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Bloch 型空间上的 Toeplitz 算子及分数阶导数刻画

贾 策[1,2]，曹广福[1]，王晓峰[1]，张艺渊[1]
(1. 广州大学数学与信息科学学院，广州 510006；2. 中国科学院生物物理研究所，北京 100101)

摘 要：令 μ 为 $\mathbf{C}^n$ 中单位球 $\mathbb{B}_n$ 上的正 Borel 测度. 本文主要刻画了 Bloch 型空间 $B^\alpha(\mathbb{B}_n)$ 上以 μ 为符号的 Toeplitz 算子 T_μ^α 的有界性和紧性，其中 $0<\alpha<1$. 当 $\alpha>1$ 时，本文利用分数阶导数给出了 $B^\alpha(\mathbb{B}_n)$ 空间上的函数刻画的充要条件.

关键词：Toeplitz 算子；分数阶导数；Bloch 型空间
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On the characterization of Toeplitz operators and fractional derivatives on Bloch-type space

JIA Ce[1,2], *CAO Guang-Fu*[1], *WANG Xiao-Feng*[1], *ZHANG Yi-Yuan*[1]
(1. School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China;
2. Institute of Biophysics, Chinese Academy of Sciences, Beijing 100101, China)

Abstract: Let μ be the positive Borel measure on the unit ball $\mathbb{B}_n$ of $\mathbf{C}^n$. We in this paper characterize the measure μ on $\mathbb{B}_n$ for which the Toeplitz operator T_μ^α is bounded or compact on the Bloch-type spaces $B^\alpha(\mathbb{B}_n)$, where $0<\alpha<1$. Additionlly, we also give a characterization for the functions on $B^\alpha(\mathbb{B}_n)$ in terms of fractional derivatives, where $\alpha>1$.

Keywords: Toeplitz operator; Fractional derivative; Bloch-type space
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1 Introduction

Let $\mathbf{C}^n$ be the complex Euclidean space of dimension n and $\mathbb{B}_n$ the unit ball of $\mathbf{C}^n$. For $\alpha>-1$, let $\mathrm{d}v_\alpha(z)=C_\alpha(1-|z|^2)^\alpha\mathrm{d}v(z)$ be the weighted volume measure, where $c_\alpha=\dfrac{\Gamma(n+\alpha+1)}{n!\ \Gamma(\alpha+1)}$ is a normalizing constant such that $v_\alpha(\mathbb{B}_n)=1$. For $\alpha>-1$ and $0<p<\infty$, the weighted Bergman space $A_\alpha^p(\mathbb{B})$ consists of all holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|_{A_\alpha^p(\mathbb{B}_n)}=\left[\int_{\mathbb{B}_n}|f(z)|^p\mathrm{d}v_\alpha(z)\right]^{1/p}<\infty.$$

When the weight $\alpha=0$, we simply write $A^p(\mathbb{B}_n)$ for $A_\alpha^p(\mathbb{B}_n)$. These are the standard Bergman spaces. When $p=2$, $A_\alpha^2(\mathbb{B})$ is a Hilbert space. It is well known that the reproducing kernel of $A_\alpha^2(\mathbb{B})$ is given by

$$K^\alpha(z,w)=1/(1-\langle z,w\rangle)^{n+1+\alpha},$$

where $\langle z,w\rangle=\sum_{i=1}^n z_i\bar{w}_i$ for $z=(z_1,\cdots,z_n)$, $w=(w_1,\cdots,w_n)$. The Bergman projection P_α is the orthogonal projection from $L^2(\mathbb{B}_n,\mathrm{d}v_\alpha)$ onto $A_\alpha^2(\mathbb{B})$ defined by

$$P_\alpha(f)(z)=f_{\mathbb{B}_n}K^\alpha(z,w)f(w)\mathrm{d}v_\alpha(w),\quad f\in L^1(\mathbb{B}_n,\mathrm{d}v_\alpha).$$

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作者简介：贾策(1984—)，天津人，博士，高级工程师，主要研究方向为算子理论. E-mail：jiace@ibp.ac.cn
通讯作者：曹广福. E-mail：guangfucao@163.com；王晓峰. E-mail：wxf@gzhu.edu.cn；张艺渊. E-mail：1053296958@qq.com

The projection P_α naturally extends to an integral operator on $L^1(\mathbb{B}_n, dv_\alpha)$, see Ref. [1, Theorem 2.11].

For $\alpha > -1$, we also define the general Bergman projection of the measure μ as

$$P_\alpha(\mu)(z) = c_\alpha \int_{\mathbb{B}_n} K^\alpha(z,w)(1-|w|^2)^\alpha d\mu(w).$$

For a measure μ on $\mathbb{B}_n$ and $\alpha > 0$, we define a Toeplitz operator as

$$T_\mu^\alpha(f)(z) = c_{\alpha-1} \int_{\mathbb{B}_n} \frac{f(w)(1-|w|^2)^{\alpha-1}}{(1-\langle z,w\rangle)^{n+\alpha}} d\mu(w),$$
$$f \in L^1(\mathbb{B}_n, dv_\alpha).$$

Thus $T_\mu^\alpha(f)(z) = P_{\alpha-1}(\mu_f)(z)$, where $d\mu_f(z) = f(z)d\mu(z)$. For $\alpha > 0$, the α-Bloch space $B^\alpha(\mathbb{B}_n)$, also known as the Bloch-type space, consists exactly of holomorphic functions f on B_n such that

$$\|f\|_{\mathbb{B}^\alpha_*(B_n)} = \sup_{z\in B_n}(1-|z|^2)^\alpha |\nabla f(z)| < \infty,$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}(z), \ldots \frac{\partial f}{\partial z_n}(z))$.

The Bloch-type space $B^\alpha(\mathbb{B}_n)$ becomes a Banach space when equipped with the norm

$$\|f\|_{B^\alpha(\mathbb{B}_n)} = |f(0)| + \sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha |\nabla f(z)|.$$

It is well known that the above norm is equivalent to $|f(0)| + \sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha |Rf(z)|$, where $Rf(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z)$ is the radial derivative of f at z.

Let $H(\mathbb{B}_n)$ be the holomorphic functions on $\mathbb{B}_n$, for any two real parameters γ and t such that neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, we define an invertible fractional differential operator $R^{\gamma,t}: H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ as follows. If $f(z) = \sum_{k=0}^\infty f_k(z)$ is the homogeneous expansion of f, then

$$R^{\gamma,t}f(z) = \sum_{k=0}^\infty \frac{\Gamma(n+1+\gamma)\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\Gamma(n+1+k+\gamma)} f_k(z).$$

The inverse of $R^{\gamma,t}$, denoted by $R_{\gamma,t}$ is given by

$$R_{\gamma,t}f(z) = \sum_{k=0}^\infty \frac{\Gamma(n+1+\gamma+t)\Gamma(n+1+k+\gamma)}{\Gamma(n+1+\gamma)\Gamma(n+1+k+\gamma+t)} f_k(z).$$

Toeplitz operators have been extensively studied on many spaces of analytic functions, see, for instance, Refs. [1-18]. A fundamental problem is to determine conditions on the measure, necessary or sufficient, for the corresponding Toeplitz operator to be either bounded or compact. There is also some previous work on the characterization of bounded and compact Toeplitz operators T_μ^α on α-Bloch spaces. In Ref. [14], the authors have completely characterized complex measure μ on the unit disk $\mathbb{D}$ under some restricted conditions for which T_μ^α is bounded or compact on Bloch-type spaces $B^\alpha(\mathbb{D})$ with $0<\alpha<\infty$. In Ref. [13], due to the limitation of technique in Ref. [16, Theorem 2], the authors have only characterized the positive Borel measure μ on $\mathbb{B}_n$ such that T_μ^α is bounded or compact on $B^\alpha(\mathbb{B}_n)$ with $1\leqslant\alpha<2$. In this paper, we will use another different technique to characterize the positive Borel measure μ on $\mathbb{B}_n$ for which the Toeplitz operator T_μ^α is bounded or compact on $B^\alpha(\mathbb{B}_n)$ with $0<\alpha<1$, which is an extension of Ref. [13]. Besides, we also give a characterization of functions on $B^\alpha(\mathbb{B}_n)$ in terms of fractional derivatives and its module with $\alpha>1$.

Our main results about the boundedness or compactness of Toeplitz operators T_μ^α on $B^\alpha(\mathbb{B}_n)$ with $0<\alpha<1$ are given in Sections 3 and 4, and the main results about characterization of functions on $B^\alpha(\mathbb{B}_n)$ in terms of fractional derivatives and its module with $\alpha>1$ are shown in Section 5.

2 Preliminaries

For $w\in\mathbb{B}_n\backslash\{0\}$, the automorphism mapping $\varphi_w: \mathbb{B}_n \to \mathbf{C}^n$ is given by

$$\varphi_w(z) = \frac{w - P_w(z) - \sqrt{1-|w|^2}Q_w(z)}{1-\langle z,w\rangle},$$

where P_w is the orthogonal projection from $\mathbf{C}_n$ onto the one dimensional subspace $[w]$ generated by w, and Q_w is the orthogonal projection from $\mathbf{C}_n$ onto $\mathbf{C}_n - [w]$ defined by $Q_w = I - P_w$. More information about the mapping φ_w is described in section 2.2 of Ref. [11] or section 1.2 of Ref. [1], where we can find the following identity

$$1-\langle\varphi_w(z),w\rangle = \frac{1-|w|^2}{1-\langle z,w\rangle} \quad (1)$$

In Ref. [3, Lemma 2.1], we can find the inequality

$$|w-\varphi_w(z)|^2\leqslant\frac{2(1-|w|^2)}{|1-\langle z,w\rangle|} \tag{2}$$

Lemma 2.1[1] For any $\alpha>-1$ and $z\in\mathbb{B}_n$, we have

$$|w-\varphi_w(z)|^2\leqslant\frac{2(1-|w|^2)}{|1-\langle z,w\rangle|}$$

if f is a holomorphic function on $\mathbb{B}_n$ with

$$\int_{B_n}(1-|w|^2)^\alpha |f(w)| \mathrm{d}v(w)<+\infty,$$

where $\mathrm{d}v$ is the normalized volume measure on $\mathbb{B}_n$.

Lemma 2.2[1] Suppose c is real and $t>-1$. Then the integrals

$$I_c(z)=\int_{S_n}\frac{\mathrm{d}\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+c}},z\in\mathbb{B}_n$$

and

$$J_{c,t}(z)=\int_{B_n}\frac{(1-|w|^2)^t\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+t+c}},\ z\in\mathbb{B}_n$$

have the following asymptotic properties:

(i) If $c<0$, then I_c and $J_{c,t}$ are both bounded in B_n;

(ii) If $c=0$, then $I_c(z)\sim J_{c,t}(z)\sim\log\frac{1}{1-|z|^2}, |z|\to1^-$;

(iii) If $c>0$, then $I_c(z)\sim J_{c,t}(z)\sim(1-|z|^2)^{-c}, |z|\to1^-$.

Lemma 2.3[16] Let $0<\alpha<2$, β be any real number satisfying the following properties:

(i) $0\leqslant\beta\leqslant\alpha$ if $0<\alpha<1$;

(ii) $0<\beta<1$ if $\alpha=1$;

(iii) $\alpha-1\leqslant\beta\leqslant1$ if $1<\alpha\leqslant2$.

Then a holomorphic function $f\in B^\alpha(\mathbb{B}_n)$ if and only if

$$F_\beta(f)=\sup_{z,w\in B_n}(1-|z|^2)^\beta(1-|w|^2)^{\alpha-\beta}\frac{|f(z)-f(w)|}{|z-w|}<\infty.$$

Moreover, for any α and β satisfying above conditions, the following two semi-norms $\sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|\nabla f(z)|$ and $F_\beta(f)$ are equivalent.

Lemma 2.4[13] Suppose that $0<\alpha<1$. If $f\in B^\alpha(\mathbb{B}_n)$, then

$$|f(z)|\leqslant\frac{1}{1-\alpha}\|f\|_{B^\alpha(\mathbb{B}_n)},\ z\in\mathbb{B}_n.$$

Lemma 2.5 For any $z,w\in\mathbb{B}_n$, the following estimate holds:

$$|w-z|\leqslant\sqrt{2|1-\langle z,w\rangle|}.$$

Proof According to inequality (2), we have

$$|w-\varphi_w(u)|^2\leqslant\frac{2(1-|w|^2)}{|1-\langle w,u\rangle|}.$$

The change of variable $u=\varphi_w(z)$ yields

$$|w-z|^2\leqslant\frac{2(1-|w|^2)}{|1-\langle\varphi_w(z),w\rangle|}.$$

This together with (1) gives the desired result.

3 Bounded Toeplitz operators

In this section, we are going to characterize bounded Toeplitz operators on $B^\alpha(\mathbb{B}_n)$ for $0<\alpha<1$. To this end, for a positive measure μ on $\mathbb{B}_n$ and $\alpha-\beta>0$, we call μ satisfies the condition $S_{\alpha,\beta}$ if

$$S_{\alpha,\beta}(\mu)(z)=(1-|z|^2)^\beta\cdot\int_{B_n}\frac{(1-|w|^2)^{\alpha-\beta-1}}{|1-\langle w,z\rangle|^{n+\alpha+1/2}}\mathrm{d}\mu(w)<\infty.$$

In fact, such a positive measure μ satisfying the condition $S_{\alpha,\beta}$ does exist and there are many. Next, we will give an example under the assumption that $\alpha-\beta>0$.

Example 3.1 Let

$$\mathrm{d}\mu(w)=(1-|w|^2)^\gamma\mathrm{d}v(w),$$

where $w\in\mathbb{B}_n$ and $\gamma>0$. If $\gamma=1/2$ and $\beta>0$, or $\gamma>\beta+1/2$, then $S_{\alpha,\beta}(\mu)(z)<\infty$ for all $z\in\mathbb{B}_n$.

Proof we have

$$S_{\alpha,\beta}(\mu)(z)=(1-|z|^2)^\beta\cdot\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-\beta-1+\gamma}}{|1-\langle w,z\rangle|^{n+\alpha+1/2}}\mathrm{d}v(w)=(1-|z|^2)^\beta\cdot\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-\beta-1+\gamma}}{|1-\langle w,z\rangle|^{n+1+\alpha-\beta-1+\gamma+\beta-\gamma+1/2}}\mathrm{d}v(w).$$

If $\gamma=1/2$ and $\beta>0$, then by (iii) of Lemma 2.2, we have

$$S_{\alpha,\beta}(\mu)(z)=(1-|z|^2)^\beta\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-\beta-1/2}}{|1-\langle w,z\rangle|^{n+1+\alpha-\beta-1/2+\beta}}\mathrm{d}v(w)\sim1.$$

If $\beta-\gamma+1/2<0$, that is, $\gamma>\beta+1/2$, then by (i) of Lemma 2.2, we get

$$\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-\beta-1+\gamma}}{|1-\langle w,z\rangle|^{n+1+\alpha-\beta-1+\gamma+\beta-\gamma+1/2}}\mathrm{d}\mu(w)<\infty,$$

hence $S_{\alpha,\beta}(\mu)(z)<\infty$ for all $z\in\mathbb{B}_n$.

Theorem 3.2 Let $0<\alpha<1$ and μ be the positive Borel measure on $\mathbb{B}_n$. If μ satisfies the condition $S_{\alpha,\beta}$ then

$$S_{\alpha,\beta}(\mu)(z) = (1-|z|^2)^{\beta}\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-\beta-1}}{|1-\langle w,z\rangle|^{n+\alpha+1/2}}d\mu(w) < \infty$$

is bounded on $B^{\alpha}(\mathbb{B}_n)$ if and only if $P_{\alpha-1}(\mu)\in B^{\alpha}(\mathbb{B}_n)$.

Proof It can be seen from Theorem 7.6 of Ref. [1] that $(A^1(\mathbb{B}_n))^*\cong B^{\alpha}(\mathbb{B}_n)$ under the integral pairing

$$\langle f,g\rangle_{\alpha-1} = \int_{\mathbb{B}_n} f(z)\overline{g(z)}(1-|z|^2)^{\alpha-1}dv(z),$$

$$f\in A^1(\mathbb{B}_n),\ g\in B^{\alpha}(\mathbb{B}_n).$$

In order to prove the boundedness of T^{α}_{μ}, we need to show

$$|\langle f,T^{\alpha}_{\mu}(g)\rangle_{\alpha-1}| \leqslant C\|f\|_{A^1(\mathbb{B}_n)}\|g\|_{B^{\alpha}(\mathbb{B}_n)}$$

for any $f\in A^1(\mathbb{B}_n)$ and $g\in B^{\alpha}(\mathbb{B}_n)$.

Applying Fubini's Theorem and the reproducing property, we obtain

$$\begin{aligned}
\langle f,T^{\alpha}_{\mu}(g)\rangle_{\alpha-1} &= c_{\alpha-1}\int_{\mathbb{B}_n} f(z)\overline{T^{\alpha}_{\mu}(g)(z)}(1-|z|^2)^{\alpha-1}dv(z) = \\
& c_{\alpha-1}\int_{\mathbb{B}_n} f(z)c_{\alpha-1}\int_{\mathbb{B}_n}\frac{\overline{g(w)}(1-|w|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+\alpha}}d\mu(w)(1-|z|^2)^{\alpha-1}dv(z) = \\
& c_{\alpha-1}\int_{\mathbb{B}_n} c_{\alpha-1}\int_{\mathbb{B}_n}\frac{f(z)(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+\alpha}}dv(z)\overline{g(w)}(1-|w|^2)^{\alpha-1}d\mu(w) = \\
& c_{\alpha-1}\int_{\mathbb{B}_n} f(w)\overline{g(w)}(1-|w|^2)^{\alpha-1}d\mu(w) = \\
& c_{\alpha-1}\int_{\mathbb{B}_n} P_{\alpha}(f\bar{g})(w)(1-|w|^2)^{\alpha-1}d\mu(w) + \\
& c_{\alpha-1}\int_{\mathbb{B}_n}(I-P_{\alpha})(f\bar{g})(w)(1-|w|^2)^{\alpha-1}d\mu(w) \triangleq I_1+I_2,
\end{aligned}$$

where

$$\begin{aligned}
(I-P_{\alpha})(f\bar{g})(w) &= f(w)\overline{g(w)} - c_{\alpha}\int_{\mathbb{B}_n}\frac{f(z)\overline{g(z)}(1-|z|^2)^{\alpha}}{(1-\langle w,z\rangle)^{n+1+\alpha}}dv(z) = \\
& c_{\alpha}\int_{\mathbb{B}_n}\frac{(\overline{g(w)}-\overline{g(z)})f(z)(1-|z|^2)^{\alpha}}{(1-\langle w,z\rangle)^{n+1+\alpha}}dv(z).
\end{aligned}$$

Choosing $\beta\geqslant 0$ such that $\alpha-\beta>0$, by Lemmas 2.3 and 2.5, we get

$$\begin{aligned}
|I_2| &= c_{\alpha-1}c_{\alpha}\left|\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(\overline{g(w)}-\overline{g(z)})(1-|z|^2)^{\alpha}(1-|w|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+1+\alpha}}dv(z)d\mu(w)\right| = \\
& c_{\alpha-1}c_{\alpha}\left|\int_{\mathbb{B}_n} f(z)(1-|z|^2)^{\alpha}\int_{\mathbb{B}_n}\frac{(\overline{g(w)}-\overline{g(z)})(1-|w|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+1+\alpha}}d\mu(w)dv(z)\right| \leqslant \\
& c_{\alpha-1}c_{\alpha}\int_{\mathbb{B}_n}|f(z)|(1-|z|^2)^{\beta}\int_{\mathbb{B}_n}(1-|z|^2)^{\alpha-\beta}(1-|w|^2)^{\beta}\frac{|g(w)-g(z)|}{|w-z|}\cdot \\
& \frac{(1-|w|^2)^{\alpha-\beta-1}|w-z|}{|1-\langle w,z\rangle|^{n+1+\alpha}}d\mu(w)dv(z).
\end{aligned}$$

Since μ satisfies the condition $S_{\alpha,\beta}$, hence $|I_2|\leqslant C\|f\|_{A^1(B_n)}\|g\|_{B^{\alpha}(B_n)}$.

Next, we consider I_1. By Fubini's Theorem, we have

$$\begin{aligned}
I_1 &= c_{\alpha-1}\int_{\mathbb{B}_n} P_{\alpha}(f\bar{g})(w)(1-|w|^2)^{\alpha-1}d\mu(w) = \\
& c_{\alpha-1}\int_{\mathbb{B}_n} c_{\alpha}\int_{\mathbb{B}_n}\frac{f(z)\overline{g(z)}(1-|z|^2)^{\alpha}}{(1-\langle w,z\rangle)^{n+1+\alpha}}dv(z)(1-|w|^2)^{\alpha-1}d\mu(w) = \\
& c_{\alpha-1}c_{\alpha}\int_{\mathbb{B}_n} f(z)\overline{g(z)\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1}}{(1-\langle z,w\rangle)^{n+1+\alpha}}d\mu(w)}(1-|z|^2)^{\alpha}dv(z).
\end{aligned}$$

Let

$$Q_\alpha(\mu)(z)=c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1}}{(1-\langle z,w\rangle)^{n+1+\alpha}}\mathrm{d}\mu(w).$$

Then we have

$$I_1=c_\alpha\int_{\mathbb{B}_n}f(z)\overline{g(z)Q_\alpha(\mu)(z)}(1-|z|^2)^\alpha\mathrm{d}v(z).$$

By some elementary calculation, we obtain the following relation between $Q_\alpha(\mu)$ and $P_{\alpha-1}(\mu)$:

$$Q_\alpha(\mu)(z)=P_{\alpha-1}(\mu)(z)+\frac{1}{n+\alpha}RP_{\alpha-1}(\mu)(z).$$

Since $g(z)$ and $P_{\alpha-1}(\mu)$ belong to $B^\alpha(\mathbb{B}_n)$, by Lemma 2.4, there exist constant C_1 and C_2 satisfying the following inequalities, respectively, $|g(z)|\leqslant C_1\|g(z)\|_{B^\alpha(B_n)}$, $|P_{\alpha-1}(\mu)|\leqslant C_2\|P_{\alpha-1}(\mu)\|_{B^\alpha(\mathbb{B}_n)}$. Then

$$\begin{aligned}&|(1-|z|^2)^\alpha g(z)Q_\alpha(\mu)(z)|=\\&|(1-|z|^2)^\alpha g(z)P_{\alpha-1}(\mu)(z)+\\&\frac{g(z)}{n+\alpha}\cdot(1-|z|^2)^\alpha RP_{\alpha-1}(\mu)(z)|<\\&(1-|z|^2)^\alpha|g(z)|\cdot|P_{\alpha-1}(\mu)|+\\&\frac{1}{n+\alpha}|g(z)\|\cdot|P_{\alpha-1}(\mu)\|_{B^\alpha(\mathbb{B}_n)}\leqslant\\&C_1C_2\|g(z)\|_{B^\alpha(\mathbb{B}_n)}\|P_{\alpha-1}(\mu)\|_{B^\alpha(\mathbb{B}_n)}+\\&\frac{C_1}{n+\alpha}\|g(z)\|_{B^\alpha(\mathbb{B}_n)}\|P_{\alpha-1}(\mu)\|_{B^\alpha(\mathbb{B}_n)}\leqslant\\&C\|g(z)\|_{B^\alpha(\mathbb{B}_n)}.\end{aligned}$$

Thus we conclude that

$$|I_1|\leqslant C\|f\|_{A^1(\mathbb{B}_n)}\|g(z)\|_{B^\alpha(\mathbb{B}_n)}.$$

Therefore, T_μ^α is bounded on $B^\alpha(\mathbb{B}_n)$.

Conversely, if T_μ^α is bounded on $B^\alpha(\mathbb{B}_n)$, then $T_\mu^\alpha(1)=P_{\alpha-1}(\mu)\in B^\alpha(\mathbb{B}_n)$. This completes the proof.

4 Compact Toeplitz operators

In this section we present our main characterization of compact Toeplitz operator on $B^\alpha(\mathbb{B}_n)$ with $0<\alpha<1$.

Theorem 4.1 Let $0<\alpha<1$. If the positive Borel measure μ satisfies $\lim\limits_{|z|\to1}S_{\alpha,\beta}(\mu)(z)=0$ then T_μ^α is compact on $B^\alpha(\mathbb{B}_n)$ if and only if $P_{\alpha-1}(\mu)\in B^\alpha(\mathbb{B}_n)$.

Proof Let $\{g_n\}$ be a sequence in $B^\alpha(\mathbb{B}_n)$ such that $\|g_n\|_{B^\alpha(\mathbb{B}_n)}\leqslant1$ and $g_n(z)\to0$ uniformly on compact subsets of $\mathbb{B}_n$. Let f be in the unit ball of $A^1(\mathbb{B}_n)$, by a similar discussion as Theorem 3.1, we have

$$\begin{aligned}&\langle f,T_\mu^\alpha(g_n)\rangle_{\alpha-1}=\\&c_{\alpha-1}\int_{\mathbb{B}_n}P_\alpha(f\bar{g}_n)(w)(1-|w|^2)^{\alpha-1}\mathrm{d}\mu(w)+\\&c_{\alpha-1}\int_{B_n}(I-P_\alpha)(f\bar{g}_n)(w)(1-|w|^2)^{\alpha-1}\mathrm{d}\mu(w)=\\&I_{1,n}+I_{2,n},\end{aligned}$$

where

$$\begin{aligned}I_{1,n}=&c_\alpha\int_{\mathbb{B}_n}f(z)\overline{g_n(z)Q_\alpha(\mu)(z)}\cdot\\&(1-|z|^2)^\alpha\mathrm{d}v(z),\end{aligned}$$

$$I_{2,n}=c_{\alpha-1}c_\alpha\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(\overline{g_n(w)}-\overline{g_n(z)})f(z)(1-|z|^2)^\alpha(1-|w|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+1+\alpha}}\mathrm{d}v(z)\mathrm{d}\mu(w).$$

Firstly, we consider $I_{2,n}$. Let $B_\delta=\{z:|z|\leqslant\delta\}$, where $0<\delta<1$. We will divide the integral into two parts, say,

$$\begin{aligned}&\lim_{n\to\infty}|I_{2,n}|=\lim_{n\to\infty}c_{\alpha-1}c_\alpha\left|\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(\overline{g_n(w)}-\overline{g_n(z)})f(z)(1-|z|^2)^\alpha(1-|w|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+1+\alpha}}\mathrm{d}v(z)\mathrm{d}\mu(w)\right|=\\&\lim_{n\to\infty}c_{\alpha-1}c_\alpha\left|\int_{\mathbb{B}_n}f(z)(1-|z|^2)^\alpha\int_{\mathbb{B}_n}\frac{(\overline{g_n(w)}-\overline{g_n(z)})(1-|w|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+1+\alpha}}\mathrm{d}\mu(w)\mathrm{d}v(z)\right|\leqslant\\&\lim_{n\to\infty}C\int_{\mathbb{B}_n\backslash\mathbb{B}_\delta}|f(z)|(1-|z|^2)^\alpha\int_{\mathbb{B}_n}\frac{|\overline{g_n(w)}-\overline{g_n(z)}|(1-|w|^2)^{\alpha-1}}{|1-\langle w,z\rangle|^{n+1+\alpha}}\mathrm{d}\mu(w)\mathrm{d}v(z)+\\&\lim_{n\to\infty}C\int_{\mathbb{B}_\delta}|f(z)|(1-|z|^2)^\alpha\int_{\mathbb{B}_n}\frac{|\overline{g_n(w)}-\overline{g_n(z)}|(1-|w|^2)^{\alpha-1}}{|1-\langle w,z\rangle|^{n+1+\alpha}}\mathrm{d}\mu(w)\mathrm{d}v(z)\triangleq J_{1,n}+J_{2,n}.\end{aligned}$$

For $J_{1,n}$, since

$$\lim_{|z|\to1}S_{\alpha,\beta}(\mu)(z) = \lim_{|z|\to1}(1-|z|^2)^\beta\int_{B_n}\frac{(1-|w|^2)^{\alpha-\beta-1}}{|1-\langle w,z\rangle|^{n+\alpha+1/2}}\mathrm{d}\mu(w) = 0,$$

where $\beta\geqslant0$ and $\alpha-\beta>0$, for a fixed $\varepsilon>0$, let δ get sufficiently close to 1 such that $S_{\alpha,\beta}(\mu)(z)<\varepsilon$, combining with Lemmas 2.3 and 2.5, we have

$$\begin{aligned}J_{1,n}\leqslant{}&\lim_{n\to\infty}C\int_{\mathbb{B}_n\setminus\mathbb{B}_\delta}|f(z)|(1-|z|^2)^\beta\int_{\mathbb{B}_n}(1-|z|^2)^{\alpha-\beta}(1-|w|^2)^\beta\cdot\\&\frac{|\overline{g_n(w)}-\overline{g_n(z)}|}{|w-z|}\frac{(1-|w|^2)^{\alpha-\beta-1}|w-z|}{|1-\langle w,z\rangle|^{n+1+\alpha}}\mathrm{d}\mu(w)\mathrm{d}v(z)\leqslant\\&\lim_{n\to\infty}C\int_{\mathbb{B}_n\setminus\mathbb{B}_\delta}|f(z)|\,\|g_n\|_{B^\alpha(\mathbb{B}_n)}(1-|z|^2)^\beta\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-\beta-1}|w-z|}{|1-\langle w,z\rangle|^{n+1+\alpha}}\mathrm{d}\mu(w)\mathrm{d}v(z)\leqslant\\&C\int_{\mathbb{B}_n\setminus\mathbb{B}_\delta}|f(z)|(1-|z|^2)^\beta\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-\beta-1}}{|1-\langle w,z\rangle|^{n+\alpha+1/2}}\mathrm{d}\mu(w)\mathrm{d}v(z)\leqslant\\&C\varepsilon\int_{\mathbb{B}_n\setminus\mathbb{B}_\delta}|f(z)|\mathrm{d}v(z)\leqslant C\varepsilon\|f\|_{A^1(\mathbb{B}_n)}\leqslant C\varepsilon.\end{aligned}$$

For $J_{2,n}$, let $\mathbb{B}_r=\{z:|z|<r\}$, where $0<r<1$, we also divide it into two parts, say,

$$\begin{aligned}J_{2,n}\leqslant{}&\lim_{n\to\infty}C\int_{\mathbb{B}_\delta}|f(z)|(1-|z|^2)^\alpha\int_{\mathbb{B}_n\setminus\mathbb{B}_r}\frac{|\overline{g_n(w)}-\overline{g_n(z)}|(1-|w|^2)^{\alpha-1}}{|1-\langle w,z\rangle|^{n+1+\alpha}}\mathrm{d}\mu(w)\mathrm{d}v(z)+\\&\lim_{n\to\infty}C\int_{\mathbb{B}_\delta}|f(z)|(1-|z|^2)^\alpha\int_{\mathbb{B}_n\setminus\mathbb{B}_r}\frac{|\overline{g_n(w)}-\overline{g_n(z)}|(1-|w|^2)^{\alpha-1}}{|1-\langle w,z\rangle|^{n+1+\alpha}}\mathrm{d}\mu(w)\mathrm{d}v(z)\triangleq K_{1,n}+K_{2,n}.\end{aligned}$$

For $K_{1,n}$, by a similar discussion as $J_{1,n}$, we obtain $K_{1,n}\leqslant C\varepsilon$. For $K_{2,n}$, since $g_n(z)\to0$ uniformly on any compact subsets of $\mathbb{B}_n$, we can choose n large enough such that $|\overline{g_n(w)}-\overline{g_n(z)}|(1-|w|^2)^{\alpha-1}\leqslant\varepsilon$ uniformly for z belongs to compact subsets of $\mathbb{B}_n$, therefore

$$\begin{aligned}K_{2,n}={}&\lim_{n\to\infty}C\int_{B_\delta}|f(z)|(1-|z|^2)^\alpha\cdot\\&\int_{\mathbb{B}_r}\frac{|\overline{g_n(w)}-\overline{g_n(z)}|(1-|w|^2)^{\alpha-1}}{|1-\langle w,z\rangle|^{n+1+\alpha}}\cdot\\&\mathrm{d}\mu(w)\mathrm{d}v(z)\leqslant\\&C\varepsilon\int_{\mathbb{B}_\delta}|f(z)|(1-|z|^2)^\alpha\cdot\\&\int_{\mathbb{B}_r}\frac{1}{|1-\langle w,z\rangle|^{n+1+\alpha}}\mathrm{d}\mu(w)\mathrm{d}v(z)\leqslant\\&C\varepsilon\|f\|_{A^1(B_n)}\leqslant C\varepsilon.\end{aligned}$$

Consequently, we have $\lim_{n\to\infty}|I_{2,n}|\leqslant C\varepsilon$, which yields that $\lim_{n\to\infty}|I_{2,n}|=0$.

For $I_{1,n}$, since $\|g_n(z)\|_{B^\alpha(B_n)}\leqslant1$, $g_n(z)\to0$ uniformly on any compact subsets of $\mathbb{B}_n$, we can choose n large enough so that $|g_n(z)|\leqslant\varepsilon$ uniformly for z belongs to compact subsets of $\mathbb{B}_n$. Combined this with what we have estimated in the proof of Theorem 3.1, we obtain

$$\begin{aligned}&\lim_{n\to\infty}|I_{1,n}|=\\&\left|\int_{\mathbb{B}_n}f(z)\overline{g_n(z)Q_\alpha(\mu)(z)}(1-|z|^2)^\alpha\mathrm{d}v(z)\right|\leqslant\\&C\varepsilon\|f\|_{A^1(\mathbb{B}_n)}\|P_{\alpha-1}(\mu)\|_{B^\alpha(\mathbb{B}_n)}\leqslant C\varepsilon.\end{aligned}$$

Thus $\lim_{n\to\infty}|I_{1,n}|=0$. Therefore, T_μ^α is compact on $B^\alpha(\mathbb{B}_n)$.

Conversely, let T_μ^α be compact on $B^\alpha(\mathbb{B}_n)$. Then T_μ^α is bounded on $B^\alpha(\mathbb{B}_n)$. By Theorem 3.1, we have $P_{\alpha-1}(\mu)\in B^\alpha(\mathbb{B}_n)$. This completes the proof.

5 Characterization fractional derivatives on Bloch-type spaces

In this section, we will give a characterization of functions on $B^\alpha(\mathbb{B}_n)$ in terms of fractional derivatives and its module with $\alpha>1$.

For $0<\alpha<1$, the Lipschitz space $\Lambda_\alpha(\mathbb{B}_n)$ consists of all holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|_{\Lambda_\alpha^*(\mathbb{B}_n)}=\sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\alpha}:z,w\in\mathbb{B}_n,z\neq w\right\}<\infty.$$

The space $\Lambda_\alpha(\mathbb{B}_n)$ is called the holomorphic Lipschitz space of order α. It is well known that each space $\Lambda_\alpha(\mathbb{B}_n)$ is contained in the ball algebra and contains the polynomials. For each $\alpha\in(0,1)$, the holomorphic Lipschitz space $\Lambda_\alpha(\mathbb{B}_n)$ is a Banach space with the norm $\|f\|_{\Lambda_\alpha(\mathbb{B}_n)}=|f(0)|+\|f\|_{\Lambda_\alpha^*(\mathbb{B}_n)}$. Please refer to Ref. [1, Theorem

7.8] for the detailed proof.

Lemma 5.1[1]　Suppose that $0<\alpha<1,\beta>1$ and f is holomorphic in $\mathbb{B}_n$. Then the following conditions are equivalent:

(i) $f\in\Lambda_\alpha(\mathbb{B}_n)$;

(ii) f is in the ball algebra and its boundary values satisfy

$$\sup\left\{\frac{|f(\zeta)-f(\xi)|}{|\zeta-\xi|^\alpha}:\zeta,\xi\in\mathbb{B}_n,\zeta\neq\xi\right\}<\infty;$$

(iii) $(1-|z|^2)^{1-\alpha}|Rf(z)|$ is bounded in $\mathbb{B}_n$;

(iv) There exists a function $g\in L^\infty(\mathbb{B}_n)$ such that

$$f(z)=\int_{\mathbb{B}_n}\frac{g(w)\mathrm{d}v_\beta(w)}{(1-\langle z,w\rangle)^{n+1+\beta-\alpha}},\ z\in\mathbb{B}_n;$$

(v) $(1-|z|^2)^{1-\alpha}|\nabla f(z)|$ is bounded in $\mathbb{B}_n$.

Lemma 5.2[1]　Suppose that $\alpha>0,\beta>1$ and f is holomorphic in $\mathbb{B}_n$. Then the following conditions are equivalent:

(i) $f\in\mathbb{B}^\alpha(B_n)$;

(ii) The function $(1-|z|^2)^\alpha\,|Rf(z)|$ is bounded in $\mathbb{B}_n$;

(iii) There exists a function $g\in L^\infty(\mathbb{B}_n)$ such that

$$f(z)=\int_{\mathbb{B}_n}\frac{g(w)\mathrm{d}v_\beta(w)}{(1-\langle z,w\rangle)^{n+\alpha+\beta}},\ z\in\mathbb{B}_n.$$

In view of Lemma 5.1 and Lemma 5.2, we clearly see that $\Lambda_{1-\alpha}(\mathbb{B}_n)=B^\alpha(\mathbb{B}_n)$. for any $0<\alpha<1$. Therefore, in order to obtain a characterization of the functions on $B^\alpha(\mathbb{B}_n)$ in terms of fractional derivatives with $0<\alpha<1$, we only need to get the corresponding result for $\Lambda_\alpha(\mathbb{B}_n)$, and Zhu in Ref. [1, Theorem 7.17] has gotten this, which is shown in the following Lemma.

Lemma 5.3　Suppose that $t>\alpha>0$. If γ is a real parameter such that neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, then a holomorphic function f in $\mathbb{B}_n$ belongs to $\mathbb{B}^\alpha(B_n)$ if and only if the function $(1-|z|^2)^{t+\alpha-1}R^{\gamma,t}f(z)$ is bounded in $\mathbb{B}_n$.

By using the relation $\Lambda_{1-\alpha}(\mathbb{B}_n)=B^\alpha(\mathbb{B}_n)$, we give the characterization of functions on $B^\alpha(\mathbb{B}_n)$ in terms of fractional derivatives with $0<\alpha<1$.

Theorem 5.4　Suppose that $0<\alpha<1$, $t+\alpha>1$. If γ is a real parameter such that neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, then a holomorphic function f in $\mathbb{B}_n$ belongs to $B^\alpha(\mathbb{B}_n)$ if and only if the function $(1-|z|^2)^{t+\alpha-1}R^{\gamma,t}f(z)$ is bounded in $\mathbb{B}_n$.

Lemma 5.5[1]　Suppose neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer. If $\beta=\gamma+N$ for some positive integer N, then there exists a one-variable polynomial h of degree N such that

$$R^{\gamma,t}\frac{1}{(1-\langle z,w\rangle)^{n+1+\beta}}=\frac{h(\langle z,w\rangle)}{(1-\langle z,w\rangle)^{n+1+\beta+t}}.$$

There also exists a polynomial $P(z,w)$ such that

$$R_{\gamma,t}\frac{1}{(1-\langle z,w\rangle)^{n+1+\beta+t}}=\frac{P(z,w)}{(1-\langle z,w\rangle)^{n+1+\beta}}.$$

Lemma 5.6[1]　Suppose neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer. Then the operator $R^{\gamma,t}$ is the unique continuous linear operator on $H(\mathbb{B}_n)$ satisfying

$$R^{\gamma,t}\frac{1}{(1-\langle z,w\rangle)^{n+1+\gamma+t}}=\frac{1}{(1-\langle z,w\rangle)^{n+1+\gamma}}$$

for all $w\in\mathbb{B}_n$. Similarly, the operator $R_{\gamma,t}$ is the unique continuous linear operator on $H(\mathbb{B}_n)$ satisfying

$$R_{\gamma,t}\frac{1}{(1-\langle z,w\rangle)^{n+1+\gamma+t}}=\frac{1}{(1-\langle z,w\rangle)^{n+1+\gamma}}$$

for all $w\in\mathbb{B}_n$.

Next we give the characterization of functions on $B^\alpha(B_n)$ in terms of fractional derivatives with $\alpha>1$.

Theorem 5.7　Suppose that $\alpha>1$ and $t>0$. If γ is a real parameter such that neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer. Then a holomorphic function f on $\mathbb{B}_n$ belongs to $B^\alpha(\mathbb{B}_n)$ if and only if $\sup_{z\in\mathbb{B}_n}(1-|z|^2)^{\alpha-1+t}R^{\gamma,t}f(z)$ is bounded on $\mathbb{B}_n$.

Proof　If $f\in B^\alpha(\mathbb{B}_n)$, then by Lemma 5.2 there exists a function $f\in B^\alpha(\mathbb{B}_n)$ such that

$$f(z)=\int_{\mathbb{B}_n}\frac{g(w)\mathrm{d}v_\beta(w)}{(1-\langle z,w\rangle)^{n+\alpha+\beta}},z\in\mathbb{B}_n,$$

here $\beta=\gamma-\alpha+N+1$ and N is a large enough positive integer such that $\beta>-1$. It follows from Lemma 5.5 that

$$R^{\gamma,t}f(z)=c_\alpha\int_{\mathbb{B}_n}\frac{h(\langle z,w\rangle)g(w)}{(1-\langle z,w\rangle)^{n+\alpha+\beta+t}}\mathrm{d}v_\beta(w),$$
$$z\in\mathbb{B}_n,$$

where h is a one-variable polynomial of degree $N-\alpha+1$. An application of Lemma 2.2 then shows the function $(1-|z|^2)^{\alpha-1+t}R^{\gamma,t}f(z)$ is bounded on $\mathbb{B}_n$.

Next, we will assume that the function $(1-|z|^2)^{\alpha-1+t}R^{\gamma,t}f(z)$ is bounded on $\mathbb{B}_n$. It follows from the remark of Ref. [1, Lemma 2.18.] that $R^{\gamma,t}f$ and $R^{\gamma+N,t}f$ are comparable for any holomorphic function f, hence the function $(c_\beta/c_{\beta+\alpha-1+t})(1-|z|^2)^{\alpha-1-t}R^{\gamma+N,t}f(z)$ is also bounded in B_n, where N is the same as the previous paragraph. By Lemma 2.1, we have

$$R^{\gamma+N,t}f(z)=c_\beta\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\beta+\alpha-1+t}R^{\gamma+N,t}f(w)}{(1-\langle z,w\rangle)^{n+1+\beta+\alpha-1+t}}dv(w)=\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1+t}R^{\gamma+N,t}f(w)}{(1-\langle z,w\rangle)^{n+1+\gamma+N+t}}dv_\beta(w),$$

where $\beta=\gamma-\alpha+N+1$ is also as in the previous paragraph. Apply the operator $R_{\gamma+N,t}$ inside the integral sign and use Lemma 5.6, we have

$$f(z)=\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1+t}R^{\gamma+N,t}f(w)}{(1-\langle z,w\rangle)^{n+1+r+N}}dv_\beta(w)=\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1+t}R^{\gamma+N,t}f(w)}{(1-\langle z,w\rangle)^{n+\beta+\alpha}}dv_\beta(w).$$

Since the function $(1-|w|^2)^{\alpha-1+t}R^{\gamma+N,t}f(w)$ belongs to $L^\infty(\mathbb{B}_n)$ by Lemma 5.1, we see that f is in $B^\alpha(\mathbb{B}_n)$ in view of Lemma 5.2.

Finally, we give the characterization of $B^\alpha(\mathbb{B}_n)$ in terms of its module with $\alpha>1$.

Theorem 5.8 Suppose that $\alpha>1$ and f is holomorphic in B_n. Then $f\in B^\alpha(\mathbb{B}_n)$ if and only if the function $(1-|z|^2)^{\alpha-1}|f(z)|$ is bounded in $\mathbb{B}_n$.

Proof If $f\in B^\alpha(\mathbb{B}_n)$, then by Lemma 5.2, there exists a function $g\in L^\infty(\mathbb{B}_n)$ such that

$$f(z)=\int_{\mathbb{B}_n}\frac{g(w)dv_\beta(w)}{(1-\langle z,w\rangle)^{n+\alpha+\beta}},\ z\in\mathbb{B}_n,$$

where $\beta>-1$. Thus, by Lemma 2.2, for every $z\in\mathbb{B}_n$, there exists a constant $C>0$ such that

$$|f(z)|=\left|\int_{\mathbb{B}_n}\frac{g(w)}{(1-\langle z,w\rangle)^{n+\alpha+\beta}}dv_\beta(w)\right|=C_\beta\left|\int_{\mathbb{B}_n}\frac{(1-|z|^2)^\beta g(w)}{(1-\langle z,w\rangle)^{n+\alpha+\beta}}dv(w)\right|\leqslant C_\beta\|g\|_\infty\int_{\mathbb{B}_n}\frac{(1-|w|^2)^\beta}{(1-\langle z,w\rangle)^{n+\alpha+\beta}}dv(w)\leqslant C(1-|z|^2)^{-(\alpha-1)}.$$

Thus $(1-|z|^2)^{\alpha-1}|f(z)|$ is bounded in $\mathbb{B}_n$.

Conversely, if $(1-|z|^2)^{\alpha-1}|f(z)|\leqslant M$ for some constant $M>0$, then by Lemma 2.1 we have

$$f(z)=c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1}f(w)}{(1-\langle z,w\rangle)^{n+\alpha}}dv(w),\quad z\in\mathbb{B}_n.$$

Thus

$$Rf(z)=\sum_{k=1}^n z_k\frac{\partial f}{\partial z_k}(z)=c_{\alpha-1}\sum_{k=1}^n z_k\frac{\partial}{\partial z_k}\left(\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1}f(w)}{(1-\langle z,w\rangle)^{n+\alpha}}dv(w)\right)=c_{\alpha-1}(n+\alpha)\sum_{k=1}^n z_k\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1}f(w)\overline{w_k}}{(1-\langle z,w\rangle)^{n+\alpha+1}}dv(w)=c_{\alpha-1}(n+\alpha)\sum_{k=1}^n z_k\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1}f(w)\langle z,w\rangle}{(1-\langle z,w\rangle)^{n+\alpha+1}}dv(w).$$

By Lemma 2.2, there exists a constant $C>0$ such that

$$|Rf(z)|\leqslant c_{\alpha-1}(n+\alpha)\cdot\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1}|f(w)|}{|1-\langle z,w\rangle|^{n+\alpha+1}}dv(w)\leqslant CM(1-|z|^2)^{-\alpha}$$

for all $z\in\mathbb{B}_n$. This shows that $f\in B^\alpha(\mathbb{B}_n)$. The proof is end.

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