一类具有正则鞍结点的平面 Filippov 系统的全局动力学

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摘　要：本文研究了一类具有正则鞍结点的平面 Filippov 系统的全局动力学. 该研究针对一般的参数，不要求参数充分小. 通过对伪平衡点、切点、无穷远平衡点及周期轨的定性分析，本文得到具有 8 条分岔曲线的全局分岔图，且在庞加莱圆盘上给出了所有的全局相图. 所得结果显示了一些没有出现在充分小参数情形下的新分岔现象.

关键词：分岔；边界平衡点；Filippov 系统；全局动力学
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Global dynamics of a planar Filippov system with a regular-SN

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Abstract: In this paper, we investigate the global dynamics of a planar Filippov system with a regular-SN for general parameters not required to be sufficiently small. By analyzing the qualitative properties of the pseudo-equilibria, tangent points, equilibria at infinity as well as all kinds of periodic orbits, we obtain the global bifurcation diagram with eight bifurcation curves and give all global phase portraits in Poincaré's disc. Some new bifurcation phenomena which do not appear in the case of small parameters is finded.

Keywords: Bifurcation; Boundary equilibrium; Filippov system; Global dynamics
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1　Introduction

Filippov system consists of a finite set of equations

$$\dot{x}=f^i(x), x\in G_i\subset\mathbf{R}^n,$$

where $G_i\,(i=1,\cdots,m)$ are open regions separated by some $(n-1)$-dimensional submanifolds in $\mathbf{R}^n$. These submanifolds are usually called switching manifolds. Many practical applications use Filippov system, such as electronic[1], mechanical systems with dry friction[2] and neural activities[3].

Consider a two-dimensional Filippov system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}f^L(x,y;\alpha) \text{ if } & h(x,y;\alpha)<0,\\ f^R(x,y;\alpha) \text{ if } & h(x,y;\alpha)>0\end{cases}\tag{1}$$

where $\alpha\in\mathbf{R}^m$ and $h(x,y;\alpha)=0$ is the unique switching manifold Σ. Usually, $f^L(x,y;\alpha)$, $f^R(x,y;\alpha)$ are called the left half vector field and the right half vector field, respectively. Bifurcation analysis is one of the most important subjects

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in qualitative analysis of differential systems. In recent years, there are many results for smooth differential systems. As in smooth systems, many interesting bifurcation phenomena[1,4,5] can be observed as the parameters of system (1) change. A special form of system (1)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}v-x^2\\ -y\end{pmatrix}, & \text{if} \quad x-\rho<0,\\ \begin{pmatrix}-1\\ 0\end{pmatrix}, & \text{if} \quad x-\rho>0\end{cases} \tag{2}$$

was investigated near the origin O: $(0,0)$, where $(\rho,v)\in\mathbf{R}^2$ and ρ,v are sufficient small[4]. Notice that the switching manifold in system (2) is a vertical line $x=\rho$, so that the projection of $(v-x^2,-y)^T$ in the direction of the normal vector of switching manifold is a constant $v-\rho^2$. Thus, it is not hard to obtain that the whole switching manifold is either a sliding region if $v-\rho^2>0$, or a crossing region if $v-\rho^2<0$, or a set of tangent points if $v-\rho^2=0$.

There are some interesting problems[6-8], for instance, how about dynamical behavior for general parameters, how about dynamical behavior for non-vertical switching line, how about the global dynamical behavior. Motivated by these interesting problems, in this paper we investigate the global dynamics for system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}v-x^2\\ -y\end{pmatrix}, & \text{if} \quad x-ky-\rho<0,\\ \begin{pmatrix}-1\\ 0\end{pmatrix}, & \text{if} \quad x-ky-\rho>0\end{cases} \tag{3}$$

where $(\rho,v,k)\in\mathbf{R}^3$ and $k\neq0$. By transformations $x\to x, y\to y/k$, system (3) is equivalently rewritten as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}v-x^2\\ -y\end{pmatrix}, & \text{if} \quad x-y-\rho<0,\\ \begin{pmatrix}-1\\ 0\end{pmatrix}, & \text{if} \quad x-y-\rho>0\end{cases} \tag{4}$$

where $(\rho,v)\in\mathbf{R}^2$. When $\rho=v=0$, the origin O: $(0,0)$ is a regular point of the right half vector filed and a saddle-node of the left half vector filed, hence, we call O as a regular-SN of system (4). We equivalently investigate the global dynamics of system (4) for general $(\rho,v)\in\mathbf{R}^2$ and obtain the following result.

Theorem 1.1 The bifurcation diagram of system (4) is shown in Fig. 1 and consists of the following curves:

(i) the boundary equilibrium bifurcation curves:

$E_{T1}:=\{(\rho,v)\in\mathbf{R}^2\mid v=\rho^2,\rho>1/2\}$,

$E_{T2}:=\{(\rho,v)\in\mathbf{R}^2\mid v=\rho^2,0<\rho<1/2\}$,

$E_A:=\{(\rho,v)\in\mathbf{R}^2\mid v=\rho^2,\rho<0\}$;

(ii) the double tangency bifurcation curves:

$DT_1:=\{(\rho,v)\in\mathbf{R}^2\mid v=\rho-1/4,\rho>1/2\}$,

$DT_2:=\{(\rho,v)\in\mathbf{R}^2\mid v=\rho-1/4,1/4<\rho<1/2\}$,

$DT_3:=\{(\rho,v)\in\mathbf{R}^2\mid v=\rho-1/4,\rho<1/4\}$;

(iii) the saddle-node bifurcation curves:

$SN_1:=\{(\rho,v)\in\mathbf{R}^2\mid v=0,\rho>1/4\}$,

$SN_2:=\{(\rho,v)\in\mathbf{R}^2\mid v=0,0<\rho<1/4\}$.

Global phase portraits of system (4) are given in Fig. 2, where A,B,C lie at $(0,0),(1/4,0),(1/2,1/4)$ respectively, and

$\mathrm{I}:=\{(\rho,v)\in\mathbf{R}^2\mid \max\{0,\rho-1/4\}<v<\rho^2, 0<\rho<1/4\}$,

$\mathrm{II}:=\{(\rho,v)\in\mathbf{R}^2\mid 0<v<\rho-14,1/4<\rho\}$,

$\mathrm{III}:=\{(\rho,v)\in\mathbf{R}^2\mid \rho-1/4<v<\rho^2,1/2<\rho\}$,

$\mathrm{IV}:=\{(\rho,v)\in\mathbf{R}^2\mid 0<v,-\sqrt{v}<\rho<\sqrt{v}\}$,

$\mathrm{V}:=\{(\rho,v)\in\mathbf{R}^2\mid \rho-1/4<v<\max\{0,-\mathrm{sgn}(\rho)\rho^2\},\rho<1/4\}$,

$\mathrm{VI}:=\{(\rho,v)\in\mathbf{R}^2\mid v<\min\{0,\rho-1/4\},-\infty<\rho<\infty\}$.

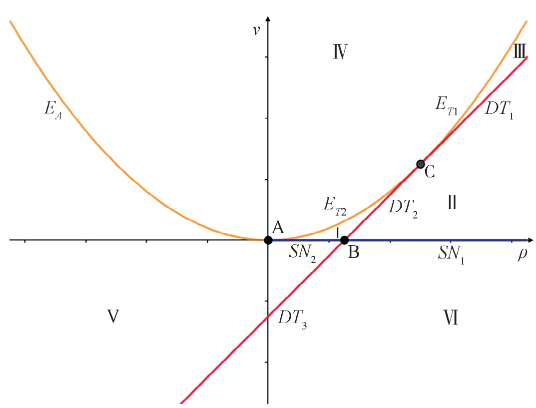

Fig. 1 Bifurcation diagram of system (4)

Note that (ρ,v) is considered generally in $\mathbf{R}^2$. We observe some dynamical behaviors which can not appear in the case that ρ,v are sufficiently small[6]. On the other hand, our result is global dynamics, not only restricted in a small neighborhood of the origin O.

This paper is organized as follows. In Section 2 we recall some definitions and theory about Filippov systems. In Section 3, we give some lemmas and finally provide a proof for our Theorem 1.1.

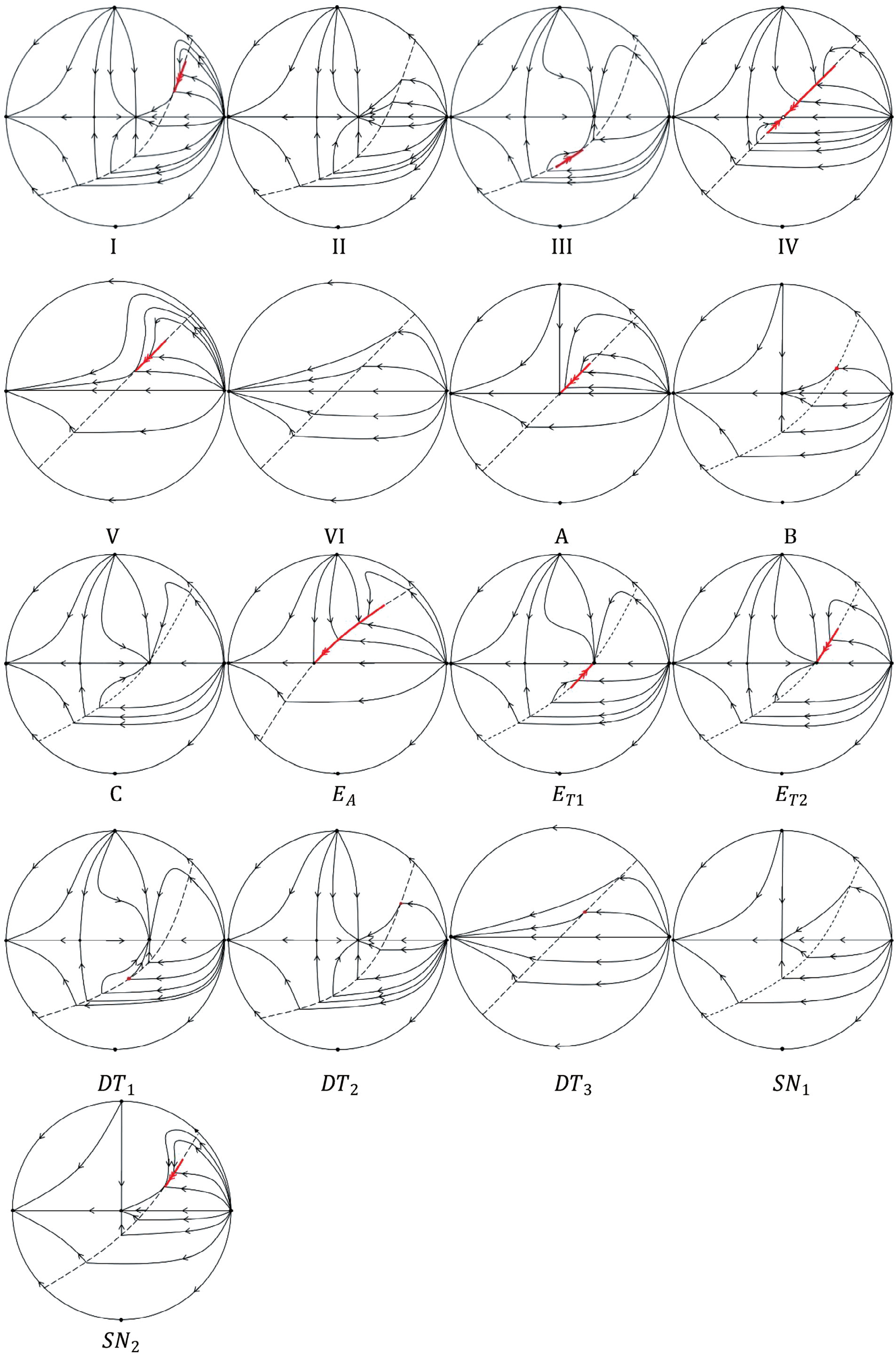

Fig. 2　Global phase portraits of system (4)

2　Preliminaries

In Filippov system, we always focus on the result of how the orbit interacts with the switching manifold Σ. Depending on the sign of normal components of left half vector field and right half vector field, we split Σ into crossing region Σ^c and sliding region Σ^s as

$$\Sigma^c :=\{(x,y)\in\Sigma \mid L_{f^L}h\cdot L_{f^R}h>0\},$$

$$\Sigma^s :=\{(x,y)\in\Sigma \mid L_{f^L}h\cdot L_{f^R}h\leqslant 0\},$$

where $L_f h=\langle \mathrm{grad}(h), f\rangle$ is the Lie derivative of h with respect to the vector field f. For $(x,y)\in\Sigma^c$, since the normal components $L_{f^L}h$ and $L_{f^R}h$ have same sign, the orbit reaching (x,y) from G_1 (G_2) concatenates with the orbit entering G_2 (G_1). That is why we call Σ^c as crossing region. For $(x,y)\in\Sigma^s$, since the normal components $L_{f^L}h$ and $L_{f^R}h$ have opposite signs, so the orbit reaching (x,y) slides along the switching manifold Σ. By Filippov convex method[9], the sliding vector field f^S that governs sliding motion is given by

$$f^S=\lambda f^L+(1-\lambda)f^R \tag{5}$$

where

$$\lambda=\frac{L_{f^R}h}{L_{f^R}h-L_{f^L}h}.$$

Denoting the numerator of f^S by $p(x,y)$, *i.e.*,

$$p(x,y):=f^R\cdot L_{f^L}h-f^L\cdot L_{f^R}h.$$

In order to figure out dynamics of Filippov system, as classic odes, we have to analyze singular points. According to Filippov's approach[11], local singularity is caused by the vanishing of some functions such as f^L, f^R, h, p, $L_{f^L}h$, $L_{f^R}h$ for system (1). Point $(x,y)\in G_L(G_R)$ is called an admissible equilibrium if $f^L=0$ (resp. $f^R=0$) is satisfied at that point and called a virtual equilibrium if $f^R=0$ (resp. $f^L=0$) is satisfied at that point. Point $(x,y)\in\Sigma$ is called an admissible pseudo-equilibrium (resp. a virtual pseudo-equilibrium) if $L_{f^L}h\cdot L_{f^R}h\neq 0$, $f^R\cdot L_{f^L}h-f^L\cdot L_{f^R}h=0$ are satisfied at $(x,y)\in\Sigma^s$ (resp. $(x,y)\in\Sigma^c$). Point $(x,y)\in\Sigma$ is called a boundary equilibrium if $f^L=0$ or $f^R=0$ and called a tangent point if $L_{f^L}h\cdot L_{f^R}h=0$ and $f^L\neq 0, f^R\neq 0$. Moreover, if at point $(x,y)\in\Sigma$ we have

$$L_{f^L}h=0,\ L^2_{f^L}h\neq 0,\ L_{f^R}h\neq 0$$

and call it a fold. A fold is visible if $L^2_{f^L}h<0$ and invisible if $L^2_{f^L}h>0$. Similar definitions hold for the vector field f^R. If at point $(x,y)\in\Sigma$ we have

$$L_{f^L}h=L^2_{f^L}h=0,\ L^3_{f^L}h\neq 0,\ L_{f^R}h\neq 0 \tag{6}$$

we call it a cusp or a double tangent point. Similar definitions hold for the vector field f^R.

For system (4), we have $\mathrm{grad}(h)=(1,-1)^T$, so that

$$L_{f^L}h=v-x^2+y, L_{f^R}h=-1,$$

where

$$f^L(x,y;v)=(v-x^2,-y)^T,$$

$$f^R(x,y;v)=(-1,0)^T,$$

$$h(x,y;\rho)=x-y-\rho.$$

According to definition of sliding region Σ^s, we obtain it by solving inequality $v-x^2+y\geqslant 0$. Let

$$K_{1,2}:=1\pm\sqrt{1-4(\rho-v)}.$$

By a straight calculation we get $\Sigma^s=\varnothing$ if $v<\rho-1/4$ and

$$\Sigma^s=\left\{(x,y)\in\mathbf{R}^2 \mid \frac{K_2}{2}\leqslant x\leqslant\frac{K_1}{2},\ x-y-\rho=0\right\} \tag{7}$$

if $v\geqslant\rho-1/4$. The boundary of the sliding region is

$$\partial\Sigma^s=\left\{(x,y)\in\mathbf{R}^2 \mid x=\frac{K_{1,2}}{2}, x-y-\rho=0\right\},$$

which consists of boundary equilibria and tangent points. By (5) we have

$$f^S=\frac{y}{v-x^2+y+1}\begin{pmatrix}-k\\-1\end{pmatrix},$$

from which we get $f^S=0$ if and only if $(x,y)=(\rho,0)$. By (7), we have the point $(\rho,0)$ is an admissible pseudo-equilibrium if and only if

$$\frac{1-\sqrt{1-4(\rho-v)}}{2}<\rho<\frac{1+\sqrt{1-4(\rho-v)}}{2},$$

$$v>\rho-1/4,$$

i.e., $v>\rho^2$.

3 Proof of the main result

In order to prove Theorem 1.1, we give some lemmas firstly. By straight calculation, the left half vector field $f^L(x,y;v)$ has no equilibria if $v<0$ and has equilibria $(\pm\sqrt{v},0)$ if $v\geqslant 0$. Furthermore, since the Jacobi matrix of $f^L(x,y;v)$ at $(\pm\sqrt{v},0)$ are

$$\begin{pmatrix}\mp 2\sqrt{v} & 0\\ 0 & -1\end{pmatrix},$$

f^L has a stable node $(\sqrt{v},0)$ and a saddle $(-\sqrt{v},0)$ for $v>0$ and a nonhyperbolic equilibrium $(0,0)$ for $v=0$.

Lemma 3.1 In the case $v>0$, system (4) has a saddle and a stable-node if $v<\rho^2$ and $\rho>0$, a saddle if $v>\rho^2$, and no equilibria if $v<\rho^2, \rho<0$. In the case $v=0$, system (4) has a saddle-node if $\rho>0$ and no equilibria if $\rho<0$. In the case $v<0$, system (4) has no equilibria.

Proof　In the case $v>0$, f^L has a saddle $(-\sqrt{v},0)$ and a node $(\sqrt{v},0)$ if $v>\rho^2$, *i. e.*, $-\sqrt{v}<\rho<\sqrt{v}$. We get

$$h(-\sqrt{v},0;\rho)=-\sqrt{v}-\rho<0,$$
$$h(\sqrt{v},0;\rho)=\sqrt{v}-\rho>0.$$

It implies that $(-\sqrt{v},0)$ is an admissible equilibrium and $(\sqrt{v},0)$ is a virtual equilibrium. Therefore, system (4) has a saddle $(-\sqrt{v},0)$ for $v>\rho^2$. If $v<\rho^2$, we obtain

$$h(-\sqrt{v},0;\rho)=-\sqrt{v}-\rho<0,$$
$$h(\sqrt{v},0;\rho)=\sqrt{v}-\rho<0$$

for $\rho>\sqrt{v}>0$, and

$$h(-\sqrt{v},0;\rho)=-\sqrt{v}-\rho>0,$$
$$h(\sqrt{v},0;\rho)=\sqrt{v}-\rho>0$$

for $\rho<-\sqrt{v}<0$. Therefore, system (4) has a saddle and a stable node if $v<\rho^2, 0<\rho$, and no equilibria if $v<\rho^2, \rho<0$.

In the case $v=0$, f^L has a saddle-node $(0,0)$. Then $h(0,0;\rho)=-\rho<0$ if $\rho>0$ and $h(0,0;\rho)=-\rho>0$ if $\rho<0$. It implies that system (4) has a saddle-node for $\rho>0$, no equilibria for $\rho<0$.

In the case $v<0$, system (4) has no equilibria because f^L has no equilibria.

Notice that if $v=\rho^2$, there is an equilibrium $(\rho,0)$ of f^L on Σ, so that $f^L=0$ and $h=0$ at $(\rho,0)$. This is to say, $(\rho,0)$ is a boundary equilibrium of system (4) if $v=\rho^2$.

In the following we analyze equilibria of system (4) at infinity. Using Poincaré transformations $x=1/z, y=u/z$, which change the infinity of the xy-plane to the u-axis of the uz-plane. We write system (4) as

$$\begin{cases}\dfrac{du}{d\tau}=u-uz-vuz^2,\\ \dfrac{dz}{d\tau}=z-vz^3\end{cases}\tag{8}$$

where $d\tau=dt/z$, and find that on the u-axis (8) has a unique unstable star node $A:(0,0)$ for each $(\rho,v)\in\mathbf{R}^2$. Thus system (4) has an unstable node at infinity I_A^+ on the positive part of x-axis, and a stable node at infinity I_A^- on the negative part of x-axis.

By Poincaré transformations $x=w/z$, $y=1/z$, which change the infinity of the xy-plane to the w-axis of the wz-plane, system (4) is transformed into

$$\begin{cases}\dfrac{dw}{d\tau}=wz-w^2+vz^2,\\ \dfrac{dz}{d\tau}=z^2\end{cases}\tag{9}$$

where $d\tau=dt/z$. Since the equilibrium at the origin of system (9) is degenerate, we use classic normal sector method to analyze it as in Ref. [10]. Let $P(w,z):=wz-w^2+vz^2, Q(w,z):=z^2$. By polar coordinates transformations $w=r\cos\theta$, $z=r\sin\theta$, we write system (9) in polar form and from which obtain an equation

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{H(\theta)}{G(\theta)},$$

where

$$G(\theta)=\sin\theta\cos^2\theta-v\sin^3\theta,$$
$$H(\theta)=\sin^3\theta+\sin\theta\cos^2\theta-\cos^3\theta+v\sin^2\theta\cos\theta.$$

As shown in Ref. [10], exceptional directions are determined by zeros of $G(\theta)$. It is not hard to check that $G(\theta)=0$ has exactly two real roots 0 and π if $v<0$, four real roots 0, $\pi/2$, π and $3\pi/2$ if $v=0$, and six real root 0, π, $\operatorname{arccot}(\pm\sqrt{v})$ and $\operatorname{arccot}(\pm\sqrt{v})+\pi$ if $v>0$.

Lemma 3.2　In the directions $\theta=0$, $\operatorname{arccot}(\sqrt{v})$, π and $\operatorname{arccot}(\sqrt{v})+\pi$, as $\tau\to+\infty$ or $\tau\to-\infty$, system (9) has a unique orbit approaching (0,0). In the directions $\theta=\pi/2$, $\operatorname{arccot}(-\sqrt{v})$, $3\pi/2$ and $\operatorname{arccot}(-\sqrt{v})+\pi$, as $\tau\to+\infty$ or $\tau\to-\infty$, system (9) has an infinite number of orbits approaching (0,0).

Proof　Since

$$G'(0)H(0)=G'(\pi)H(\pi)=-1,$$

by Theorem 3.7 of Chapter 2 of Ref. [10], system (9) has a unique orbit approaching (0,0) as $\tau\to+\infty$ or $\tau\to-\infty$ in the directions $\theta=0,\pi$. Then we investigate $\theta=\pi/2$, which only appear in the case $v=0$. By the transformation $\theta'=\theta-\pi/2$, we obtain that $G(\theta')=\cos\theta'\sin^2\theta'$. Since the Taylor expansion of $G(\theta')$ is

$$G(\theta')=\left[1-\frac{(\theta')^2}{2!}+h.o.t\right]\left[\theta'-\frac{(\theta')^3}{3!}+h.o.t\right]^2,$$

i.e., $G(\theta')=(\theta')^2+h.o.t$. As shown in Ref. [12], in direction $\theta=\pi/2$, system (9) either has an infinite number of orbits approaching (0,0) as $\tau\to+\infty$ or $\tau\to-\infty$, or no orbits approaching (0,0) as $\tau\to\pm\infty$. We obtain

$$\begin{cases}\dfrac{dw}{d\tau}=wz-w^2,\\ \dfrac{dz}{d\tau}=z^2\end{cases}\qquad(10)$$

in the case $v=0$, and system (10) has a solution $w=0$. Because of $dz/d\tau>0(z\neq0)$, we find an orbit leaving(0,0) as $\tau\to+\infty$ in direction $\theta=\pi/2$. Therefore, system (9) has an infinite number of orbits approaching (0,0) as $\tau\to-\infty$, in direction $\theta=\pi/2$. Similarly, we can prove the conclusion of $\theta=3\pi/2$ in this lemma.

Let

$\theta_1:=\operatorname{arccot}(\sqrt{v})\in(0,\pi/2)$,

$\theta_2:=\operatorname{arccot}(-\sqrt{v})\in(\pi/2,\pi)$,

$\theta_3:=\operatorname{arccot}(\sqrt{v})+\pi\in(\pi,3\pi/2)$,

$\theta_4:=\operatorname{arccot}(-\sqrt{v})+\pi\in(3\pi/2,2/\pi)$.

Clearly, $\theta_1=\operatorname{arccot}(\sqrt{v})$ implies $\cot\theta_1=\sqrt{v}$, since

$$G'(\theta)=\cos^3\theta-2\sin^2\theta\cos\theta-3v\sin^2\theta\cos\theta$$

and $\sin\theta_1>0, v>0$, we have

$$G'(\theta_1)=-\frac{2\sqrt{v}}{\sin\theta_1}(v+1)<0,$$

$$H(\theta_1)=\sin^3\theta_1(v+1)>0.$$

Since $G'(\theta_1)H(\theta_1)<0$, by theorem 3.7 of Chapter 2 of Ref. [10] system (9) has a unique orbit approaching (0,0) as $\tau\to+\infty$ or $\tau\to-\infty$ in direction $\theta=\theta_1$. We consider the case θ_2, in the following, which implies that $\cot\theta_2=-\sqrt{v}$. Similarly,

$$G'(\theta_2)=\frac{2\sqrt{v}}{\sin\theta_2}(v+1)>0,$$

$$H(\theta_2)=\sin^3\theta_2(v+1)<0,$$

where $\sin\theta_2>0$. Because $G'(\theta_2)H(\theta_2)<0$, in the directions $\theta_2=\operatorname{arccot}(-\sqrt{v})$ as $\tau\to+\infty$ or $\tau\to-\infty$, system (9) has an infinite number of orbits approaching (0,0). We can also prove the conclusion of $\theta=\theta_3$, $\theta=\theta_4$ in this lemma by the same way.

Lemma 3.3 For the case $v>\rho-1/4$, $v\neq\rho^2$, the fold $(K_1/2, K_1/2-\rho)$ of f^L is visible if $\rho>1/2$ $v<\rho^2$, and is invisible either $\rho<1/2$ or $\rho>1/2$, $v>\rho^2$; the fold $(K_2/2, K_2/2-\rho)$ of f^L is visible if $\rho<1/2$, $v<\rho^2$, and is invisible either $\rho>1/2$ or $\rho<1/2, \rho^2<v$. There is a visible fold $(1-\rho, 1-2\rho)$ for $v=\rho^2, \rho\neq1/2$, and a cusp $(1/2, 1/2-\rho)$ for $v=\rho-1/4, \rho\neq1/2$. There are no tangent points for $v<\rho-1/4$.

Proof Let $T_{1,2}$, T_3, T_4 be $(K_{1,2}/2, K_{1,2}/2-\rho)$, $(1-\rho, 1-2\rho)$, $(1/2, 1/2-\rho)$, respectively. We obtain that system (4) has tangent points $T_{1,2}$ for $v>\rho-1/4, v\neq\rho^2$, T_3 for $v=\rho^2, \rho\neq1/2$, and T_4 for $v=\rho-1/4$ by solving equation $L_{f^L}h=0$. Then we need to judge the sign of $L^2_{f^L}h$ at T_i, $i=1,\cdots,4$, where $L^2_{f^L}h=-2xv+2x^3-y$.

Firstly, we analyze the tangent point T_1 with conditions $v>\rho-1/4, v\neq\rho^2$. By straight calculation, we have

$$L^2_{f^L}h\,|_{T_1}=\frac{t^2}{2}-(\rho-\frac{1}{2})t,$$

where $t=\sqrt{1-4(\rho-v)}$. Since $L^2_{f^L}h\,|_{T_1}=0$ has two real roots $t=0, 2\rho-1$, if $\rho<1/2$ we obtain $L^2_{f^L}h\,|_{T_1}>0$ for $t>0$. This is to say, T_1 is an invisible fold of f^L. And if $\rho>1/2$, we have $L^2_{f^L}h\,|_{T_1}>0$ for $t>1-2\rho$ and $L^2_{f^L}h\,|_{T_1}<0$ for $0<t<1-2\rho$. Moreover, because

$$v<\rho^2\Rightarrow4v<4\rho^2\Rightarrow1-4\rho+4v<(1-2\rho)^2\Rightarrow\sqrt{1-4\rho+4v}=t<1-2\rho,$$

we get $L^2_{f^L}h\,|_{T_1}<0$ and T_1 is a visible fold for $v<\rho^2$. On the other hand, since

$$v>\rho^2\Rightarrow4v>4\rho^2\Rightarrow1-4\rho+4v>(1-2\rho)^2\Rightarrow\sqrt{1-4\rho+4v}=t>1-2\rho,$$

$L^2_{f^L}h\,|_{T_1}>0$ and T_1 is an invisible fold for $v>\rho^2$.

We can prove the conclusion of T_2 by the same strategy.

System (4) has a boundary equilibrium $(\rho,0)\in\partial\Sigma^s$ for $v=\rho^2$. Further, we obtain that another boundary of switching manifold $T_3:(1-\rho, 1-2\rho)$

is a tangent point if $v=\rho^2, \rho\neq 1/2$. The visibility of T_3 is determined by the sign of $L_{f^L}^2 h$. Since

$$L_{f^L}^2 h\mid_{T_3}=(1-2\rho)^2,$$

$L_{f^L}^2 h\mid_{T_3}>0$ for $\rho\neq 1/2$, i. e., T_3 is an invisible fold for $v=\rho^2, \rho\neq 1/2$. We have $L_{f^L}^2 h\mid_{T_4}=0$ if $v=\rho-1/4, \rho\neq 1/2$. Furthermore, by

$$L_{f^L}^3 h=-2v^2+8x^2v-6x^4+y,$$

we get $L_{f^L}^3 h=-2(\rho-1/2)^2<0$ at T_4. Therefore, according to (6) T_4 is a cusp for $v=\rho-1/4$, $\rho\neq 1/2$. Finally, since $\Sigma^s=$ for $v<\rho-1/4$, there is no tangent points.

Lemma 3.4　System (4) has no periodic orbits.

Proof　Suppose that $G_1:=\{(x,y)\mid h(x,y;v)<0\}$ and $G_2:=\{(x,y)\mid h(x,y;v)>0\}$, which denote left half plane and right half plane respectively. Based on the regions where periodic orbits exist, we need to consider three scenarios as follows.

At first, periodic orbits exist in whole G_1 or G_2 without sliding segment, *i. e.*, system (4) has standard periodic orbits. By well-known consequence of index theory, which figures out periodic orbits in a continuous, planar vector field must encircle at last one equilibrium. Since there are no equilibria in right half vector field f^R, standard periodic orbits exist in G_1 if they exist. As in the proof of Lemma 3.1, there are three scenarios about equilibria of system (4). By index theory we get that the standard periodic orbits of system (4) can not encircle a saddle. Since the number of hyperbolic equilibria inside a periodic orbit is odd, it cans not encircle a saddle and a node simultaneously. By Lemma 3.1, system (4) has a saddle-node (0,0) for $v=0, \rho>0$. Notice that $x\equiv 0$ is an orbit of system $\dot{x}=-x^2$, $\dot{y}=-y$. Thus, there is no periodic orbits surrounding the saddle-node (0,0) for system (4). Since the right half vector field $f^R=(-1,0)^T$, any orbit starting in G_1 can not go back to G_1 after entering into G_2. Therefore, system (4) has no periodic orbits existing in both G_1 and G_2.

Then, suppose that system (4) has a sliding periodic orbit Γ, i. e., $\Gamma\cap\Sigma^s\neq\varnothing$, and the interior of Γ is denoted by $int(\Gamma)\subset G_1$. Since $L_{f^R}h=-1<0$, we get $L_{f^L}h>0$ for each point on Σ^s. Therefore, orbits which start at $\Gamma\cap\Sigma^s$ are unique in the forward time, and leave Σ^s at a tangent point T. Since $f(T,I^-)\subset int(\Gamma)$ is bounded, where $f(T,I^-)$ denotes the negative half orbit which is started at the point T, the α-limit set A_T of $f(T,I)$ is either an equilibrium or a periodic orbit. Since system (4) has no standard periodic orbits in G_1, the α-limit set A_T of $f(T,I)$ is an equilibrium which can be a saddle, a node or a saddle-node, *i. e.*, the sliding periodic orbit Γ must surround an equilibrium. Notice that $x\equiv\sqrt{v}$ is an orbit of left half system, therefore, Γ can not encircles a node.

Finally, the results of saddle and saddle-node can be proved similarly.

Having Lemmas 3.1～3.4, in the following we give a proof for Theorem 1.1.

Proof of Theorem 1.1　We obtain boundary equilibrium bifurcation curve $v=\rho^2$ by Lemma 3.1. Indeed, if $\rho>0$, system (4) has a boundary node for $v=\rho^2$. As parameters ρ, v change, system (4) has a stable node for $v<\rho^2$ and a stable pseudo-node $(\rho,0)$ for $v>\rho^2$. If $\rho>0$, system (4) has a saddle $(-\sqrt{v},0)$ and a stable pseudo-node $(\rho,0)$ for $v>\rho^2$. Those two equilibria become a boundary saddle when $v=\rho^2$ and both vanish when $v<\rho^2$. By Lemma 3.3, we obtain double tangency bifurcation curve $v=\rho-1/4$. System (4) has a cusp$(1/2, 1/2-\rho)$, which divides into a visible fold and an invisible fold as parameters change into $v>\rho-1/4$. There are no tangent points for $v<\rho-1/4$. Thus, we obtain the boundary equilibrium bifurcation curves, the double tangency bifurcation curves and the saddle-node bifurcation curves in bifurcation diagram as shown in Fig. 1.

As shown in Lemma 3.2, f^L has an unstable node I_A^+ on the positive part of x-axis, a stable node I_A^- on the negative part of x-axis, and two degenerate equilibria $I_B^{\pm}$ on the y-axis. $I_A^{\pm}$, $I_B^{\pm}$ are shown in Fig. 3.

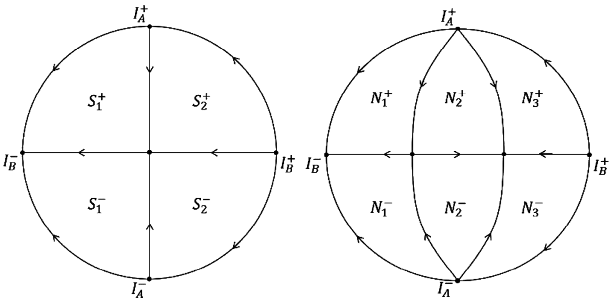

Fig. 3　Equilibria $I_A^\pm$, $I_B^\pm$ at infinity

If $v<0$, according to the direction of vector field and Lemma 3.2, we get that the orbits of left half system go into I_A^+ as $t\to-\infty$ and into I_A^- as $t\to+\infty$. If $v=0$, the orbit $f(p,I)$ goes into I_A^- as $t\to-\infty$ and into $I_B^+(I_B^-)$ as $t\to+\infty$, where $p\in S_1^+(S_1^-)$. Solving differential equation $\mathrm{d}x/\mathrm{d}y=x^2/y$, we obtain $y=C\mathrm{e}^{-1/x}$. Since

$$\lim_{x\to\infty} C\mathrm{e}^{-1/x}=C,$$

$f(p,I)$ enters into I_A^+ as $t\to-\infty$, where $p\in S_2^+(S_2^-)$. If $v>0$, by Lemma 3.2 we obtain the following results. Orbits starting from $N_1^+(N_1^-)$ go into I_A^- as $t\to+\infty$ and into $I_B^+(I_B^-)$ as $t\to-\infty$. Orbits starting from $N_2^+(N_2^-)$ go into $I_B^+(I_B^-)$ as $t\to-\infty$. Orbits starting from $N_3^\pm$ go into I_A^+ as $t\to-\infty$. According to these results, we obtain all global phase portraits on Poincaré disc as shown in Fig. 2.

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