部分可观测耦合随机抛物方程组的参数估计

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摘　要：本文研究加性时空白噪声驱动下的部分可观测耦合随机抛物方程组的参数极大似然估计量. 在观测时间和噪声强度不变的条件下，本文证明了估计量的强相合性和渐近正态性. 数值算例验证了理论结果.

关键词：部分可观测；耦合随机抛物方程；极大似然估计；渐近正态；强相合
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Parameter estimation in partially observable coupled stochastic parabolic equations

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Abstract: In this paper, we investigate the maximum likelihood estimator of the parameter of a partially observable coupled stochastic parabolic equations driven by the additive white Gaussian noises in time and space. For fixed observation time and noise intensity, the estimator is proved to be asymptotically consistent and with asymptotic normality. A numerical example is provided to illustrate the theoretic results.

Keywords: Partially observable; Coupled stochastic parabolic equations; Maximum likelihood estimator; Asymptotic normality; Strongly consistent
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1 Introduction

The general analytical theory of (linear and nonlinear) stochastic partial differential equations (SPDE) has made great progress and become a mature mathematical field in the past few decades. Stochastic partial differential equations (SPDEs) generalize deterministic partial differential equations (PDEs) by introducing driving noise processes into dynamics. Not only the theory of SPDEs, but also the statistics for SPDEs have recently seen a significant development, paving the way for a realistic modeling of complex phenomena.

The SPDE interested in this paper belongs to a general class of activator-inhibitor models, which can be described by two coupled stochastic reaction-diffusion equations $X=(A,I)$ of the following form:

$$\begin{cases} \dfrac{\partial}{\partial t}A(t,x)=D_A\dfrac{\partial^2}{\partial x^2}A(t,x)+ \\ \quad f_A(X(t,x),x)+\sigma_A\xi_A(t,x), \\ \dfrac{\partial}{\partial t}I(t,x)=D_I\dfrac{\partial^2}{\partial x^2}I(t,x)+f_I(X(t,x),x)+ \\ \quad \sigma_I\xi_I(t,x), \end{cases}$$

where $X=(A,I)$, with space-time white noise

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processes ξ_A, ξ_I, and σ_A, σ_I are the noise levels. In cell dynamics, we think of A as a hypothetical signalling molecule in response to an external signal gradient becoming enriched on one side of the cell with diffusion D_A, yielding polarity. I counter-acts A so that removal of the signal results in loss of polarity with diffusion D_I. f_A and f_I represent the interaction functions of activator A and inhibitor I. More detailed information about f_A and f_I can be found in Ref. [1]. Meinhardt[2] has been the first to apply such models to cell polarisation in the context of cell migration, where the ratio of activator-inhibitor diffusion can be tuned to either obtain a single stable cell front, or multiple independent fronts associated with non-directed random cell motility.

In the current paper we present a simple version of the modified Meinhardt two variable model. We set $f_A(X(t,x),x)=I(t,x)$, $f_I(X(t,x),x)=A(t,x)$, $\sigma_A=\sigma_I=1$, $D_I=1$ and focus on the following specific coupled stochastic parabolic equations problem:

$$\begin{cases} \mathrm{d}u(t,x)=\theta u_{xx}(t,x)\mathrm{d}t+v(t,x)\mathrm{d}t+ \\ \quad \mathrm{d}W_1(t,x),\ t\in[0,T],\ x\in\Lambda, \\ \mathrm{d}v(t,x)=u(t,x)\mathrm{d}t+v_{xx}(t,x)\mathrm{d}t+\mathrm{d}W_2(t,x), \\ \quad t\in[0,T],\ x\in\Lambda, \\ u(0,x)=v(0,x)=0,\ x\in\Lambda, \\ u(t,0)=u(t,\pi)=0,\ t\in[0,T], \\ v(t,0)=v(t,\pi)=0,\ t\in[0,T] \end{cases} \tag{1}$$

where $\Lambda=(0,\pi)$, $\{W_1(t)\}_{t\geqslant 0}$ and $\{W_2(t)\}_{t\geqslant 0}$ are two cylindrical Brownian motions on $L^2(\Lambda)$, ($\mathrm{d}W_i(t,x)$, $i=1,2$ is also referred to as *space-time* white noise) on a complete filtered probability space $\mathbb{F}=(\Omega,F,\{F_t\}_{t\geqslant 0},\mathbb{P})$ with the usual assumptions (completeness of F_0 and right-continuity of F_t), $T<\infty$ is fixed, moreover, $\theta\in(\alpha,\beta)$, $\alpha>0$ is the unknown parameter to be estimated based on finite dimensional approximations to solutions of such systems.

To the best of our knowledge, this problem has not been studied for the partially observable coupled stochastic parabolic equations, that is to say, we can only observe one process in the coupled equations. In this paper, we extend the estimation method introduced by Huebner and Khasminskii[3], where the parameter estimation in a single equation is discussed and first introduced by using the spectral method to study the consistency, asymptotic normality and asymptotic efficiency of maximum likelihood estimator of a parameter in the drift coefficient of an SPDE. Recently, Cialenco[4] showed some attractive methods to estimate parameters in stochastic partial differential equations. Most approaches focus on estimating coefficients for the linear part of the equation, either from discrete[5] or spectral observations[3, 6], but also the aspects of the driving noise[7, 8], have been analysed.

The number N of the Fourier coefficients used to calculate the maximum likelihood estimator $\widehat{\theta}_N$ is a natural asymptotic parameter. The asymptotic properties of $\widehat{\theta}_N$ as $N\to\infty$ is a focal point of this work and we prove that the maximum likelihood estimator $\widehat{\theta}_N$ is strongly consistent and asymptotic normality which is also our main result. Throughout this work, N is the only parameter of asymptotic.

The paper is organized as follows. In Section 2 we set the stage: starting with notations, continuing with the solution of Eq. (1), ending with the description of the statistical experiment and presenting the main object of this study: the maximum likelihood estimator. Section 3 is dedicated to the main results. Numerical example which illustrate the efficiency of the proposed estimator are presented in Section 4.

2 Preliminaries

For a random variable ξ, $\mathbb{E}\xi$ and Varξ denote the expectation and the variance, respectively. $\mathbf{R}^n$ is an n-dimensional Euclidean space, $N(m,\sigma^2)$ is a Gaussian random variable with mean m and variance σ^2. B^* denote the transpose of matrix B. Notation $a_n \sim b_n$ for two sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, with $a_n>0, b_n>0$, means that $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=1$. For example, $n^2-2n\sim n^2$ and $\sum\limits_{k=1}^{n} k^2 \sim \frac{n^3}{3}$.

Let us look for the solution of Eq. (1) as a Fourier series.

Definition 2.1 A solution of Eq. (1) are two random element with values in $L^2((0,T)\times\Lambda)$, such that, for every twice continuously differentiable on $[0,\pi]$ function $\varphi(x)$ satisfying $\varphi(0)=\varphi(\pi)=0$ and every $t\in[0,T]$, the following equality holds with probability 1:

$$\begin{cases}(u,\varphi)_{L^2(\Lambda)}(t)=\theta\int_0^t(u,\varphi_{xx})_{L^2(\Lambda)}(s)\mathrm{d}s+\\ \int_0^t(v,\varphi)_{L^2(\Lambda)}(s)\mathrm{d}s+W_{1,\varphi}(t),\\ (v,\varphi)_{L^2(\Lambda)}(t)=\int_0^t(v,\varphi_{xx})_{L^2(\Lambda)}(s)\mathrm{d}s+\\ \int_0^t(u,\varphi)_{L^2(\Lambda)}(s)\mathrm{d}s+W_{2,\varphi}(t)\end{cases}\tag{2}$$

Proposition 2.2 There exists a unique solution of Eq. (1).

Proof We solve Eq. (1) by using the classical method of separation of variables. Let $h_k(x)=\sqrt{2/\pi}\cdot\sin(kx), k\geqslant 1$. Taking $\varphi=h_k$ in Eq. (2), due to $W_i(t)$, $i=1,2$ is the noise term, a cylindrical Brownian motion on $L^2(\Lambda)$, at this point we interpret $\mathrm{d}W_i(t,x)$, $i=1,2$ as a formal sum

$$\mathrm{d}W_i(t,x)=\sum_{k\geqslant 1}h_k(x)\mathrm{d}W_{i,k}(t),$$

where $\{W_{i,k}(t)\}_{k\geqslant 1}$, $i=1,2$ are independent standard Brownian motions for different k. We find that $u_k(t)=(u,h_k)_{L^2(\Lambda)}(t)$ and $v_k(t)=(v,h_k)_{L^2(\Lambda)}(t)$ satisfy

$$\begin{cases}u_k(t)=-\theta k^2\int_0^t u_k(s)\mathrm{d}s+\int_0^t v_k(s)\mathrm{d}s+W_{1,k}(t),\\ v_k(t)=-k^2\int_0^t v_k(s)\mathrm{d}s+\int_0^t u_k(s)\mathrm{d}s+W_{2,k}(t)\end{cases}\tag{3}$$

Denote

$$Y_k(t)=\begin{bmatrix}u_k(t)\\ v_k(t)\end{bmatrix},\ A_k=\begin{bmatrix}-k^2\theta & 1\\ 1 & -k^2\end{bmatrix}$$

and

$$V_k(t)=\begin{bmatrix}W_{1,k}(t)\\ W_{2,k}(t)\end{bmatrix}.$$

We know the solution of this $Y_k(t)$ is

$$Y_k(t)=\int_0^t \mathrm{e}^{A_k(t-s)}\mathrm{d}V_k(s)\tag{4}$$

here V_k is a two-dimensional standard Brownian motion. Therefore,

$$\mathbb{E}Y_k(t)Y_k^*(t)=\begin{bmatrix}\mathbb{E}u_k^2(t) & \mathbb{E}u_k(t)v_k(t)\\ \mathbb{E}v_k(t)u_k(t) & \mathbb{E}v_k^2(t)\end{bmatrix}=\int_0^t \mathrm{e}^{A_k^*(t-s)}\mathrm{e}^{A_k(t-s)}\mathrm{d}s=$$
$$\frac{2}{\lambda_{1,k}-\lambda_{2,k}}\begin{bmatrix}\left(\frac{2k^2+\lambda_{1,k}}{2}\right)\chi_{1,k}(t)-\left(\frac{2k^2+\lambda_{2,k}}{2}\right)\chi_{2,k}(t) & \left(\frac{2k^2+\lambda_{2,k}}{2}\right)\chi_{1,k}(t)-\left(\frac{2k^2+\lambda_{2,k}}{2}\right)\chi_{2,k}(t)\\ \left(\frac{2k^2+\lambda_{1,k}}{2}\right)\chi_{2,k}(t)-\left(\frac{2k^2+\lambda_{1,k}}{2}\right)\chi_{1,k}(t) & \left(\frac{2k^2+\lambda_{1,k}}{2}\right)\chi_{2,k}(t)-\left(\frac{2k^2+\lambda_{2,k}}{2}\right)\chi_{1,k}(t)\end{bmatrix}\tag{5}$$

where

$$\lambda_{1,k}=-(k^2\theta+k^2)+\sqrt{(k^2\theta-k^2)^2+4}$$

and

$$\lambda_{2,k}=-(k^2\theta+k^2)-\sqrt{(k^2\theta-k^2)^2+4}$$

are the eigenvalues of $A_k+A_k^*$,

$$\chi_{1,k}(t)=-\frac{1}{\lambda_{1,k}}(1-\mathrm{e}^{\lambda_1 t}),$$

$$\chi_{2,k}(t)=-\frac{1}{\lambda_{2,k}}(1-\mathrm{e}^{\lambda_{2,k}t}).$$

By calculation, we have

$$\mathbb{E}u_k^2(t)=\frac{2}{\lambda_{1,k}-\lambda_{2,k}}\left[\left(\frac{2k^2+\lambda_{1,k}}{2}\right)\chi_{1,k}(t)-\left(\frac{2k^2+\lambda_{2,k}}{2}\right)\chi_{2,k}(t)\right]\sim\frac{1}{2\theta k^2}\tag{6}$$

and

$$\mathbb{E}v_k^2(t)=\frac{2}{\lambda_1-\lambda_2}\left[\left(\frac{2k^2+\lambda_1}{2}\right)\chi_{2,k}(t)-\left(\frac{2k^2+\lambda_2}{2}\right)\chi_{1,k}(t)\right]\sim\frac{1}{2k^2}\tag{7}$$

which implies $u(t),v(t)\in L_F^2(\Omega;L^2(\Lambda))$ for all $0\leqslant t\leqslant T$.

Since h_k, $k\geqslant 1$ is an orthogonal basis in $L^2(\Lambda)$, we conclude that $u=\sum_{k\geqslant 1}u_k h_k$ and $v=\sum_{k\geqslant 1}v_k h_k$ are solutions of Eq. (1). Uniqueness of solutions of Eq. (1) follows from the uniqueness of solution of Eq. (3) for every k. Proposition 2.2 is proved.

Assume that only $\{u_k(t)\}_{1\leqslant k\leqslant N}$ is observed. Neglecting the contribution of $v_k(t)$ in Eq. (3) leads to a parametric estimation problem for θ with respect to the scalar processes $\{u_k(t)\}_{0\leqslant t\leqslant T}$ for $k=1,\ldots,N$. It is obvious that $u_k(t)$ is a diffusion process. Then, from the result of Ref. [9, Theorem 7.15 on page 279] for each θ and each k, the process $u_k^T:=\{u_k(t),0\leqslant t\leqslant T\}$ generates the measure $\mathbb{P}_{\theta,k}$ in the space of continuous real-valued functions on $[0,T]$. Let us denote the space by (C_T,B_T). We can get the following likelihood function:

$$L_k(\theta)=\frac{\mathrm{d}\,\mathbb{P}_{\theta,k}}{\mathrm{d}\,\mathbb{P}_{0,k}}(u_k^T)=\exp\Big(-\int_0^T\theta k^2\,u_k(t)\mathrm{d}\,u_k(t)-\int_0^T\theta^2\,k^4\,u_k^2(t)\mathrm{d}t\Big).$$

Similarly, the vector $u^N=\{u_1^T,u_2^T,\ldots,u_N^T\}$, generates a probability measure $\mathbb{P}_\theta^N$ on the space of continuous $\mathbf{R}^N$-valued functions on $[0,T]$. Since the random processes $u_k(t)$ are independent, $\mathbb{P}_\theta^N$ is a product measure, say, $\mathbb{P}_\theta^N=\prod_{k=1}^N\mathbb{P}_{\theta,k}$, and thus the measures $\mathbb{P}_\theta^N$ are equivalent for different values of θ. In particular, we have

$$L^N(\theta)=\frac{\mathrm{d}\,\mathbb{P}_\theta^N}{\mathrm{d}\,\mathbb{P}_0^N}(u^N)=\exp\Big(-\sum_{k=1}^N k^2\int_0^T\theta\,u_k(t)\mathrm{d}\,u_k(t)-\sum_{k=1}^N k^4\int_0^T\theta^2\,u_k^2(t)\mathrm{d}t\Big)\tag{8}$$

Maximising $L^N(\theta)$ with respect to θ, leads to the following MLE:

$$\hat{\theta}_N=-\frac{\sum_{k=1}^N k^2\int_0^T u_k(t)\mathrm{d}\,u_k(t)}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)\mathrm{d}t}\tag{9}$$

3 The main results

In this section, we use $\hat{\theta}_N$ to estimate the unknown drift parameter θ in Eq. (1) and present the main results of this paper as follows.

Theorem 3.1 As $N\to\infty$, $\hat{\theta}_N$ is a strongly consistent estimator. Namely

$$\lim_{N\to\infty}N^{\frac{3}{2}}(\hat{\theta}_N-\theta)=0,\mathbb{P}\text{-a.s}\tag{10}$$

Also it is a asymptotically normal estimator of θ, that is,

$$\lim_{N\to\infty}N^{\frac{3}{2}}(\hat{\theta}_N-\theta)\overset{d}{=}N\left(0,\frac{6\theta}{T}\right)\tag{11}$$

where "$\overset{d}{=}$" represents convergence in distribution.

Lemma 3.2[10] Let ξ_n, $n\geqslant1$ be independent random variables such that $\xi_n\geqslant0$, $\sum_{k\geqslant1}\mathbb{E}\xi_k=+\infty$, and

$$\sum_{n\geqslant1}\frac{\operatorname{Var}\xi_n}{\big(\sum_{k=1}^n\mathbb{E}\xi_k\big)^2}<\infty.$$

Then

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n\xi_k}{\sum_{k=1}^n\mathbb{E}\xi_k}=1$$

with probability one.

Lemma 3.3[10] Let $w_k=w_k(t)$, $k=1,\ldots,n$ be independent standard Brownian motions and let $f_k=f_k(t)$, $k=1,\ldots,n$ be adapted, continuous, square-integrable processes such that

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n\int_0^T f_k^2(t)\mathrm{d}t}{\sum_{k=1}^n\mathbb{E}\int_0^T f_k^2(t)\mathrm{d}t}=1$$

in probability. Then

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n\int_0^T f_k(t)\mathrm{d}w_k(t)}{\Big(\sum_{k=1}^n\mathbb{E}\int_0^T f_k^2(t)\mathrm{d}t\Big)^{1/2}}\overset{d}{=}N(0,1).$$

Proof of Theorem 3.1 We borrow some ideas from Lototsky[10]. From Eq. (8), we can get

$$\hat{\theta}_N-\theta=\frac{\sum_{k=1}^N k^2\int_0^T u_k(t)\mathrm{d}W_{1,k}(t)}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)\mathrm{d}t}+\frac{\sum_{k=1}^N k^2\int_0^T u_k(t)\,v_k(t)\mathrm{d}t}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)\mathrm{d}t}\tag{12}$$

For discussion, we divide $u(t,x)$ into $\bar{u}(t,x)$ and $\tilde{u}(t,x)$ as

$$\begin{cases}\mathrm{d}\,\bar{u}(t,x)=\theta\,\bar{u}_{xx}(t,x)\mathrm{d}t+\mathrm{d}W_1(t,x),\\ \mathrm{d}\,\tilde{u}(t,x)=\theta\,\tilde{u}_{xx}(t,x)\mathrm{d}t+v(t,x)\mathrm{d}t\end{cases}\tag{13}$$

Correspondingly, we have $\bar{u}_k(t)$ and $\tilde{u}_k(t)$

$$\begin{cases} \mathrm{d}\,\bar{u}_k(t) = -\theta k^2 \bar{u}_k(t)\mathrm{d}t + \mathrm{d}\,W_{1,k}(t), \\ \mathrm{d}\,\tilde{u}_k(t) = -\theta k^2 \tilde{u}_k(t)\mathrm{d}t + v_k(t)\mathrm{d}t \end{cases} \tag{14}$$

From now on, we divide the proof into two parts. At first, we prove that

$$\frac{\sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N} k^4\ \mathbb{E}\int_0^T \bar{u}_k^2(t)\mathrm{d}t} = 1 \tag{15}$$

in probability. Next, we prove

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} k^2 \int_0^T u_k(t)\ v_k(t)\mathrm{d}t}{\sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)\mathrm{d}t} = 0 \tag{16}$$

in probability.

(i) In order to estimate Eq. (14), we firstly do some computations. we have

$$\mathbb{E}\int_0^T \bar{u}_k^2(t)\mathrm{d}t \sim \frac{T}{2\theta k^2} \tag{17}$$

$$\mathrm{Var}\left(\int_0^T \bar{u}_k^2(t)\mathrm{d}t\right) \sim \frac{T}{2\,\theta^3\ k^6} \tag{18}$$

and

$$\mathbb{E}\left(k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t\right) \sim \frac{Tk^2}{2\theta} \tag{19}$$

$$\mathrm{Var}\left(k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t\right) \sim \frac{Tk^2}{2\,\theta^3} \tag{20}$$

Next by applying Lemma 3. 2, the following equality

$$\frac{\mathbb{E}\sum_{k=1}^{N} k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N} k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t} = 1 \tag{21}$$

holds in probability. Further, combining Eq. (5) and Eq. (16), we have

$$\frac{\mathbb{E}\sum_{k=1}^{N} k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t}{\mathbb{E}\sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)\mathrm{d}t} = 1 \tag{22}$$

Now we estimate

$$\frac{\sum_{k=1}^{N} \int_0^T k^4\ u_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N} \int_0^T k^4\ \bar{u}_k^2(t)\mathrm{d}t} =$$

$$\frac{\sum_{k=1}^{N} k^4 \int_0^T (\bar{u}_k(t) + \tilde{u}_k(t)\ )^2\mathrm{d}t}{\sum_{k=1}^{N} k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t} =$$

$$1 + \frac{\sum_{k=1}^{N} k^4 \left(\int_0^T \tilde{u}_k^2(t)\ + 2\ \tilde{u}_k(t)\ \bar{u}_k(t)\mathrm{d}t\right)}{\sum_{k=1}^{N} k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t} \tag{23}$$

from Eq. (13) and obtain

$$\tilde{u}_k(t) = \int_0^t \mathrm{e}^{-\theta k^2(t-s)}\ v_k(s)\mathrm{d}s =$$

$$\int_0^t \mathrm{e}^{-\theta k^2 t}\ \mathrm{e}^s\ v_k(s)\mathrm{d}s \leqslant$$

$$\int_0^t \frac{1}{\theta k^2 t}\ \mathrm{e}^s\ v_k(s)\mathrm{d}s \leqslant$$

$$\frac{\mathrm{e}^t}{\sqrt{2}\theta k^2 t}\left(\int_0^t v_k^2(s)\mathrm{d}s\right)^{1/2} \approx O(\frac{1}{k^3}) \tag{24}$$

The last equality can be derived from Eq. (6). On the other hand, the Cauchy-Schwarz inequality shows that

$$\int_0^T \tilde{u}_k(t)\,\bar{u}_k(t)\mathrm{d}t \leqslant \left(\int_0^T \tilde{u}_k^2(t)\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \bar{u}_k^2(t)\mathrm{d}t\right)^{\frac{1}{2}} \tag{25}$$

Eq. (23) and Eq. (24) lead to

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} k^4 \left(\int_0^T (\tilde{u}_k^2(t)\ + 2\ \tilde{u}_k(t)\ \bar{u}_k(t))\mathrm{d}t\right)}{\sum_{k=1}^{N} k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t} = 0 \tag{26}$$

Finally, it holds that

$$\frac{\sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N} k^4 \int_0^T \bar{u}_k^2(t)\mathrm{d}t} = 1 \tag{27}$$

Combining Eq. (20), Eq. (21) and Eq. (26), Eq. (14) holds in probability.

(ii) According to Eq. (22)～Eq. (26), one can also get

$$\frac{\sum_{k=1}^{N} k^2 \int_0^T u_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N} k^2 \int_0^T \bar{u}_k^2(t)\mathrm{d}t} = 1 \tag{28}$$

From Eq. (16), we have

$$\mathbb{E}\left(k^2 \int_0^T \bar{u}_k^2(t)\mathrm{d}t\right) \sim \frac{T}{2\theta} \tag{29}$$

and

$$\mathrm{Var}\left(k^2 \int_0^T \bar{u}_k^2(t)\mathrm{d}t\right) \sim \frac{T}{2\,\theta^3\ k^2} \tag{30}$$

By applying Lemma 3. 2, the following equality

$$\frac{\mathbb{E}\sum_{k=1}^{N}k^2\int_0^T \bar{u}_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N}k^2\int_0^T \bar{u}_k^2(t)\mathrm{d}t}=1 \tag{31}$$

holds in probability. Moreover, for $v(t,x)$, we can take the same separation as $u(t,x)$, then we have

$$\begin{cases}\mathrm{d}\,\bar{v}_k(t)=-k^2\bar{v}_k(t)\mathrm{d}t+\mathrm{d}W_{2,k}(t),\\ \mathrm{d}\,\tilde{v}_k(t)=-k^2\tilde{v}_k(t)\mathrm{d}t+u_k(t)\mathrm{d}t\end{cases} \tag{32}$$

As the same as $u_k(t)$, for $v_k(t)$, we have

$$\frac{\sum_{k=1}^{N}k^4\int_0^T v_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N}k^4\int_0^T \bar{v}_k^2(t)\mathrm{d}t}=1,$$

$$\frac{\sum_{k=1}^{N}k^2\int_0^T v_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N}k^2\int_0^T \bar{v}_k^2(t)\mathrm{d}t}=1,$$

and

$$\frac{\sum_{k=1}^{N}k^2\int_0^T \bar{v}_k^2(t)\mathrm{d}t}{\sum_{k=1}^{N}k^2\,\mathbb{E}\int_0^T \bar{v}_k^2(t)\mathrm{d}t}=1 \tag{33}$$

Substituting these equalities and Eq. (14), Eq. (27), Eq. (30) into the inequality below, it is direct to obtain that

$$\frac{\sum_{k=1}^{N}k^2\int_0^T u_k(t)\,v_k(t)\mathrm{d}t}{\sum_{k=1}^{N}k^4\int_0^T u_k^2(t)\mathrm{d}t}\leqslant \frac{\sum_{k=1}^{N}k^2\left(\int_0^T u_k^2(t)\mathrm{d}t+\int_0^T v_k^2(t)\mathrm{d}t\right)}{\sum_{k=1}^{N}k^4\int_0^T u_k^2(t)\mathrm{d}t}=$$

$$\frac{\sum_{k=1}^{N}k^2\left(\int_0^T \bar{u}_k^2(t)\mathrm{d}t+\int_0^T \bar{v}_k^2(t)\mathrm{d}t\right)}{\sum_{k=1}^{N}k^4\int_0^T \bar{u}_k^2(t)\mathrm{d}t}=\frac{\sum_{k=1}^{N}k^2\left(\int_0^T \mathbb{E}\bar{u}_k^2(t)\mathrm{d}t+\int_0^T \mathbb{E}\bar{v}_k^2(t)\mathrm{d}t\right)}{\sum_{k=1}^{N}k^4\int_0^T \mathbb{E}\bar{u}_k^2(t)\mathrm{d}t}.$$

Then, by

$$\int_0^T \mathbb{E}\bar{v}_k^2(t)\mathrm{d}t\sim\frac{T}{2k^2},$$

together with Eq. (16), one has

$$\frac{\sum_{k=1}^{N}k^2\left(\int_0^T \mathbb{E}\bar{u}_k^2(t)\mathrm{d}t+\int_0^T \mathbb{E}\bar{v}_k^2(t)\mathrm{d}t\right)}{\sum_{k=1}^{N}k^4\int_0^T \mathbb{E}\bar{u}_k^2(t)\mathrm{d}t}=0$$

in probability. Thus Eq. (15) holds. Substituting Eq. (14) and Eq. (15) into Eq. (11), we obtain

$$\hat{\theta}_N-\theta=\frac{\sum_{k=1}^{N}k^2\int_0^T u_k(t)\mathrm{d}W_{1,k}(t)}{\sum_{k=1}^{N}k^4\int_0^T u_k^2(t)\mathrm{d}t}+\frac{\sum_{k=1}^{N}k^2\int_0^T u_k(t)\,v_k(t)\mathrm{d}t}{\sum_{k=1}^{N}k^4\int_0^T u_k^2(t)\mathrm{d}t}=\frac{\sum_{k=1}^{N}k^2\int_0^T u_k(t)\mathrm{d}W_{1,k}(t)}{\sum_{k=1}^{N}k^4\int_0^T \mathbb{E}u_k^2(t)\mathrm{d}t} \tag{34}$$

in probability. By the strong law of large number, we obtain $\lim\limits_{N\to\infty}(\hat{\theta}_N-\theta)=0$ with probability one, strongly consistent is proved. Applying Lemma 3.3 to Eq. (33) with $f_k(t)=u_k(t)$, $k=1,\ldots,n$, we see

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N}k^2\int_0^T u_k(t)\mathrm{d}W_{1,k}(t)}{\sqrt{\sum_{k=1}^{N}k^4\int_0^T \mathbb{E}u_k^2(t)\mathrm{d}t}}\overset{d}{=}N(0,1).$$

By direct calculating with Eq. (5), we obtain

$$I(\theta):=\lim_{N\to\infty}\sum_{k=1}^{N}k^4\int_0^T \mathbb{E}u_k^2(t)\mathrm{d}t\sim\frac{TN^3}{6\theta}.$$

Therefore,

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N}k^2\int_0^T u_k(t)\mathrm{d}W_{1,k}(t)}{\sum_{k=1}^{N}k^4\int_0^T \mathbb{E}u_k^2(t)\mathrm{d}t}\overset{d}{=}N(0,I(\theta)^{-1}).$$

Furthermore, one also has

$$\lim_{N\to\infty}N^{\frac{3}{2}}(\hat{\theta}_N-\theta)\overset{d}{=}N\left(0,\frac{6\theta}{T}\right).$$

Eq. (10) holds. The proof is end.

4 A numerical example

In this section we present a numerical illustrations to exhibit the performance of the MLE $\hat{\theta}_N$ given by Theorem 3.1. We first simulate the sample paths of Eq. (1) by using the Monte Carlo method with the classic Euler-Maruyama scheme.

In each numerical experiment, we generate 10^3 sample paths with step size $\Delta t=10^{-2}$ and $T=10$. We examine the following three different settings, respectively: (i) $\theta=0.5$; (ii) $\theta=1$; (iii) $\theta=2$.

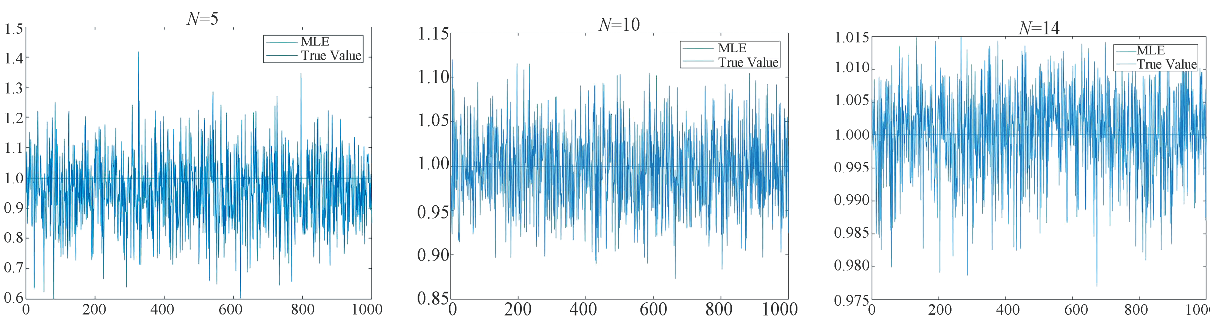
Tab. 1　The ME and MSE of MLE $\widehat{\theta}_N$

		$\mathbb{E}(\widehat{\theta}_N-\theta)$	$\mathbb{E}(\widehat{\theta}_N-\theta)^2$
$3*\theta=0.5$	$N=5$	-0.0616	0.0108
	$N=10$	-0.0134	0.0020
	$N=14$	-0.0056	6.37e−4
$3*\theta=1$	$N=5$	-0.0425	0.0160
	$N=10$	-0.0034	0.0021
	$N=14$	5.20e−4	4.75e−5
$2*\theta=2$	$N=5$	-0.0142	0.0204
	$N=10$	2.43e−4	1.73e−5

Tab. 1 reports some statistics related to the $\widehat{\theta}_N$, which include the Mean Error (ME) $\mathbb{E}(\widehat{\theta}_N-\theta)$ and the Mean Square Error (MSE) $\mathbb{E}(\widehat{\theta}_N-\theta)^2$. The data shows the error between the $\widehat{\theta}_N$ and the true value is very small and tell us the $\widehat{\theta}_N$ is strongly consistent.

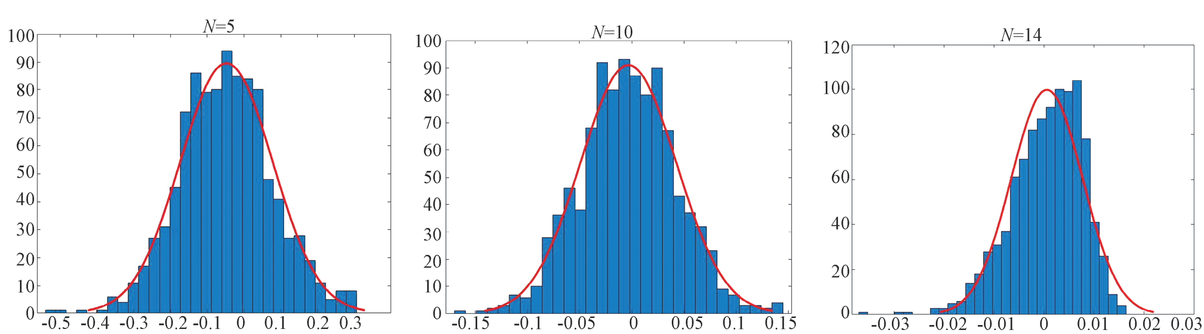
Fig. 1～Fig. 4 are plotted under the setting (ii). Fig. 1 shows the MLE $\widehat{\theta}_N$ and the true value on one picture with $N=5$, $N=10$ and $N=14$. We find that as N increases, MLE $\widehat{\theta}_N$ is closer to the true value. Fig. 2 depicts the histogram of the statistic $(\widehat{\theta}_N-\theta)$ with $N=5$, $N=10$ and $N=14$. The dashed curve is the normal density. Fig. 3 depicts the histogram of the statistic $N^{\frac{3}{2}}\sqrt{\frac{T}{6\theta}}(\widehat{\theta}_N-\theta)$ with $N=14$. Fig. 4 we use the normplot function in matlab to compares the distribution of the data of $N^{\frac{3}{2}}\sqrt{\frac{T}{6\theta}}(\widehat{\theta}_N-\theta)$ to the normal distribution with $N=5$, $N=10$ and $N=14$, we can find that the coincidence degree is very high.

From this example we can conclude that the MLE works quite well (from Theorem 3.1), and we have $\lim\limits_{N\to\infty}N^{\frac{3}{2}}\sqrt{\frac{T}{6\theta}}(\widehat{\theta}_N-\theta)\overset{d}{=}N(0,1)$.



(a)　(b)　(c)

Fig. 1　The comparision the MLE $\widehat{\theta}_N$ and the true value



(a)　(b)　(c)

Fig. 2　Histogram of $(\widehat{\theta}_N-\theta)$ with (a) $N=5$, (b) $N=10$ and (c) $N=14$. The curves are the plots of the normal density: (a) $\mu=-0.047\,835\,8\,[-0.055\,624\,4, -0.040\,047\,3]$, $\sigma=0.125\,511\,[0.120\,241, 0.131\,268]$; (b) $\mu=-0.003\,259\,[-0.006\,115\,56, -0.000\,402\,448]$, $\sigma=0.046\,032\,9\,[0.044\,100\,1, 0.048\,144\,2]$; (c) $\mu=0.000\,551\,763\,[0.000\,110\,152, 0.000\,993\,374]$, $\sigma=0.007\,116\,48\,[0.006\,817\,68, 0.007\,442\,88]$, where μ is the mean, and σ is the standard deviation, and the intervals next to the parameter estimates are the 95% confidence intervals for the distribution parameters

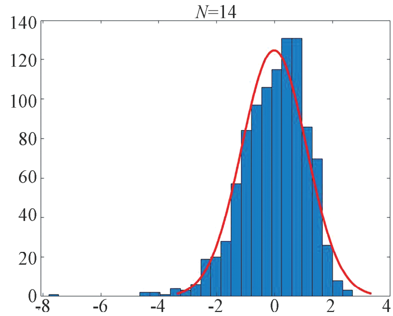


Fig. 3　Histogram of $N^{\frac{3}{2}}\sqrt{\frac{T}{6\theta}}(\widehat{\theta}_N-\theta)$ with $N=14$. The curve is plot of the normal density: $\mu=-0.0134751$ [−0.082 977 3, 0.056 027 2], $\sigma=1.12002$ [1.072 99, 1.171 39]

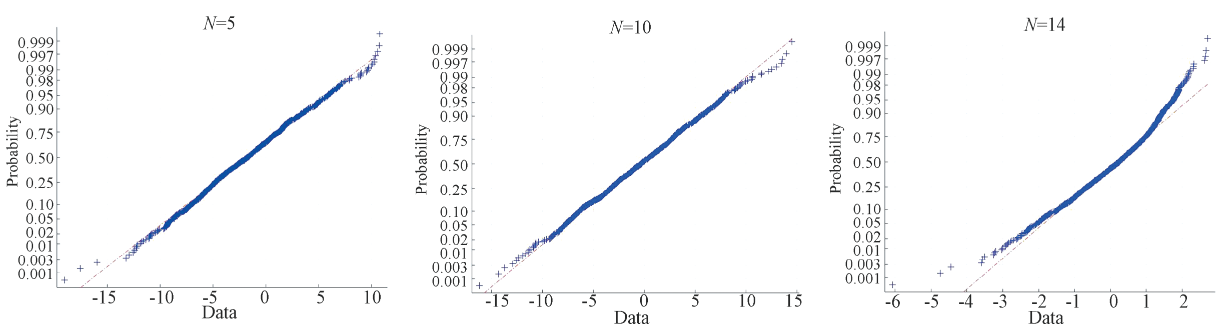


Fig. 4　The coincidence between the distribution of $N^{\frac{3}{2}}\sqrt{\frac{T}{6\theta}}(\widehat{\theta}_N-\theta)$ and the normal distribution, the plus signs ('+') marker is the data sampled form $N^{\frac{3}{2}}\sqrt{\frac{T}{6\theta}}(\widehat{\theta}_N-\theta)$ and the straight lines represent the normal distribution

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