)$$

is the coordinate of the unique equilibrium of the right subsystem.

In the following, we introduce the Poincaré map, which is our main tool in looking for periodic orbits. Clearly, the right subsystem in (6) has a unique tangency point $(0,b)$. Since the foci is admissible, the orbit starting from this tangency point intersects y-axis for the first time at a point when $A>0$. Thus the orbit starting from $(0,y_0)$ intersects y-axis for the first time at $(0,y_2)$ for $y_0\leqslant b$. We denote the time by t_R and define a right

Poincaré map P_R as $y_2=P_R(y_0)$ for all $y_0\leqslant b$. From (11), we obtain a parametric representation of the right Poincaré map P_R as

$$\begin{cases} y_0=b+\dfrac{e^{-At_R}\psi_+(t_R)}{\sin Bt_R}Bx_R, \\ y_2=P_R(y_0)=b-\dfrac{e^{At_R}\psi_-(t_R)}{\sin Bt_R}Bx_R \end{cases} \tag{12}$$

where $t_R\in(\frac{\pi}{B},\hat{t}_R]$ and

$$\psi_\pm(t)\coloneqq 1-e^{\pm At}\left(\cos Bt\mp\frac{A}{B}\sin Bt\right) \tag{13}$$

is defined in Ref. [6]. Here $\hat{t}_R$ is the unique zero of $\psi_+(t)$ in $(\pi/B, 2\pi/B)$. Clearly, $\psi_+(t)$ is strictly decreasing in $(\pi/B,2\pi/B)$. We have $\psi_+(\pi/B)>0$ and $\psi_+(2\pi/B)<0$ by calculation. Thus $\hat{t}_R$ is the unique zero of $\psi_+(t)$ in $(\pi/B,2\pi/B)$. By $y_0\leqslant b$ we have $t_R\in(\frac{\pi}{B},\hat{t}_R]$.

When $A<0$, the orbit starting from this tangency point intersects y-axis for the first time at a point as the time is reversed. We denote the coordinate of this point by $(0, y_b)$. Thus the orbit starting from $(0,y_0)$ intersects y-axis for the first time at $(0,y_2)$ for $y_0\leqslant y_b$. We denote the time by t_R and define a right Poincaré map P_R as $y_2=P_R(y_0)$ for all $y_0\leqslant y_b$. From (11) we obtain a parametric representation of the right Poincaré map P_R as (12), where $t_R\in(\frac{\pi}{B},\hat{t}_R]$ since $P_R(y_0)\geqslant b$ and $\psi_\pm(t)$ is defined in (13). Here $\hat{t}_R$ is the unique zero of $\psi_+(t)$ in $(\pi/B,2\pi/B)$ for $A>0$.

From (12), we get

$$P_R{}'(y_0)=-\frac{\psi_+(t_R)}{\psi_-(t_R)}=\frac{y_0-b}{P_R(y_0)-b}e^{2At_R}<0 \tag{14}$$

$$P_R{}''(y_0)=2B^2x_R^2\left(1+\frac{A^2}{B^2}\right)\frac{\sinh At_R-\dfrac{A}{B}\sin Bt_R}{(P_R(y_0)-b)^3}e^{3At_R} \tag{15}$$

Since

$\operatorname{sign}(B\sinh At_R-A\sin Bt_R)=\operatorname{sign} A$,

it follows from (15) that

$$\operatorname{sign} P_R''(y_0)=\operatorname{sign} A \tag{16}$$

Lemma 3.1 (a) The reverse map P_R^{-1} of P_R has the asymptote

$A_R^{-1}(y)=-e^{-A\pi/B}y+(1+e^{-A\pi/B})(b+2Ax_R)$, satisfying $\operatorname{sign}(P_R^{-1}(y)-A_R^{-1}(y))=\operatorname{sign} A$.

(b) If $\alpha>0$, then P_R is a surjection from $(-\infty,b]$ to $[\hat{y}_2,+\infty)$, where $\hat{y}_2\coloneqq y_2(\hat{t}_R)$.

(c) If $\alpha<0$, then P_R is a surjection from $(-\infty,\hat{y}_0]$ to $[b,+\infty)$, where $\hat{y}_0\coloneqq y_0(\hat{t}_R)$.

Proof By (12), the reverse P_R^{-1} of P_R takes a parametric form

$$y_2=b-\frac{e^{At_R}\psi_-(t_R)}{\sin Bt_R}Bx_R,$$

$$P_R^{-1}(y_2)=b+\frac{e^{-At_R}\psi_+(t_R)}{\sin Bt_R}Bx_R.$$

Then

$$\lim_{y_2\to+\infty}\frac{P_R^{-1}(y_2)}{y_2}=\lim_{t_R\to(\pi/B)^+}\frac{b+\dfrac{e^{-At_R}\psi_+(t_R)}{\sin Bt_R}Bx_R}{b-\dfrac{e^{At_R}\psi_-(t_R)}{\sin Bt_R}Bx_R}=-e^{-A\pi/B},$$

and

$$\lim_{y_2\to+\infty}[P_R^{-1}(y_2)-(-e^{-A\pi/B})y_2]=\lim_{t_R\to(\pi/B)^+}b+\frac{e^{-At_R}\psi_+(t_R)}{\sin Bt_R}Bx_R+e^{-A\pi/B}\left(b-\frac{e^{At_R}\psi_-(t_R)}{\sin Bt_R}Bx_R\right)=(1+e^{-A\pi/B})(b+2Ax_R).$$

Proposition 3.2[5] Assume that $x_L<0$. Then the following statements are true for the left Poincaré map P_L:

(a) If $\delta\neq0$, then P_L is defined by

$$y=\frac{e^{-\delta t_L}\varphi_\delta(t_L)}{\sin t_L}x_L,\ P_L(y)=-\frac{e^{\delta t_L}\varphi_{-\delta}(t_L)}{\sin t_L}x_L$$

for $\pi<t_L\leqslant\hat{t}_L$ being $\varphi_{|\delta|}(\hat{t}_L)=0$;

(b) If $\delta<0$, then we define $\hat{y}=y(\hat{t}_L)>0$, and we have $P_L(\hat{y})=0$,

$P_L:[\hat{y},\infty)\to(-\infty,0]$ and $\lim_{y\to\hat{y}^+}P_L'(y)=-\infty$;

(c) If $\delta>0$, then we define $\hat{y}_1=y_1(\hat{t}_L)<0$, and we have $P_L(0)=\hat{y}_1$,

$P_L:[0,\infty)\to(-\infty,\hat{y}_1]$ and $\lim_{y\to0^+}P_L'(y)=0$;

(d) If $\delta\neq0$, then we have $P_L'(y)<0$ for all y where map P_L is defined,

$\lim_{y\to+\infty}P_L'(y)=-e^{\delta\pi}$, $\operatorname{sign} P_L''(y)=-\operatorname{sign}\delta$.

Finally, we define the Poincaré map P as the composition $P=P_R\circ P_L$, where the left Poincaré map P_L is defined in Ref. [5]. From Ref. [5], the Poincaré map P is well defined for $y \geqslant y_P$, where $y_P := \hat{y}$ for $\delta < 0$ and $y_P := 0$ for $\delta > 0$. By the proof of Lemma 3.1,

$$\lim_{y \to -\infty} P_R'(y) = \lim_{t \to \frac{\pi}{B}^+} P_R'(y) = -e^{-A\pi/B}.$$

We have

$$\lim_{y \to \infty} P_L'(y) = -e^{\delta\pi}.$$

Then

$$\lim_{y \to \infty} P'(y) = \lim_{y \to \infty} (P_R' \circ P_L(y)) \cdot P_L{}'(y) = e^{(A/B+\delta)\pi}.$$

If $\bar{y}$ is a fixed point of Poincaré map P, then $P(\bar{y}) = P_R(P_L(\bar{y})) = \bar{y}$, and so $P_L(\bar{y}) = P_R^{-1}(\bar{y})$. Hence the existence of crossing periodic orbits is equivalent to the existence of zeroes of the function

$$\Psi(y) := P_R^{-1}(y) - P_L(y) \tag{17}$$

well defined when both $y \geqslant y_P$ and $y \geqslant y_{P1}$, where $y_{P1} := b$ for $\alpha < 0$ and $y_{P1} := \hat{y}_2$ for $\alpha > 0$.

For $y \neq \hat{y}$ and $y \neq b$,

$$\Psi'(y) = \frac{1}{(P_R' \circ P_R^{-1})(y)} - P_L'(y), \Psi''(y) = -\frac{P_R{}'' \circ P_R^{-1}(y)}{(P_R' \circ P_R^{-1})^3(y)} - P_L{}''(y) \tag{18}$$

Lemma 3.3 (a) If $\Psi(\bar{y}) = 0$, then $P(\bar{y}) = \bar{y}$. Furthermore,

$$C_\infty = \lim_{y \to +\infty} \Psi'(y) = -e^{-A\pi/B} + e^{\delta\pi} \tag{19}$$

(b) If $\delta A > 0$, $y \neq \hat{y}$ and $y \neq b$, then we have sign $\Psi''(y) = \mathrm{sign}(A+\delta)$ and so function Ψ has at most two zeroes.

Proof If $\Psi(\bar{y}) = 0$, then it directly follows that $P(\bar{y}) = \bar{y}$ and

$$\lim_{y \to +\infty} \Psi'(y) = \lim_{y \to +\infty} \frac{1}{(P_R' \circ P_R^{-1})(y)} - P_L'(y) = -e^{-A\pi/B} + e^{\delta\pi}.$$

By (14) (16) and Ref. [5], we have sign $P_L''(y) = -\mathrm{sign}\,\delta$, sign $P_R''(y) = \mathrm{sign}\,A$, and $P_R'(y) < 0$. Associated with (18), (b) holds.

Lemma 3.4 When $\delta < 0$, $A < 0$, $b = 0$, System (6) with (8) has no crossing periodic orbits.

Proof A crossing periodic orbit Γ has exactly two points at the y-axis, namely, $(0, y_L)$ and $(0, y_U)$ with $y_L < 0 \leqslant b < y_U = y_L + h$, where $h > 0$. By removing the two crossing points, we define the left open arc $\Gamma_L := \Gamma \cap \{(x,y): x < 0\}$, the right open arc $\Gamma_R := \Gamma \cap \{(x,y): x > 0\}$, and the oriented segments

$$I_L := \{(x,y): x = 0, y = (1-\mu)y_L + \mu y_U, 0 \leqslant \mu \leqslant 1\},$$

$$I_R := \{(x,y): x = 0, y = \mu y_L + (1-\mu)y_U, 0 \leqslant \mu \leqslant 1\}.$$

Since $\Gamma_L \cup I_L$ is a closed Jordan curve, we define its interior $\Omega_L := \mathrm{int}\{\Gamma_L \cup I_L\}$ and $\sigma^- := \mathrm{area}(\Omega_L)$. Analogously, we define $\Omega_R := \mathrm{int}\{\Gamma_R \cup I_R\}$ and $\sigma^+ := \mathrm{area}(\Omega_R)$. According to Proposition 3.6 of Ref. [3], if System (6) with (8) has a crossing periodic orbit passing through the points $(0, y_L)$ and $(0, y_L + h)$, then we have

$$2\delta\sigma^- + 2A\sigma^+ + bh = 0 \tag{20}$$

Since (20) cannot be fulfilled when $\delta < 0, A < 0$, $b = 0$, System (6) with (8) has no crossing periodic orbits.

Proof of Theorem 2.2 The two foci of the left and right subsystems of (6) lie at (x_L, y_L) and (x_R, y_R), defined below (10) and (11), respectively. Since the two foci of System (6) is admissible, we get $a < 0$ and $\rho > 0$. Note that $\beta > 0$ by (8). Because of the symmetry of these two admissible foci with respect to y-axis, we get

$$a/(1+\delta^2) = -\rho/\beta,\ 2\delta a/(1+\delta^2) = \alpha\rho/\beta + b.$$

Then

$$\rho = -\beta a/(1+\delta^2),\ b = (2\delta+\alpha)a/(1+\delta^2) \tag{21}$$

For the case $\delta\alpha < 0$, we first consider the case $\delta < 0, \alpha > 0$. By (10), (12), Lemma 3.1 (b) and Proposition 3.2 (b), we can obtain that $\hat{y} = a e^{-\delta \hat{t}_L} \sin \hat{t}_L$, $\hat{y}_2 = b - (\rho/B) e^{A\hat{t}_R} \sin B\hat{t}_R$.

If $\delta^2 + 1 - \beta = b = 0$, then $B = 1, \delta = -A$ and $\hat{y}_2 = a e^{-\delta \hat{t}_R} \sin \hat{t}_R$. Since $\hat{t}_R$ is the unique zero of $\psi_+(t)$ in $(\pi, 2\pi)$, from (13) and Proposition 3.2 (a) we get $\hat{t}_R = \hat{t}_L$, which implies that $\hat{y} = \hat{y}_2$. By (17) we have $\Psi(\hat{y}) = P_R^{-1}(\hat{y}) - P_L(\hat{y}) = 0$. That is, System (6) has at least one crossing periodic orbits. For the case $\delta > 0, \alpha < 0$, by (10), (12), Lemma 3.1 (c) and Proposition 3.2 (c), we get

$$\hat{y}_0 = b + (\rho/B) e^{-A\hat{t}_R} \sin B\hat{t}_R$$

and $\hat{y}_1 = -a e^{\delta \hat{t}_L} \sin \hat{t}_L$.

If $\delta^2 + 1 - \beta = b = 0$, then $B = 1, \delta = -A$ and $\hat{y}_0 = -a e^{\delta \hat{t}_R} \sin \hat{t}_R$. Similarly to the case $\delta < 0, \alpha > 0$, by (13) and Proposition 3.2 (a), we get $\hat{t}_R = \hat{t}_L$, which implies that $\hat{y}_0 = \hat{y}_1$. By (17) we get

$$\Psi(0)=P_R^{-1}(0)-P_L(0)=\hat{y}_0-\hat{y}_1=0,$$

which means that system (6) has at least one crossing periodic orbits. (a) holds.

For the case $\delta\alpha>0$, since $b\geqslant 0$, we have $\delta<0$, $\alpha<0$ by (21). From Lemma 3.4, we know that $\Psi(y)\neq 0$ for $b=0$, so by continuity we get $\Psi(y)\neq 0$ for $0<b\ll 1$. Clearly, $b\to 0^+$ as $|\delta|\to+\infty$ by (21). That is, there exists $M>0$ such that for all $|\delta|>M$ we have $0<b\ll 1$. Thus System (6) has no crossing periodic orbits. Since $\delta<0,\alpha<0$, we have $C_\infty<0$ and $\Psi''(y)<0$ by (19) and Lemma 3.3(b), which means that the function Ψ has at most two zeroes. Thus System (6) has at most two crossing periodic orbits. (b) holds. The proof is end.

In Theorem 2.2, we prove the existence of at least one crossing periodic orbits for System (6). The non-existence of crossing periodic orbits for some systems is proved for the case $\delta\alpha>0$. In the following, for the case $\delta\alpha<0$ we provide two examples to show that there may be no crossing periodic orbits no matter $b=0$ or not.

Example 3.5 Let $\alpha=1,\beta=5/16,\rho=5/32$, $\delta=-1,a=-1,b=1/2$ in System (6), *i.e.*,

$$\dot{X}=\begin{cases}\begin{pmatrix}-2 & -1\\ 2 & 0\end{pmatrix}X-\begin{pmatrix}0\\ -1\end{pmatrix}, & \text{if } x<0,\\ \begin{pmatrix}1 & -1\\ 5/16 & 0\end{pmatrix}X-\begin{pmatrix}-1/2\\ 5/32\end{pmatrix}, & \text{if } x>0\end{cases}\qquad(22)$$

It is easy to check that System (22) has two admissible foci at $(-1/2,1)$ and $(1/2,1)$. Since $\delta<0$, by Proposition 3.2 (b) we obtain

$$P_L(y)=P_L(y)-P_L(\hat{y})=$$
$$P_L'(\xi)(y-\hat{y})<-\mathrm{e}^{-\pi}(y-\hat{y})=G(y),\ \xi\in(\hat{y},y).$$

By Lemma 3.1 (a), the intersection y_A of A_R^{-1} of the right Poincaré map with the y-axis is given by

$$y_A=(1+\mathrm{e}^{A\pi/B})(b+2Ax_R)=1+\mathrm{e}^{2\pi}.$$

Since the slope of the A_R^{-1} is greater than the slope of the $G(y)$, and

$$y_A=1+\mathrm{e}^{2\pi}>1-\mathrm{e}^{\hat{t}_L}\cos\hat{t}_L=$$
$$-\mathrm{e}^{\hat{t}_L}\sin\hat{t}_L=\hat{y},\hat{t}_L\in(\pi,2\pi)$$

we have $A_R^{-1}(y)>G(y)$. Associated with Lemma 3.1 (a),

$$P_R^{-1}(y)>A_R^{-1}(y)>G(y)>P_L(y).$$

Hence $\Psi(y)>0$ and System (22) has no crossing periodic orbits.

Example 3.6 Let $\alpha=2$, $\beta=10/9,\rho=5/9$, $\delta=-1,a=-1,b=0$ in System (6), *i.e.*,

$$\dot{X}=\begin{cases}\begin{pmatrix}-2 & -1\\ 2 & 0\end{pmatrix}X-\begin{pmatrix}0\\ -1\end{pmatrix}, & \text{if } x<0,\\ \begin{pmatrix}2 & -1\\ 10/9 & 0\end{pmatrix}X-\begin{pmatrix}0\\ 5/9\end{pmatrix}, & \text{if } x>0.\end{cases}$$

Similarly to Example 3.5, we can prove that there is no crossing periodic orbits by similar analysis. Thus we omit its proof here.

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