关于高斯最小值猜测的一个注记

钟扬帆，马　婷，胡泽春
(四川大学数学学院，成都 610064)

摘　要：高斯分布又叫正态分布，在数学、统计学、物理及工程等领域有重要应用.与高斯分布相关的不等式一直吸引着众多学者的关注，其中一个著名的例子是“高斯最小值猜想”：如果 $n\geqslant 2$，$(X_i, 1\leqslant i\leqslant n)$ 为中心化高斯随机向量，则不等式 $E(\min_{1\leqslant i\leqslant n}|X_i|)\geqslant E(\min_{1\leqslant i\leqslant n}|Y_i|)$ 成立，其中 $Y_1,\cdots,Y_n$ 为相互独立的中心化高斯随机向量，满足 $E(X_i^2)=E(Y_i^2)$，$i=1,\cdots,n$. 本文证明该猜测成立当且仅当 $n=2$.

关键词：高斯最小值猜测；高斯随机向量
中图分类号：O211.3　　**文献标识码**：A　　DOI：10.19907/j.0490-6756.2021.061003

A note on the Gaussian minimum conjecture

ZHONG Yang-Fan, *MA Ting*, *HU Ze-Chun*
(School of Mathematics, Sichuan University, Chengdu 610064, China)

Abstract: Gaussian distribution, also called normal distribution, plays an important role in mathematics, statistics, physics, engineering, etc. Inequalities on Gaussian distribution attract many attentions, in which a famous example is the Gaussian minimum conjecture, which says that if $n\geqslant 2$, and $(X_i, 1\leqslant i\leqslant n)$ is a centered Gaussian random vector, then the inequality $E(\min_{1\leqslant i\leqslant n}|X_i|)\geqslant E(\min_{1\leqslant i\leqslant n}|Y_i|)$ holds, where $Y_1,\cdots,Y_n$ are independent centered Gaussian random variables with $E(X_i^2)=E(Y_i^2)$ for any $i=1,\cdots,n$. In this note, we show that this conjecture holds if and only if $n=2$.

Keywords: Gaussian minimum conjecture; Gaussian random vector

1 Introduction

Let $n\geqslant 2$ and $(X_i, 1\leqslant i\leqslant n)$ be a centered Gaussian random vector. The well-known Šidák's Inequality[1-2] says that

$$E(\max_{1\leqslant i\leqslant n}|X_i|)\leqslant E(\max_{1\leqslant i\leqslant n}|Y_i|) \quad (1)$$

where $Y_1,\cdots,Y_n$ are independent centered Gaussian random variables with $E(X_i^2)=E(Y_i^2)$ for any $i=1,\cdots,n$. Hereafter, $E(\cdot)$ denotes the expectation.

If we replace "max" by "min" in Šidák's inequality, Gordon *et al*[3-4]. proved among other things that

$$E(\min_{1\leqslant i\leqslant n}|X_i|)\geqslant\frac{1}{2}E(\min_{1\leqslant i\leqslant n}|Y_i|) \quad (2)$$

Note that in Refs. [3] and [4], the authors proved the inequality (2) without any condition on the joint distribution of $(X_1,\cdots,X_n)$. Furthermore, Li and Shao[5-6] conjectured that when $(X_i, 1\leqslant i\leqslant n)$ is a centered Gaussian random vector, $\frac{1}{2}$ in the inequality (2) can be removed, *i.e.*,

$$E(\min_{1\leqslant i\leqslant n}|X_i|)\geqslant E(\min_{1\leqslant i\leqslant n}|Y_i|) \quad (3)$$

收稿日期：2021-02-23
基金项目：国家自然科学基金(11771309)
作者简介：钟扬帆(1995—)，女，四川内江人，硕士研究生，主要研究方向为概率不等式. E-mail：1817663293@qq.com
通讯作者：胡泽春. E-mail：zchu@scu.edu.cn

which is called the Gaussian minimum conjecture. As to the minimum value point of $E(\min_{1\leqslant i\leqslant n}|X_i|)$ with $E(X_i^2)=1, \forall i=1,\cdots,n$, for $n\geqslant 3$, we refer to Ref. [7, Conjecture 5.1], which says that if $n\geqslant 3$ and $p>0$, then among all Gaussian random vector $(X_1,\cdots,X_n)$ with $X_i\sim N(0,1)$ for all $i\leqslant n$, the expectation $E(\min_{i\leqslant n}|X_i|^p)$ is minimal if and only if all the off-diagonal elements of the covariance matrix equal $-1/(n-1)$.

Now we state the main result of this paper.

Theorem 1.1 The Gaussian minimum conjecture holds if and only if $n=2$.

The rest of this paper is organized as follows. In Section 2 and Section 3, we give the necessity proof and the sufficiency proof of Theorem 1.1, respectively. In the final section, we give some remarks.

2 Necessity proof of Theorem 1.1

In this part, we will show that if $n\geqslant 3$, the inequality (3) does not hold for some n-dimensional centered Gaussian random vector $(X_1,\cdots,X_n)$ and independent centered Gaussian random variables $Y_1,\cdots,Y_n$ with $E(X_i^2)=E(Y_i^2), \forall i=1,\cdots,n$.

(i) $n=3$. Let Y_1,Y_2,Y_3 be three independent standard Gaussian random variables. Define $X_1=(Y_1-Y_2)/\sqrt{2}, X_2=(Y_2-Y_3)/\sqrt{2}, X_3=(Y_3-Y_1)/\sqrt{2}$. Ref. [8] mentioned that van Handel checked numerically that $0.17\sim E(\min_{1\leqslant i\leqslant 3}|X_i|^2)<E(\min_{1\leqslant i\leqslant 3}|Y_i|^2)\sim 0.19$. This example also shows that (3) does not hold for $n=3$, which means that $E(\min_{1\leqslant i\leqslant 3}|X_i|)<E(\min_{1\leqslant i\leqslant 3}|Y_i|)$. In the following, we will give the exact values of $E(\min_{1\leqslant i\leqslant 3}|X_i|)$ and $E(\min_{1\leqslant i\leqslant 3}|Y_i|)$.

At first, we calculate $E(\min_{1\leqslant i\leqslant 3}|Y_i|)$. The density function $p_1(x,y,z)$ of (Y_1,Y_2,Y_3) can be expressed by

$$p_1(x,y,z)=\frac{1}{(2\pi)^{3/2}}e^{-\frac{x^2+y^2+z^2}{2}}.$$

By the symmetry, we have

$$E(\min_{1\leqslant i\leqslant 3}|Y_i|)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|x|\wedge|y|\wedge|z|)\frac{1}{(2\pi)^{3/2}}e^{-\frac{x^2+y^2+z^2}{2}}dxdydz=$$
$$\frac{8}{(2\pi)^{3/2}}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}(x\wedge y\wedge z)e^{-\frac{x^2+y^2+z^2}{2}}dxdydz=$$
$$\frac{8\cdot 3!}{(2\pi)^{3/2}}\int_0^{\infty}\left(\int_0^z e^{-\frac{y^2+z^2}{2}}\left(\int_0^y xe^{-\frac{x^2}{2}}dx\right)dy\right)dz=$$
$$\frac{48}{(2\pi)^{3/2}}\int_0^{\infty}\left(\int_0^z e^{-\frac{y^2+z^2}{2}}(1-e^{-\frac{y^2}{2}})dy\right)dz=$$
$$\frac{12\sqrt{2}}{\pi^{3/2}}\left[\int_0^{\infty}\left(\int_0^z e^{-\frac{y^2+z^2}{2}}dy\right)dz-\int_0^{\infty}\left(\int_0^z e^{-\frac{2y^2+z^2}{2}}dy\right)dz\right] \quad (4)$$

Define a function

$$F(a):=\int_0^{\infty}\left(\int_0^z e^{-\frac{ay^2+z^2}{2}}dy\right)dz,\ a>0$$

and a set

$$D:=\{(y,z)\in\mathbf{R}^2: 0\leqslant y\leqslant z\}.$$

Define a transformation

$$(y,z):=T(u,v)=(u/\sqrt{a},v).$$

Denote by D_T the original image of D under T. Then we have

$$D_T=\{(u,v)\in\mathbf{R}^2: T(u,v)\in D\}=\{(u,v)\in\mathbf{R}^2: 0\leqslant u/\sqrt{a}\leqslant v\}.$$

Now, we have

$$F(a)=\int_D e^{-\frac{ay^2+z^2}{2}}dydz=$$
$$\frac{1}{\sqrt{a}}\int_{D_T}e^{-\frac{u^2+v^2}{2}}dudv=$$
$$\frac{1}{\sqrt{a}}\cdot\left(\frac{\pi}{2}-\arctan\frac{1}{\sqrt{a}}\right)\cdot\int_0^{\infty}e^{-\frac{r^2}{2}}rdr=$$
$$\frac{1}{\sqrt{a}}\left(\frac{\pi}{2}-\arctan\frac{1}{\sqrt{a}}\right)=$$
$$\frac{1}{\sqrt{a}}\arctan\sqrt{a} \quad (5)$$

By (4) and (5), we get

$$E(\min_{1\leqslant i\leqslant 3}|Y_i|)=\frac{12\sqrt{2}}{\pi^{3/2}}(F(1)-F(2))=$$
$$\frac{12\sqrt{2}}{\pi^{3/2}}\left(\arctan 1-\frac{1}{\sqrt{2}}\arctan\sqrt{2}\right)=$$
$$\frac{12}{\pi^{3/2}}\left(\frac{\sqrt{2}\pi}{4}-\arctan\sqrt{2}\right) \quad (6)$$

Next, we calculate $E(\min_{1\leqslant i\leqslant 3}|X_i|)$. Note that $X_3=-(X_1+X_2)$, and the covariance matrix of (X_1,X_2) is $\Sigma=\begin{pmatrix}1 & -\frac{1}{2}\\ -\frac{1}{2} & 1\end{pmatrix}$. It follows that

$$\det\Sigma=\frac{3}{4},\ \Sigma^{-1}=\begin{pmatrix}4/3 & 2/3\\ 2/3 & 4/3\end{pmatrix}.$$

Then the density function $p_2(x,y)$ of (X_1,X_2) can be expressed by

$$p_2(x,y)=\frac{1}{2\pi\sqrt{3/4}}e^{-\frac{\frac{4}{3}(x^2+y^2+xy)}{2}}=\frac{1}{\sqrt{3}\pi}e^{-\frac{2(x^2+y^2+xy)}{3}}.$$

By the symmetry, we have

$$E(\min_{1\leqslant i\leqslant 3}|X_i|)=\frac{1}{\sqrt{3}\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|x|\wedge|y|\wedge|x+y|)e^{-\frac{2(x^2+y^2+xy)}{3}}dxdy=$$
$$\frac{1}{\sqrt{3}\pi}\Big[\int_0^{\infty}\int_0^{\infty}(x\wedge y)e^{-\frac{2(x^2+y^2+xy)}{3}}dxdy+\int_{-\infty}^{0}\int_{-\infty}^{0}((-x)\wedge(-y))e^{-\frac{2(x^2+y^2+xy)}{3}}dxdy+$$
$$\int_{-\infty}^{0}\int_0^{\infty}((-x)\wedge y\wedge|x+y|)e^{-\frac{2(x^2+y^2+xy)}{3}}dxdy+\int_0^{\infty}\int_{-\infty}^{0}(x\wedge(-y)\wedge|x+y|)e^{-\frac{2(x^2+y^2+xy)}{3}}dxdy\Big]=$$
$$\frac{2}{\sqrt{3}\pi}\Big[\int_0^{\infty}\int_0^{\infty}(x\wedge y)e^{-\frac{2(x^2+y^2+xy)}{3}}dxdy+\int_0^{\infty}\int_0^{\infty}(x\wedge y\wedge|x-y|)e^{-\frac{2(x^2+y^2-xy)}{3}}dxdy\Big]=$$
$$\frac{4}{\sqrt{3}\pi}\Big[\int_0^{\infty}\Big(\int_0^{y}xe^{-\frac{2(x^2+y^2+xy)}{3}}dx\Big)dy+\int_0^{\infty}\Big(\int_0^{y}(x\wedge(y-x))e^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy\Big]=$$
$$\frac{4}{\sqrt{3}\pi}\Big[\int_0^{\infty}\Big(\int_0^{y}xe^{-\frac{2(x^2+y^2+xy)}{3}}dx\Big)dy+\int_0^{\infty}\Big(\int_0^{y/2}xe^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy+\int_0^{\infty}\Big(\int_{y/2}^{y}(y-x)e^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy\Big]=$$
$$\frac{4}{\sqrt{3}\pi}\Big[\int_0^{\infty}\Big(\int_0^{y}xe^{-\frac{2(x^2+y^2+xy)}{3}}dx\Big)dy+\int_0^{\infty}\Big(\int_0^{y/2}xe^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy+$$
$$\int_0^{\infty}\Big(\int_{y/2}^{y}ye^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy-\int_0^{\infty}\Big(\int_{y/2}^{y}xe^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy\Big]:=\frac{4}{\sqrt{3}\pi}[G_1+G_2+G_3-G_4]\qquad(7)$$

where

$$G_1:=\int_0^{\infty}\Big(\int_0^{y}xe^{-\frac{2(x^2+y^2+xy)}{3}}dx\Big)dy,$$
$$G_2:=\int_0^{\infty}\Big(\int_0^{y/2}xe^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy,$$
$$G_3:=\int_0^{\infty}\Big(\int_{y/2}^{y}ye^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy,$$
$$G_4:=\int_0^{\infty}\Big(\int_{y/2}^{y}xe^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy.$$

We have

$$G_1=\int_0^{\infty}\Big(\int_0^{y}\Big(x+\frac{y}{2}\Big)e^{-\frac{2(x^2+y^2+xy)}{3}}dx\Big)dy-$$
$$\frac{1}{2}\int_0^{\infty}\Big(\int_0^{y}\Big(y+\frac{x}{2}\Big)e^{-\frac{2(x^2+y^2+xy)}{3}}dx\Big)dy+$$
$$\frac{1}{4}\int_0^{\infty}\Big(\int_0^{y}xe^{\frac{2(x^2+y^2+xy)}{3}}dx\Big)dy=$$
$$\frac{G_1}{4}+\int_0^{\infty}\Big(\int_0^{y}\Big(x+\frac{y}{2}\Big)e^{-\frac{2\left(x+\frac{y}{2}\right)^2+\frac{3}{2}y^2}{3}}dx\Big)dy-$$
$$\frac{1}{2}\int_0^{\infty}\Big(\int_x^{\infty}\Big(y+\frac{x}{2}\Big)e^{-\frac{2\left(y+\frac{x}{2}\right)^2+\frac{3}{2}x^2}{3}}dy\Big)dx=$$
$$\frac{G_1}{4}+\int_0^{\infty}e^{-\frac{y^2}{2}}\Big(\int_{y/2}^{3y/2}ue^{-\frac{2u^2}{3}}du\Big)dy-$$
$$\frac{1}{2}\int_0^{\infty}e^{-\frac{x^2}{2}}\Big(\int_{3x/2}^{\infty}ve^{-\frac{2v^2}{3}}dv\Big)dx=$$
$$\frac{G_1}{4}+\frac{3}{4}\Big[\int_0^{\infty}e^{-\frac{y^2}{2}}(e^{-\frac{y^2}{6}}-e^{-\frac{3y^2}{2}})dy-$$
$$\frac{1}{2}\int_0^{\infty}e^{-\frac{x^2}{2}}\cdot e^{-\frac{3x^2}{2}}dx\Big]=$$
$$\frac{G_1}{4}+\frac{3}{8}\Big(2\int_0^{\infty}e^{-\frac{2y^2}{3}}dy-3\int_0^{\infty}e^{-2y^2}dy\Big).$$

It follows that

$$G_1=\frac{1}{2}\Big(2\int_0^{\infty}e^{-\frac{2y^2}{3}}dy-3\int_0^{\infty}e^{-2y^2}dy\Big)\qquad(8)$$

We have

$$G_2=\int_0^{\infty}\Big(\int_0^{y/2}\Big(x-\frac{y}{2}\Big)e^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy+$$
$$\frac{1}{2}\int_0^{\infty}\Big(\int_0^{y/2}\Big(y-\frac{x}{2}\Big)e^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy+$$
$$\frac{1}{4}\int_0^{\infty}\Big(\int_0^{y/2}xe^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy=$$
$$\frac{G_2}{4}+\int_0^{\infty}\Big(\int_0^{y/2}\Big(x-\frac{y}{2}\Big)e^{-\frac{2\left(x-\frac{y}{2}\right)^2+\frac{3y^2}{2}}{3}}dx\Big)dy+$$
$$\frac{1}{2}\int_0^{\infty}\Big(\int_{2x}^{\infty}\Big(y-\frac{x}{2}\Big)e^{-\frac{2\left(y-\frac{x}{2}\right)^2+\frac{3x^2}{2}}{3}}dy\Big)dx=$$
$$\frac{G_2}{4}+\int_0^{\infty}e^{-\frac{y^2}{2}}\Big(\int_{-y/2}^{0}ue^{-\frac{2u^2}{3}}du\Big)dy+$$
$$\frac{1}{2}\int_0^{\infty}e^{-\frac{x^2}{2}}\Big(\int_{3x/2}^{\infty}ve^{-\frac{2v^2}{3}}dv\Big)dx=$$
$$\frac{G_2}{4}+\frac{3}{4}\Big[\int_0^{\infty}e^{-\frac{y^2}{2}}(e^{-\frac{y^2}{6}}-1)dy+$$

$$\frac{1}{2}\int_0^\infty e^{-\frac{x^2}{2}}\cdot e^{-\frac{3x^2}{2}}dx\Big]=\frac{G_2}{4}+\frac{3}{8}\Big(2\int_0^\infty e^{-\frac{2y^2}{3}}dy-2\int_0^\infty e^{-\frac{y^2}{2}}dy+\int_0^\infty e^{-2x^2}dx\Big).$$

It follows that

$$G_2=\frac{1}{2}\Big(2\int_0^\infty e^{-\frac{2y^2}{3}}dy-2\int_0^\infty e^{-\frac{y^2}{2}}dy+\Big(\int_0^\infty e^{-2x^2}dx\Big) \tag{9}$$

We have

$$G_3=\int_0^\infty\Big(\int_{y/2}^y\Big(y-\frac{x}{2}\Big)e^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy+\frac{1}{2}\int_0^\infty\Big(\int_{y/2}^y\Big(x-\frac{y}{2}\Big)e^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy+\frac{1}{4}\int_0^\infty\Big(\int_{y/2}^y ye^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy=$$

$$\frac{G_3}{4}+\int_0^\infty\Big(\int_x^{2x}\Big(y-\frac{x}{2}\Big)e^{-\frac{2\left(y-\frac{x}{2}\right)^2+\frac{3x^2}{2}}{3}}dy\Big)dx+\frac{1}{2}\int_0^\infty\Big(\int_{y/2}^y\Big(x-\frac{y}{2}\Big)e^{-\frac{2\left(x-\frac{y}{2}\right)^2+\frac{3y^2}{2}}{3}}dx\Big)dy=$$

$$\frac{G_3}{4}+\int_0^\infty e^{-\frac{x^2}{2}}\Big(\int_{x/2}^{3x/2}ue^{-\frac{2u^2}{3}}du\Big)dx+\frac{1}{2}\int_0^\infty e^{-\frac{y^2}{2}}\Big(\int_0^{y/2}ve^{-\frac{2v^2}{3}}dv\Big)dy=$$

$$\frac{G_3}{4}+\frac{3}{4}\Big[\int_0^\infty e^{-\frac{x^2}{2}}(e^{-\frac{x^2}{6}}-e^{-\frac{3x^2}{2}})dx+\frac{1}{2}\int_0^\infty e^{-\frac{y^2}{2}}(1-e^{-\frac{y^2}{6}})dy\Big]=\frac{G_3}{4}+\frac{3}{8}\Big(\int_0^\infty e^{-\frac{y^2}{2}}dy+\int_0^\infty e^{-\frac{2x^2}{3}}dx-2\int_0^\infty e^{-2x^2}dx\Big).$$

It follows that

$$G_3=\frac{1}{2}\Big(\int_0^\infty e^{-\frac{y^2}{2}}dy+\int_0^\infty e^{-\frac{2x^2}{3}}dx-2\int_0^\infty e^{-2x^2}dx\Big) \tag{10}$$

We have

$$G_4=\int_0^\infty\Big(\int_{y/2}^y\Big(x-\frac{y}{2}\Big)e^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy+\frac{1}{2}\int_0^\infty\Big(\int_{y/2}^y\Big(y-\frac{x}{2}\Big)e^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy+\frac{1}{4}\int_0^\infty\Big(\int_{y/2}^y xe^{-\frac{2(x^2+y^2-xy)}{3}}dx\Big)dy=$$

$$\frac{G_4}{4}+\int_0^\infty\Big(\int_{y/2}^y\Big(x-\frac{y}{2}\Big)e^{-\frac{2\left(x-\frac{y}{2}\right)^2+\frac{3y^2}{2}}{3}}dx\Big)dy+\frac{1}{2}\int_0^\infty\Big(\int_x^{2x}\Big(y-\frac{x}{2}\Big)e^{-\frac{2\left(y-\frac{x}{2}\right)^2+\frac{3x^2}{2}}{3}}dy\Big)dx=$$

$$\frac{G_4}{4}+\int_0^\infty e^{-\frac{y^2}{2}}\Big(\int_0^{y/2}ue^{-\frac{2u^2}{3}}du\Big)dy+\frac{1}{2}\int_0^\infty e^{-\frac{x^2}{2}}\Big(\int_{x/2}^{3x/2}ve^{-\frac{2v^2}{3}}dv\Big)dx=$$

$$\frac{G_4}{4}+\frac{3}{4}\Big[\int_0^\infty e^{-\frac{y^2}{2}}(1-e^{-\frac{y^2}{6}})dy+\frac{1}{2}\int_0^\infty e^{-\frac{x^2}{2}}(e^{-\frac{x^2}{6}}-e^{-\frac{3x^2}{2}})dx\Big]=\frac{G_4}{4}+\frac{3}{8}\Big(2\int_0^\infty e^{-\frac{y^2}{2}}dy-\int_0^\infty e^{-\frac{2y^2}{3}}dy-\int_0^\infty e^{-2y^2}dy\Big).$$

It follows that

$$G_4=\frac{1}{2}\Big(2\int_0^\infty e^{-\frac{y^2}{2}}dy-\int_0^\infty e^{-\frac{2y^2}{3}}dy-\int_0^\infty e^{-2y^2}dy\Big) \tag{11}$$

By (7)～(11), we obtain

$$E(\min_{1\leqslant i\leqslant 3}|X_i|)=\frac{6}{\sqrt{3}\pi}\Big(2\int_0^\infty e^{-\frac{2y^2}{3}}dy-\int_0^\infty e^{-2y^2}dy-\int_0^\infty e^{-\frac{y^2}{2}}dy\Big)=\frac{6}{\sqrt{3}\pi}\cdot\frac{\sqrt{2\pi}}{2}\Big(2\sqrt{\frac{3}{4}}-\sqrt{\frac{1}{4}}-1\Big)=\frac{3(2-\sqrt{3})}{\sqrt{2\pi}} \tag{12}$$

Hence we get

$$E(\min_{1\leqslant i\leqslant 3}|Y_i|)-E(\min_{1\leqslant i\leqslant 3}|X_i|)=\frac{12}{\pi^{3/2}}\Big(\frac{\sqrt{2\pi}}{4}-\arctan\sqrt{2}\Big)-\frac{3(2-\sqrt{3})}{\sqrt{2\pi}}=\frac{12}{\pi^{3/2}}\Big(\frac{\sqrt{6}\pi}{8}-\arctan\sqrt{2}\Big)>0,$$

since $\tan(\arctan\sqrt{2})\sim 1.414$, $\tan\frac{\sqrt{6}\pi}{8}\sim 1.434$.

(ii) $n\geqslant 4$. Without loss of generality, we only consider the case that $n=4$. We use proof by contradiction. Suppose that (3) holds for $n=4$. Let $Y_i, X_i, i=1,2,3$ be the same as in the above example. Let Y_4 be a standard Gaussian random variable independent of (Y_1, Y_2, Y_3). Then, by the assumption, for any $a>0$, we have

$$E((\min_{1\leqslant i\leqslant 3}|X_i|)\wedge|aY_4|)\geqslant E((\min_{1\leqslant i\leqslant 3}|Y_i|)\wedge|aY_4|).$$

Letting $a\to\infty$, by the monotone convergence theorem, we obtain that

$$E(\min_{1\leqslant i\leqslant 3}|X_i|)\geqslant E(\min_{1\leqslant i\leqslant 3}|Y_i|).$$

It is a contradiction. Hence for any $M>0$, there exists $a_0>M$ such that

$$E((\min_{1\leqslant i\leqslant 3}|X_i|)\wedge|a_0Y_4|)< E((\min_{1\leqslant i\leqslant 3}|Y_i|)\wedge|a_0Y_4|).$$

3 Sufficiency proof of Theorem 1.1

In this part, we will show that the inequality

(3) holds if $n=2$. Write $X_1=x_1f_1, X_2=x_2f_2$, where both f_1 and f_2 have the standard normal distribution $N(0,1)$. Without loss of generality, we can assume that $x_1, x_2>0$. Further we can assume that $x_1=1, x_2=a\in(0,1]$.

Denote by Σ_1 the covariance matrix of (f_1, f_2). By the symmetry of the distribution $N(0,1)$, we can assume that $\Sigma_1=\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$, where $0\leqslant\rho\leqslant 1$. Then the covariance matrix Σ_2 of (f_1, af_2) can be expressed by $\Sigma_2=\begin{pmatrix}1 & a\rho\\ a\rho & a^2\end{pmatrix}$. It follows that if $\rho\in[0,1)$, then

$$\Sigma_2^{-1}=\frac{1}{a^2(1-\rho^2)}\begin{pmatrix}a^2 & -a\rho\\ -a\rho & 1\end{pmatrix}$$

and thus the density function of (f_1, af_2) is

$$p(x,y)=\frac{1}{2\pi\sqrt{a^2(1-\rho^2)}}\exp\left(-\frac{a^2x^2+y^2-2a\rho xy}{2a^2(1-\rho^2)}\right).$$

At first, we assume that $\rho\in[0,1)$. By the symmetry, we have

$$E(|f_1|\wedge|af_2|)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|x|\wedge|y|)\frac{1}{2\pi\sqrt{a^2(1-\rho^2)}}\cdot e^{-\frac{a^2x^2+y^2-2a\rho xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y=\frac{1}{2\pi\sqrt{a^2(1-\rho^2)}}\cdot\Big[\int_0^{\infty}\int_0^{\infty}(x\wedge y)e^{-\frac{a^2x^2+y^2-2a\rho xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y+\int_{-\infty}^{0}\int_{-\infty}^{0}((-x)\wedge(-y))e^{-\frac{a^2x^2+y^2-2a\rho xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y+\int_0^{\infty}\int_{-\infty}^{0}(x\wedge(-y))e^{-\frac{a^2x^2+y^2-2a\rho xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y+\int_{-\infty}^{0}\int_0^{\infty}((-x)\wedge y)e^{-\frac{a^2x^2+y^2-2a\rho xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y\Big]=\frac{1}{\pi\sqrt{a^2(1-\rho^2)}}\Big[\int_0^{\infty}\int_0^{\infty}(x\wedge y)e^{-\frac{a^2x^2+y^2-2a\rho xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y+\int_0^{\infty}\int_0^{\infty}(x\wedge y)e^{-\frac{a^2x^2+y^2+2a\rho xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y\Big].$$

Define

$$I(\vartheta):=\int_0^{\infty}\int_0^{\infty}(x\wedge y)e^{-\frac{a^2x^2+y^2+2a\vartheta xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y.$$

Then

$$E(|f_1|\wedge|af_2|)=\frac{1}{\pi\sqrt{a^2(1-\rho^2)}}[I(\rho)+I(-\rho)].$$

We have

$$I(\vartheta)=\int_0^{\infty}\int_0^{\infty}(x\wedge y)e^{-\frac{a^2x^2+y^2+2a\vartheta xy}{2a^2(1-\rho^2)}}\mathrm{d}x\mathrm{d}y=\int_0^{\infty}\left(\int_0^{y}xe^{-\frac{a^2x^2+y^2+2a\vartheta xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y+\int_0^{\infty}\left(\int_0^{x}ye^{-\frac{a^2x^2+y^2+2a\vartheta xy}{2a^2(1-\rho^2)}}\mathrm{d}y\right)\mathrm{d}x.$$

Define

$$J(\alpha,\beta,\gamma):=\int_0^{\infty}\left(\int_0^{y}xe^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y,$$

where $\alpha>0, \beta>0, \alpha\beta-\gamma^2>0$. Then we have

$$I(\vartheta)=J(a^2,1,a\vartheta)+J(1,a^2,a\vartheta),$$

thus

$$E(|f_1|\wedge|af_2|)=\frac{1}{\pi\sqrt{a^2(1-\rho^2)}}\cdot[J(a^2,1,a\rho)+J(1,a^2,a\rho)+J(a^2,1,-a\rho)+J(1,a^2,-a\rho)] \tag{13}$$

In the following, we come to calculate the function $J(\alpha,\beta,\gamma)$. We have

$$J(\alpha,\beta,\gamma)=\int_0^{\infty}\left(\int_0^{y}xe^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y=\int_0^{\infty}\left(\int_0^{y}\left(x+\frac{\gamma}{\alpha}y\right)e^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y-\frac{\gamma}{\alpha}\int_0^{\infty}\left(\int_0^{y}ye^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y:=J_1(\alpha,\beta,\gamma)-J_2(\alpha,\beta,\gamma) \tag{14}$$

where

$$J_1(\alpha,\beta,\gamma):=\int_0^{\infty}\left(\int_0^{y}\left(x+\frac{\gamma}{\alpha}y\right)e^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y,$$

$$J_2(\alpha,\beta,\gamma):=\frac{\gamma}{\alpha}\int_0^{\infty}\left(\int_0^{y}ye^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y.$$

We have

$$J_1(\alpha,\beta,\gamma)=\int_0^{\infty}\left(\int_0^{y}\left(x+\frac{\gamma}{\alpha}y\right)e^{-\frac{\alpha\left(x+\frac{\gamma}{\alpha}y\right)^2+\frac{\alpha\beta-\gamma^2}{\alpha}y^2}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y=\int_0^{\infty}e^{-\frac{(\alpha\beta-\gamma^2)y^2}{2a^2(1-\rho^2)\alpha}}\left(\int_{\frac{\gamma}{\alpha}y}^{\left(1+\frac{\gamma}{\alpha}\right)y}ue^{-\frac{\alpha u^2}{2a^2(1-\rho^2)}}\mathrm{d}u\right)\mathrm{d}y=\frac{a^2(1-\rho^2)}{\alpha}\int_0^{\infty}e^{-\frac{(\alpha\beta-\gamma^2)y^2}{2a^2(1-\rho^2)\alpha}}\cdot\left(e^{-\frac{\gamma^2y^2}{2a^2(1-\rho^2)\alpha}}-e^{-\frac{(\alpha+\gamma)^2y^2}{2a^2(1-\rho^2)\alpha}}\right)\mathrm{d}y=\frac{a^2(1-\rho^2)}{\alpha}\int_0^{\infty}\left(e^{-\frac{\beta y^2}{2a^2(1-\rho^2)}}-e^{-\frac{(\alpha+\beta+2\gamma)y^2}{2a^2(1-\rho^2)}}\right)\mathrm{d}y=\frac{a^2(1-\rho^2)}{\alpha}\cdot\frac{\sqrt{2\pi}}{2}\left(\frac{a\sqrt{1-\rho^2}}{\sqrt{\beta}}-\frac{a\sqrt{1-\rho^2}}{\sqrt{\alpha+\beta+2\gamma}}\right)=\frac{\sqrt{2\pi}a^3(1-\rho^2)^{\frac{3}{2}}}{2\alpha}\left(\frac{1}{\sqrt{\beta}}-\frac{1}{\sqrt{\alpha+\beta+2\gamma}}\right) \tag{15}$$

$$J_2(\alpha,\beta,\gamma)=$$

$$\frac{\gamma}{\alpha}\int_0^\infty\left(\int_0^y\left(y+\frac{\gamma}{\beta}x\right)\mathrm{e}^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y-$$
$$\frac{\gamma}{\alpha}\int_0^\infty\left(\int_0^y\left(\frac{\gamma}{\beta}x\right)\mathrm{e}^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y=$$
$$-\frac{\gamma^2}{\alpha\beta}\int_0^\infty\left(\int_0^y x\mathrm{e}^{-\frac{\alpha x^2+\beta y^2+2\gamma xy}{2a^2(1-\rho^2)}}\mathrm{d}x\right)\mathrm{d}y+$$
$$\frac{\gamma}{\alpha}\int_0^\infty\left(\int_x^\infty\left(y+\frac{\gamma}{\beta}x\right)\mathrm{e}^{-\frac{\beta\left(y+\frac{\gamma}{\beta}x\right)^2+\frac{\alpha\beta-\gamma^2}{\beta}x^2}{2a^2(1-\rho^2)}}\mathrm{d}y\right)\mathrm{d}x=$$
$$-\frac{\gamma^2}{\alpha\beta}J(\alpha,\beta,\gamma)+\frac{\gamma}{\alpha}\int_0^\infty\mathrm{e}^{-\frac{(\alpha\beta-\gamma^2)x^2}{2a^2(1-\rho^2)\beta}}$$
$$\left(\int_{\frac{(\beta+\gamma)x}{\beta}}^\infty u\,\mathrm{e}^{-\frac{\beta u^2}{2a^2(1-\rho^2)}}\mathrm{d}u\right)\mathrm{d}x=$$
$$-\frac{\gamma^2}{\alpha\beta}J(\alpha,\beta,\gamma)+\frac{\gamma}{\alpha}\int_0^\infty e^{-\frac{(\alpha\beta-\gamma^2)x^2}{2a^2(1-\rho^2)\beta}}\cdot$$
$$\frac{a^2(1-\rho^2)}{\beta}\mathrm{e}^{-\frac{(\beta+\gamma)^2x^2}{2a^2(1-\rho^2)\beta}}\mathrm{d}x=$$
$$-\frac{\gamma^2}{\alpha\beta}J(\alpha,\beta,\gamma)+\frac{a^2(1-\rho^2)\gamma}{\alpha\beta}\int_0^\infty\mathrm{e}^{-\frac{(\alpha+\beta+2\gamma)x^2}{2a^2(1-\rho^2)}}\mathrm{d}x=$$
$$-\frac{\gamma^2}{\alpha\beta}J(\alpha,\beta,\gamma)+\frac{\sqrt{2\pi}a^3(1-\rho^2)^{\frac{3}{2}}\gamma}{2\alpha\beta\sqrt{\alpha+\beta+2\gamma}}\quad(16)$$

By (14)～(16), we get

$$J(\alpha,\beta,\gamma)=\frac{1}{1-\frac{\gamma^2}{\alpha\beta}}\left[\frac{\sqrt{2\pi}a^3(1-\rho^2)^{\frac{3}{2}}}{2\alpha}\cdot\left(\frac{1}{\sqrt{\beta}}-\frac{1}{\sqrt{\alpha+\beta+2\gamma}}\right)-\frac{\sqrt{2\pi}a^3(1-\rho^2)^{\frac{3}{2}}\gamma}{2\alpha\beta\sqrt{\alpha+\beta+2\gamma}}\right]=$$
$$\frac{\sqrt{2\pi}a^3(1-\rho^2)^{\frac{3}{2}}}{2}\cdot\frac{\beta}{\alpha\beta-\gamma^2}\left(\frac{1}{\sqrt{\beta}}-\frac{\beta+\gamma}{\beta\sqrt{\alpha+\beta+2\gamma}}\right),$$

which together with (13) implies that

$$E(|f_1|\wedge|af_2|)=\frac{1}{\pi\sqrt{a^2(1-\rho^2)}}\cdot$$
$$[J(a^2,1,a\rho)+J(1,a^2,a\rho)+J(a^2,1,-a\rho)+J(1,a^2,-a\rho)]=\frac{1}{\pi\sqrt{a^2(1-\rho^2)}}\cdot$$
$$\frac{\sqrt{2\pi}a^3(1-\rho^2)^{\frac{3}{2}}}{2}\left[\frac{1}{a^2(1-\rho^2)}\left(1-\frac{1+a\rho}{\sqrt{a^2+1+2a\rho}}\right)+\frac{a^2}{a^2(1-\rho^2)}\left(\frac{1}{a}-\frac{a^2+a\rho}{a^2\sqrt{1+a^2+2a\rho}}\right)+\right.$$
$$\left.\frac{1}{a^2(1-\rho^2)}\left(1-\frac{1-a\rho}{\sqrt{a^2+1-2a\rho}}\right)+\frac{a^2}{a^2(1-\rho^2)}\left(\frac{1}{a}-\frac{a^2-a\rho}{a^2\sqrt{a^2+1-2a\rho}}\right)\right]=\frac{a^2(1-\rho^2)}{\sqrt{2\pi}}\cdot$$
$$\frac{2(1+a)-\sqrt{1+a^2+2a\rho}-\sqrt{1+a^2-2a\rho}}{a^2(1-\rho^2)}=$$
$$\frac{2(1+a)-\sqrt{1+a^2+2a\rho}-\sqrt{1+a^2-2a\rho}}{\sqrt{2\pi}}\quad(17)$$

If $\rho=1$, then $f_2=f_1$ a. s.. Note that $a\in(0,1]$. Then we have

$$E(|f_1|\wedge|af_2|)=aE(|f_1|)=a\int_{-\infty}^\infty|x|\cdot\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x=\frac{2a}{\sqrt{2\pi}}\int_0^\infty x\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x=\frac{2a}{\sqrt{2\pi}}.$$

In addition, if $\rho=1$, we have

$$\frac{2(1+a)-\sqrt{1+a^2+2a\rho}-\sqrt{1+a^2-2a\rho}}{\sqrt{2\pi}}=\frac{2a}{\sqrt{2\pi}}.$$

Hence (17) holds for any $\rho\in[0,1]$.

For any $a\in(0,1]$ and any $\rho\in(0,1)$, we have

$$\frac{\mathrm{d}\left[\frac{2(1+a)-\sqrt{1+a^2+2a\rho}-\sqrt{1+a^2-2a\rho}}{\sqrt{2\pi}}\right]}{\mathrm{d}\rho}=\frac{a}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{1+a^2-2a\rho}}-\frac{1}{\sqrt{1+a^2+2a\rho}}\right)>0.$$

Hence for any $a\in(0,1]$,

$$\frac{2(1+a)-\sqrt{1+a^2+2a\rho}-\sqrt{1+a^2-2a\rho}}{\sqrt{2\pi}}$$

is a strictly increasing function in $\rho\in[0,1]$. Hence it reaches its minimum value at $\rho=0$, *i. e.* the inequality (3) holds.

4 Remarks

Remark 1 (i) Prof. Shao Qi-Man[9] told us that the Gaussian minimum conjecture for $n=2$ can be proved based on the following fact:

$$P(\min(|X_1|,|X_2|)>x)=1-P(|X_1|\leqslant x)-P(|X_2|\leqslant x)+P(|X_1|\leqslant x,|X_2|\leqslant x)\geqslant$$
$$1-P(|X_1|\leqslant x)-P(|X_2|\leqslant x)+P(|X_1|\leqslant x)P(|X_2|\leqslant x)=$$
$$1-P(|Y_1|\leqslant x)-P(|Y_2|\leqslant x)+P(|Y_1|\leqslant x)P(|Y_2|\leqslant x)=$$
$$1-P(|Y_1|\leqslant x)-P(|Y_2|\leqslant x)+P(|Y_1|\leqslant x,|Y_2|\leqslant x)=$$
$$P(\min(|Y_1|,|Y_2|)>x)\quad(18)$$

where the Gaussian correlation inequality was used.

(ii) In Section 3, we gave the proof of the Gaussian minimum conjecture for $n=2$ based on an explicit formula for $E(\min(|X_1|,|X_2|))$. We hope that this method may to be used to explore the minimum value point of $E(\min_{1\leqslant i\leqslant n}|X_i|)$ with $E(X_i^2)=1, \forall i=1,\cdots,n$, for $n\geqslant 3$.

Remark 2 For $n\geqslant 3$, we can't obtain the corresponding inequality similar to (18) by using the Gaussian correlation inequality. In fact, these inequalities do not hold by the necessity result of Theorem 1. 1.

参考文献：

[1] Gluskin E D. Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces [J]. Math USSR Sbornik, 1989, 64: 85.

[2] Šidák's Z. Retangular confidence regions for the means of multivariate normal distributions [J]. J Am Stat Assoc, 1967, 62: 626.

[3] Gordon Y, Litvak A, Schütt C, *et al*. Minima of sequences of Gaussian random variables [J]. C R Acad Paris Ser I, 2005, 340: 445.

[4] Gordon Y, Litvak A, Schütt C, *et al*. On the minimum of several random variables [J]. Proc Amer Math Soc, 2006, 134: 3665.

[5] Li W B. L1: introduction, overview and applications [EB/OL]. [2021-02-20]. http://jamesyli. com/wenboli_backup/papers/CBMS—10Lectures. pdf.

[6] Shao Q M. In memory of Wenbo V. Li's contribution [C]//High dimensional probability VII, Progress in Probability 71. Berlin/New York: Springer, 2016.

[7] Litvak A E. Around the simplex mean width conjecture [C]//Analytic aspects of convexity. Berlin/New York: Springer, 2018.

[8] Mallat S, Zeitouni O. A conjecture converning optimality of the Karhunen-Loeve basis in nonlinear reconstruction [EB/OL]. [2021-02-16]. https://arxiv. org/pdf/1109. 0489v2. pdf.

[9] A private correspondence with Prof. Shao Qi-Man from The South University of Science and Technology.

引用本文格式：

中　文：钟扬帆，马婷，胡泽春. 关于高斯最小值猜测的一个注记[J]. 四川大学学报：自然科学版，2021，58：061003.

英　文：Zhong Y F, Ma T, Hu Z C. A note on the Gaussian minimum conjecture [J]. J Sichuan Univ: Nat Sci Ed, 2021, 58: 061003.