Logistic 扩散过程中漂移系数的序列极大似然估计

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摘 要: 本文研究了 Logistic 扩散过程中未知漂移参数的序列极大似然估计量的性质, 给出了序列估计量及其均方误差的显式表达式, 并证明了该估计量是闭的、无偏的、一致正态分布且强相合的. 最后, 本文用数值实验验证了理论结果.

关键词: 序列极大似然估计; Logistic 扩散过程; 无偏; 均方误差
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Sequential maximum likelihood estimation of the drift parameter of Logistic diffusion process

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Abstract: This paper investigates the properties of a sequential maximum likelihood estimator of the unknown drift parameter for a Logistic diffusion process. We derive the explicit formulas for the sequential estimator and its mean squared error. The estimator is proved to be closed, unbiased, uniformly normally distributed and strongly consistent. Finally, a numerical experiment is provided to illustrate our theory.

Keywords: Sequential maximum likelihood estimator; Logistic diffusion process; Unbiased; Mean squared error
(2010 MSC 93B30)

1 Introduction

In this paper, we study a sequential maximum likelihood estimator (SMLE) of the unknown drift parameter for the following Logistic diffusion process

$$dX_t = \alpha X_t (1-\beta X_t) dt + \sigma X_t dW_t, \quad X_0 = x_0 > 0 \tag{1}$$

where $\{W_t, t\geqslant 0\}$ is a standard Wiener process on a complete filtered probability space $(\Omega, F, (F_t, t\geqslant 0), P)$ with the filtration $(F_t, t\geqslant 0)$ satisfying the usual conditions. Suppose $X(t)$ represents a population density at time t. The intrinsic growth rate α is the unknown parameter to be estimated on the basis of continuous observation of the process X up to a certain predetermined level of precision. The known parameter β is called the carrying capacity of the environment and usually represents the maximum population that can be supported by the resources of the environment. The known parameter σ is the noise intensity which represents the effect of the noise on the dynamics of X.

The Logistic diffusion process is useful for

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modeling the population systems under environmental noise, which have recently been studied by many authors[1-5]. However, if the noise is sufficiently large then the population will become extinct. Therefore, it is reasonable to assume that $\alpha>0$, $\beta>0$, $\sigma^2<2\alpha$.

In this paper, we use the SMLE to estimate the unknown parameter of Logistic diffusion process. Sequential maximum likelihood estimation in discrete time has been studied in Lai and Siegmund[6], and this type of sampling plan dates back to Anscombe[7]. In continuous time processes, SMLE was first studied by Novikov[8]. He proved the SMLE of the drift parameter for a linear stochastic differential equation is unbiased, normally distributed and effective. Since then this method has been applied to estimate the drift parameter of other kinds of stochastic differential equations[9-10]. To the best of our knowledge, the probelm has not been studied for the Logistic diffusion process. In this paper, we are concerned with this topic.

The aim of this paper is to study the statistical inference for the Logistic diffusion process X given by Eq. (1). More precisely, we would estimate the unknown parameter α based on a continuous observation of the state process X up to a certain predetermined level of precision by proposing to use an SMLE. We prove that the SMLE associated with the unknown parameter α is closed, unbiased, normally distributed and strongly consistent. However, it is very difficult to obtain the upper bound of the average observation time by the similar method adopted by Lee *et al.*[10]. To overcome this difficulty, a useful computing method is proposed in this paper, based on the upper bound of $E\alpha[1/X_t]$.

The rest of the paper is organized as follows. In Section 2, we introduce a sequential estimation plan for the Logistic diffusion process, and present the main theoretical results in Theorems 2.1～2.3. Section 3 is devoted to numerical studies which illustrate the efficiency of the proposed estimator.

2 Sequential maximum likelihood estimation

In this section, we use $(\tau(H),\hat{\alpha}_{\tau(H)})$ to estimate the unknown drift parameter α in Eq. (1). Here, $\tau(H)$ is defined to be a stopping time which is the first time such that the observed Fisher informatin of the Logistic diffusion process exceeds the previously determined level H and $\hat{\alpha}_{\tau(H)}$ is a sequential estimator of the drift parameter α tracking at $\tau(H)$. More specifically, the observation stopping time $\tau(H)$ is defined to be

$$\tau(H)=\inf\left\{t>0:\int_0^t(1-\beta X_s)^2\mathrm{d}s=H\right\} \tag{2}$$

where the predetermined level of precision $0<H<\infty$, which is $F_{\tau(H)}$-measurable[11]. In the following theorem we enumerate the fundamental properties of this sequential estimation plan $(\tau(H),\hat{\alpha}_{\tau(H)})$.

Theorem 2.1 Let the random pair $(\tau(H),\hat{\alpha}_{\tau(H)})$ be the sequential estimation plan with the observation stopping time $\tau(H)$ defined as Eq. (2). Then we obtain the SMLE of the unknown drift parameter α given by

$$\hat{\alpha}_{\tau(H)}=\frac{1}{H}\left(\int_0^{\tau(H)}\frac{1-\beta X_t}{X_t}\mathrm{d}X_t\right) \tag{3}$$

Moreover, we have

(i) The sequential estimator is unbiased, *i.e.*, for each $H\in(0,\infty)$,

$$E_\alpha(\hat{\alpha}_{\tau(H)})=\alpha,\ \forall\alpha\in(0,\infty),$$

where E_α denotes the expectation operator corresponding to the probability measure P_α;

(ii) For each $H>0$ fixed, it holds that

$$\frac{\sqrt{H}(\hat{\alpha}_{\tau(H)}-\alpha)}{\sigma}\sim N(0,1),$$

where $N(0,1)$ denotes the standard normal distribution. In particular, we have

$$E_\alpha(\hat{\alpha}_{\tau(H)}-\alpha)^2=\frac{\sigma^2}{H};$$

(iii) The sequential estimator is strongly consistent. Namely

$$\hat{\alpha}_{\tau(H)}\to\alpha,\ P-\text{a.s.}$$

when $H\to\infty$.

Proof We first show that the $F_{\tau(H)}$-measurable random variable $\hat{\alpha}_{\tau(H)}$ is indeed the SMLE as follows. Let θ,α be any two real numbers. Suppose that X_θ and X_α satisfy the Logistic diffusion processes

$$dX_t^\theta=\theta X_t^\theta(1-\beta X_t^\theta)dt+\sigma X_t^\theta dW_t$$

and

$$dX_t^\alpha=\alpha X_t^\alpha(1-\beta X_t^\alpha)dt+\sigma X_t^\alpha dW_t,$$

respectively. Then the probability measures $P^\theta_{\tau(H),X}$ and $P^\alpha_{\tau(H),X}$ corresponding to the processes $\{X_t^\theta: 0\leqslant t\leqslant\tau(H)\}$ and $\{X_t^\alpha: 0\leqslant t\leqslant\tau(H)\}$ respectively, are equivalent and their Radon-Nikodym derivative is given by[12]

$$\left.\frac{dP^\alpha_{\tau(H),X}}{dP^\theta_{\tau(H),X}}\right|_{F^{X^\alpha}_{\tau(H)}}=\exp\left\{\frac{\alpha-\theta}{\sigma^2}\int_0^{\tau(H)}\frac{1-\beta X_t}{X_t}dX_t-\frac{(\alpha-\theta)^2}{2\sigma^2}\int_0^{\tau(H)}(1-\beta X_t)^2dt\right\}\tag{4}$$

where $F^{X^\alpha}_{\tau(H)}$ is the natural filtration generated by $\{X_t^\alpha: 0\leqslant t\leqslant\tau(H)\}$. Then, by solving the equation

$$\frac{d}{d\alpha}\left(\sigma^2\log\left.\frac{dP^\alpha_{\tau(H),X}}{dP^\theta_{\tau(H),X}}\right|_{F^{\alpha\alpha}_{\tau(H)}}\right)=0,$$

we obtain the SMLE given by

$$\hat\alpha_{\tau(H)}=\frac{1}{H}\left(\int_0^{\tau(H)}\frac{1-\beta X_t}{X_t}dX_t\right)\tag{5}$$

To verify (i), it is enough notice that for any $T>0$

$$P_\alpha(\tau(H)\geqslant T)=P_\alpha\left(\int_0^T(1-\beta X_t)^2dt<H\right)$$

and therefore, we have

$$P_\alpha(\tau(H)=\infty)=P_\alpha\left(\int_0^\infty(1-\beta X_t)^2dt<H\right).$$

Thus it suffices to show

$$P_\alpha\left(\int_0^\infty(1-\beta X_t)^2dt=\infty\right)=1,$$

which is to say that

$$\int_0^T(1-\beta X_t)^2dt\to\infty\text{ ,a. s. },T\to\infty.$$

Since $\alpha>0.5\sigma^2$, the Logistic diffusion process X given by Eq. (1) has a unique stationary distribution $\mu(\cdot)$ on $(0,\infty)$ (see Ref. [3, Theorem 3.2]). By the ergodic theorem, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T(1-\beta X_t)^2dt=\int_0^\infty(1-\beta x)^2\mu(dx)>0,$$

which implies that

$$P_\alpha\left(\int_0^\infty(1-\beta X_t)^2dt=\infty\right)=1.$$

The proof of (ii) follows directly from noting that

$$\hat\alpha_{\tau(H)}=\frac{1}{H}\left(\int_0^{\tau(H)}\frac{1-\beta X_t}{X_t}dX_t\right)=\frac{1}{H}\left\{\int_0^{\tau(H)}\frac{1-\beta X_t}{X_t}[\alpha X_t(1-\beta X_t)dt+\sigma X_tdW_t]\right\}=\frac{1}{H}\left[\alpha\int_0^{\tau(H)}(1-\beta X_t)^2dt+\sigma\int_0^{\tau(H)}(1-\beta X_t)dW_t\right]=\alpha+\frac{\sigma}{H}\int_0^{\tau(H)}(1-\beta X_t)dW_t,$$

and the fact that the process $\int_0^{\tau(H)}(1-\beta X_t)dW_t$ is a Wiener process with variance $H>0$ (see Ref. [13, Theorem 7.14 on page 234]). Hence

$$E_\alpha(\hat\alpha_{\tau(H)})=E_\alpha\left(\alpha+\frac{\sigma}{H}\int_0^{\tau(H)}(1-\beta X_t)dW_t\right)=\alpha.$$

Next, the random variable

$$\int_0^{\tau(H)}(1-\beta X_t)dW_t\sim N(0,H),$$

since $\int_0^{\tau(H)}(1-\beta X_t)^2dt=H$, which proves that $\frac{\sqrt{H}(\hat\alpha_{\tau(H)}-\alpha)}{\sigma}\sim N(0,1)$ for each $H>0$ fixed, *i. e.*, (iii) is valid.

Lastly, the conclusion (iv) follows from $\frac{1}{H}\int_0^{\tau(H)}(1-\beta X_t)dW_t\to0$, as $H\to\infty$ by the law of large number.

In the next theorem we get the upper and lower bounds of the average observation time under some assumptions.

Theorem 2.2 Suppose that $0<x_0<\frac{1}{\beta}$. Then the average observation time $E_\alpha[\tau(H)]$ of the sequential estimation plan satisfies

$$E_\alpha[\tau(H)]\geqslant\frac{2}{\sigma^2}\left[1-\frac{1}{\beta}+E_\alpha(\ln x_0-\beta x_0)+\alpha H\right]\tag{6}$$

In the case $\alpha>\sigma^2$, the following upper bound estimate holds for $E_\alpha[\tau(H)]$

$$E_\alpha[\tau(H)]\leqslant\frac{2}{\sigma^2}\left[\frac{\alpha\beta}{\alpha-\sigma^2}+E_\alpha\left(\ln x_0-\beta x_0+\frac{1}{x_0}\right)+\alpha H\right]\tag{7}$$

Proof By Itô formula, we have

$$\ln X_t-\beta X_t=\int_0^t\frac{1-\beta X_s}{X_s}dX_s-\frac{\sigma^2}{2}t+$$

$$(\ln x_0 - \beta x_0) = \int_0^t \frac{1-\beta X_s}{X_s}[\alpha X_s(1-\beta X_s)ds + \sigma X_s dW_s] - \frac{\sigma^2}{2}t + (\ln x_0 - \beta x_0) = \left[\alpha\int_0^t (1-\beta X_s)^2 ds + \sigma\int_0^t (1-\beta X_s)dW_s\right] - \frac{\sigma^2}{2}t + (\ln x_0 - \beta x_0) \tag{8}$$

Setting $t=\tau(H)$ in Eq. (8), we get

$$\frac{\sigma^2}{2}\tau(H) = -[\ln X_{\tau(H)} - \beta X_{\tau(H)}] + (\ln x_0 - \beta x_0) + \left[\alpha\int_0^{\tau(H)} (1-\beta X_s)^2 ds + \sigma\int_0^{\tau(H)} (1-\beta X_s)dW_s\right] \geqslant [1 - X_{\tau(H)}] + (\ln x_0 - \beta x_0) + \alpha H + \sigma\int_0^{\tau(H)} (1-\beta X_s)dW_s \tag{9}$$

Since $x_0 < \frac{1}{\beta}$, by Lemm 2. 3 of Jiang *et al.*[14], we have

$$E_\alpha X_{\tau(H)} \leqslant \frac{1}{\beta} \tag{10}$$

Hence,

$$\frac{\sigma^2}{2}E_\alpha[\tau(H)] \geqslant 1 - \frac{1}{\beta} + E_\alpha(\ln x_0 - \beta x_0) + \alpha H.$$

Eq. (6) holds.

To deduce Eq. (7), notice that by Eq. (9) we have

$$\frac{\sigma^2}{2}\tau(H) \leqslant \beta X_{\tau(H)} + \left(\frac{1}{X_{\tau(H)}} - 1\right) + \alpha H + (\ln x_0 - \beta x_0) + \sigma\int_0^{\tau(H)} (1-\beta X_s)dW_s \tag{11}$$

Finally, since $\alpha > \sigma^2$, we obtain an upper bound by the Lemma 2. 2 of Jiang *et al.*[14]

$$E_\alpha\left[\frac{1}{X_{\tau(H)}}\right] \leqslant \frac{1}{x_0} + \frac{\alpha\beta}{\alpha - \sigma^2},$$

from which the desired result Eq. (7) follows using Eq. (10) and Eq. (11).

Now a result about the efficiency of the SMLE can be obtained by the following theorem.

Theorem 2. 3 Let the sequential plan $(\tau(H), \hat{\alpha}_\tau)$ be an arbitrary unbiased estimation plan for the Logistic diffusion process $\{X_t\}$ with the unknown parameter $\alpha \in (0, \infty)$, namely,

$$E_\alpha(\hat{\alpha}_\tau) = \alpha, \text{ for all } \alpha \in (0, \infty) \tag{12}$$

Suppose also that $0 < E_\alpha\left(\int_0^\tau (1-\beta X_s)^2 ds\right) < \infty$. Then

$$Var_\alpha(\hat{\alpha}_\tau) = E_\alpha(\hat{\alpha}_\tau - \alpha)^2 \geqslant \frac{\sigma^2}{E_\alpha\left(\int_0^\tau (1-\beta X_s)^2 ds\right)} \tag{13}$$

Proof The proof is adapted from that of Theorem 7. 22 in Ref. [13]. Differentiating both sides of Eq. (12) with respect to α yields that

$$E_\alpha\left\{\hat{\alpha}_\tau\left[\frac{1}{\sigma^2}\int_0^\tau \frac{1-\beta X_t}{X_t}dX_t - \frac{\alpha}{\sigma^2}\int_0^\tau (1-\beta X_t)^2 dt\right]\right\} = 1$$

Then, since

$$E_\alpha\left[\int_0^\tau \frac{1-\beta X_t}{X_t}dX_t - \alpha\int_0^\tau (1-\beta X_t)^2 dt\right] = E_\alpha\left\{\int_0^\tau \frac{1-\beta X_t}{X_t}[\alpha X_t(1-\beta X_t)dt + \sigma X_t dW_t] - \alpha\int_0^\tau (1-\beta X_t)^2 dt\right\} = E_\alpha\left\{\left[\alpha\int_0^\tau (1-\beta X_t)^2 dt + \sigma\int_0^\tau (1-\beta X_t)dW_t\right] - \alpha\int_0^\tau (1-\beta X_t)^2 dt\right\} = E_\alpha\left[\sigma\int_0^\tau (1-\beta X_t)dW_t\right] = 0,$$

it follows that

$$E_\alpha\left\{(\hat{\alpha}_\tau - \alpha)\left[\frac{1}{\sigma^2}\int_0^\tau \frac{1-\beta X_t}{X_t}dX_t - \frac{\alpha}{\sigma^2}\int_0^\tau (1-\beta X_t)^2 dt\right]\right\} = 1,$$

namely,

$$E_\alpha\left\{(\hat{\alpha}_\tau - \alpha)\left[\int_0^\tau \frac{1-\beta X_t}{X_t}dX_t - \alpha\int_0^\tau (1-\beta X_t)^2 dt\right]\right\} = \sigma^2 \tag{14}$$

Applying Cauchy-Schwarz inequality in Eq. (14), we obtain

$$\sigma^4 \leqslant E_\alpha(\hat{\alpha}_\tau - \alpha)^2 \cdot E_\alpha\left[\int_0^\tau \frac{1-\beta X_t}{X_t}dX_t - \alpha\int_0^\tau (1-\beta X_t)^2 dt\right]^2 = E_\alpha(\hat{\alpha}_\tau - \alpha)^2 \cdot E_\alpha\left[\sigma\int_0^\tau (1-\beta X_t)dW_t\right]^2 = \sigma^2 E_\alpha(\hat{\alpha}_\tau - \alpha)^2 \cdot E_\alpha\left[\int_0^\tau (1-\beta X_t)^2 dt\right].$$

Therefore, Eq. (13) holds.

A sequential estimator $\hat{\alpha}_\tau$ is said to be efficient in the Mean Square Error (MSE) sense if

Eq. (13) becomes an equality for all $\alpha \in (0, \infty)$. Since property (iii) of Theorem 2.1 holds for the SMLE $\hat{\alpha}_{\tau(H)}$ for all $\alpha \in (0, \infty)$, by the above definition, the SMLE $\hat{\alpha}_{\tau(H)}$ is efficient in the MSE sense.

3 Numerical illustrations

In this section, we present some numerical illustrations to exhibit the performance of the SMLE $\hat{\alpha}_{\tau(H)}$ given by Theorem 2.1. We first simulate the sample paths of the Logistic diffusion process Eq. (1) by using the Monte Carlo method with the classic Euler-Maruyama scheme. In each numerical experiment, we generate 10^3 sample paths with step size $\Delta t = 10^{-2}$. We examine the following four different settings, respectively:

(a) set $\alpha=2$, $\beta=1$, $\sigma=1.5$;

(b) set $\alpha=4$, $\beta=1$, $\sigma=1.5$;

(c) set $\alpha=4$, $\beta=1$, $\sigma=1.5$;

(d) set $\alpha=2$, $\beta=0.5$, $\sigma=1.5$.

Tab. 1 The ME of the SMLE $\hat{\alpha}_{\tau(H)}$, the MSE of the SMLE $\hat{\alpha}_{\tau(H)}$ and the SD of $[\hat{\alpha}_{\tau(H)} - \alpha]^2$

	$E_\alpha[\tau(H)]$	$E_\alpha[\hat{\alpha}_{\tau(H)} - \alpha]$	$E_\alpha[\hat{\alpha}_{\tau(H)} - \alpha]^2$	SD of $[\hat{\alpha}_{\tau(H)} - \alpha]^2$
$\alpha=2$, $\beta=1$, $\sigma=1.5$				
$H=10$	18.770 9	−0.000 4	0.222 4	0.309 9
$H=50$	88.077 0	−0.005 7	0.043 9	0.063 6
$H=100$	175.249 6	−0.001 7	0.024 2	0.030 9
$\alpha=4$, $\beta=1$, $\sigma=1.5$				
$H=10$	34.852 9	0.000 1	0.222 4	0.315 8
$H=50$	172.619 5	0.001 5	0.047 3	0.072 4
$H=100$	345.085 1	0.007 5	0.020 2	0.029 3
$\alpha=1$, $\beta=0.5$, $\sigma=0.75$				
$H=10$	35.939 3	−0.015 3	0.054 1	0.071 9
$H=50$	176.483 8	−0.006 1	0.011 8	0.016 5
$H=100$	351.796 3	−0.006 2	0.005 6	0.008 2
$\alpha=2$, $\beta=0.5$, $\sigma=1.5$				
$H=10$	18.414 0	−0.013 5	0.237 0	0.349 0
$H=50$	87.794 1	−0.009 8	0.046 8	0.066 3
$H=100$	174.747 0	−0.003 3	0.021 4	0.031 5

Tab. 1 reports some statistics related to the SMLE $\hat{\alpha}_{\tau(H)}$, which include the Mean Error (ME) $E_\alpha[\hat{\alpha}_{\tau(H)} - \alpha]$, the MSE $E_\alpha[\hat{\alpha}_{\tau(H)} - \alpha]^2$, and the Standard Deviation (SD) of $[\hat{\alpha}_{\tau(H)} - \alpha]^2$. The mean time $E_\alpha[\tau(H)]$ needed to achieve three different H-levels are also reported. Notice also that the mean observation time $E_\alpha[\tau(H)]$, which is approximated from Monte Carlo simulation, shows a linear growth in H.

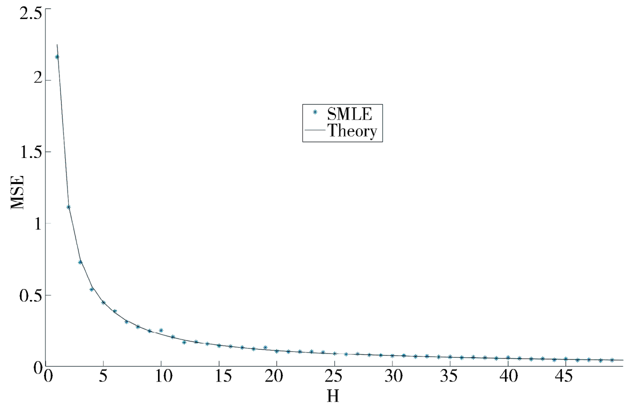

Fig. 1 The star points are the MSE plot for the SMLE $\hat{\alpha}_{\tau(H)}$, and the solid line is the corresponding theoretical MSE of the SMLE. The SMLE shows performance well

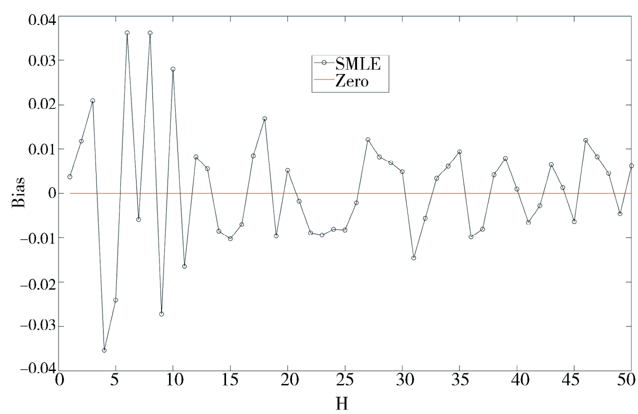

Fig. 2 Bias plot for the SMLE $\hat{\alpha}_{\tau(H)}$ against H

Figs. 1, 2 and 3 are plotted under the setting (a). Fig. 1 displays the MSE (the star points) of the SMLE $\hat{\alpha}_{\tau(H)}$ against $H>0$. The solid line in this plot is the theoretical MSE curve of the SMLE, *i.e.*, $\frac{\sigma^2}{H}$. We observe that they almost coincide. Fig. 2 shows the bias of the SMLE (from Theorem 2.1 (iii)), we have the bias $\hat{\alpha}_{\tau(H)} - \alpha \sim N(0, \frac{\sigma^2}{H})$. Fig. 3 depicts the histogram of the statistic $\frac{\sqrt{H}(\hat{\alpha}_{\tau(H)} - \alpha)}{\sigma}$, with $H=10$ and $H=50$.

The dashed curve is the standard normal density. We find that the SMLE works quite well (from Theorem 2.1 (iii)), we have

$$\frac{\sqrt{H}\,(\hat{\alpha}_{\tau(H)}-\alpha)}{\sigma}\sim N(0,1).$$

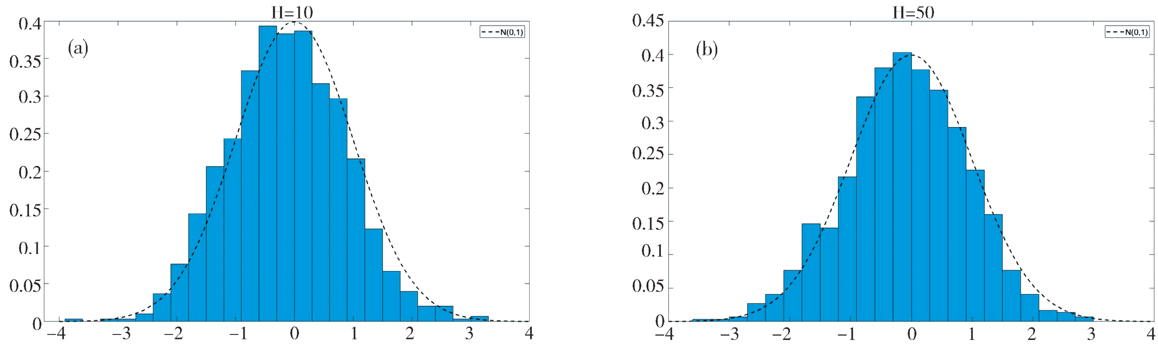

Fig. 3 Histogram of $\frac{\sqrt{H}\,(\hat{\alpha}_{\tau(H)}-\alpha)}{\sigma}$ with (a) $H=10$ and (b) $H=50$, the dashed lines are the plots of the standard normal density

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