GCD 封闭集上的幂矩阵行列式间的整除性

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摘 要：设 a,b,n 为正整数，$S=\{x_1,\cdots,x_n\}$ 是由 n 个不同正整数$x_1,\cdots,x_n$构成的集合. 以 (S^a) ($[S^a]$)表示 $n\times n$ 矩阵，其中第 i 行 j 列元为x_i和x_j的最大公因子(x_i,x_j)(最小公倍数 $[x_i,x_j]$)的 a 次幂. 本文给出以下结果：若 $a|b,n\leqslant 3$，则 $\det(S^a)|\det(S^b)$，$\det[S^a]|\det[S^b]$，$\det(S^a)|\det[S^b]$；若 $a|b,n\geqslant 4$，S 是n 个不同正整数构成的 $n-3$ 重最大公因子闭集，则 $\det(S^a)|\det(S^b)$，$\det[S^a]|\det[S^b]$，$\det(S^a)|\det[S^b]$；对任意正整数 $n\geqslant 4$，存在 $n-4$ 重最大公因子闭集 S，使得 $\det(S)\nmid\det(S^2)$，$\det[S]\nmid\det[S^2]$，$\det(S)\nmid\det[S^2]$. 所得结果加强和推广了 Hong 在 2003 年及 Chen 和 Hong 在 2020 年得到的结果.

关键词：整除；最大公因子幂矩阵；最小公倍数幂矩阵；最大公因子闭集；r 重最大公因子闭集

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Divisibility among determinants of power matrices on gcd-closed sets

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Abstract: Let a,b,n be positive integers and $S=\{x_1,\cdots,x_n\}$ be a set of n distinct positive integers. We denote by (S^a) (resp. $[S^a]$) the $n\times n$ matrix having the ath power of the greatest common divisor (resp. the least common multiple) of x_i and x_j as its (i,j)-entry. In this paper, we prove the following results: If $a|b$ and $n\leqslant 3$, then $\det(S^a)|\det(S^b)$, $\det[S^a]|\det[S^b]$, $\det(S^a)|\det[S^b]$; If $a|b,n\geqslant 4$ and S is an$(n-3)$-fold gcd-closed set of n distinct positive integers, then $\det(S^a)|\det(S^b)$, $\det[S^a]|\det[S^b]$, $\det(S^a)|\det[S^b]$; For any integer $n\geqslant 4$, there is an $(n-4)$-fold gcd-closed set$S$ of n distinct positive integers such that $\det(S)\nmid\det(S^2)$, $\det[S]\nmid\det[S^2]$, $\det(S)\nmid\det[S^2]$. Our results extend Hong's theorem in 2003 and the theorems of Chen and Hong in 2020.

Keywords: Divisibility; Power GCD matrix; Power LCM matrix; GCD-closed set; r-fold gcd-closed set
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1 Introduction

Throughout this paper, we denote by (x,y) (resp. $[x,y]$) the greatest common divisor (resp. least common multiple) of integers x and y. Let $\mathbf{Z}$ denote the set of integers and $|T|$ stand for the cardinality of a finitie set T of integers. Let f be an arithmetical function and $S=\{x_1,\cdots,x_n\}$ be a set of n distinct positive integers. Let $(f(x_i,x_j))$ (abbreviated by $(f(S))$) denote the $n\cdot n$ matrix having f evaluated at the greatest common divisor (x_i,x_j) of x_i and x_j as its (i,j)-entry. Let $(f[x_i,x_j])$ (abbreviated by $(f[S])$) denote the $n\cdot n$ matrix having f evaluated at the least common multiple $[x_i,x_j]$ of x_i and x_j as its (i,j)-entry. Let ξ_a be the arithmetical function defined by $\xi_a = x^a$ for any positive integer x, where a is a positive integer. The $n\cdot n$ matrix $(\xi_a(x_i,x_j))$ (abbreviated by (S^a)) and $(\xi_b[x_i,x_j])$ (abbreviated by $[S^b]$) are called power GCD matrix and power LCM matrix, respectively. A set S is called *factor closed* (FC) if the conditions $x\in S$ and $d|x$ imply that $d\in S$. We say that the set S is *gcd closed* if $(x_i,x_j)\in S$ for all $1\leqslant i,j\leqslant n$. Evidently, any FC set is gcd closd but not conversely.

In 1875, Smith[1] showed that

$$\det(f(i,j)) = \prod_{k=1}^{n}(f*\mu)(k),$$

where μ is the Möbius function and $f*\mu$ is the Dirichlet convolution of f and μ. Apostol[2] extended Smith's result by showing that if f and g are arithmetical functions and if β is defined for positive integers t and r by

$$\beta(t,r) = \sum_{d|(t,r)} f(d)g\left(\frac{r}{d}\right),$$

then

$$\det(\beta(i,j)) = g(1)^n f(1)\cdots f(n).$$

He noted that, as a consequence, $\det(C(i,j)) = n!$, where $C(t,r)$ is Ramanujan's sum. McCarthy[3] generalized Smith's and Apostol's results to the class of even functions (mod r). He evaluated the determinants of $n\cdot n$ matrices of the form $(\beta(i,j))$, where $\beta(t,r)$ is an even function of t (mod r). A complex-valued function $\beta(t,r)$ of the positive integral variables t and r is said to be an even function of t (mod r) if $\beta(t,r)=\beta((t,r),r)$ for all values of t. The functions considered by Smith and Apostol are even functions of t (mod r) for every r. Bourque and Ligh[4] evaluated the determinants of $n\cdot n$ matrices of the form $\beta(x_i,x_j)$, where the set S is FC and $\beta(t,r)$ is an even function of t (mod r). In 1993, Bourque and Ligh[5] extended the Beslin-Ligh result[6] and Smith's theorem by showing that the determinant of the matrix $(\xi_a(x_i,x_j))$ defined on a gcd-closed set $S=\{x_1,\cdots,x_n\}$ is equal to $\prod_{x\in S}\alpha_{S,\xi_a}(x)$, where

$$\alpha_{S,\xi_a}(x) = \sum_{\substack{d|x \\ d\notin E_S(x)}} (\xi_a*\mu)(d)$$

and

$$E_S(x):=\{z\in\mathbf{Z}^+ : \exists\, y\in S, y<x, z|y\}.$$

Divisibility is one of the most important topics in the field of Smith matrices. Bourque and Ligh[7] showed that if S is FC, then $(S)|[S]$ holds in the ring $M_n(\mathbf{Z})$ of $n\times n$ matrices over the integers. That is, there exists a matrix $A\in M_n(\mathbf{Z})$ such that $[S]=(S)A$ or $[S]=A(S)$. Hong[8] showed that such factorization is no longer true in general if S is gcd closed. Let $x,y\in S$ with $x<y$. If $x|y$ and the conditions $x|d|y$ and $d\in S$ imply that $d\in\{x,y\}$, we say that x is a *greatest-type divisor* of y in S. For $x\in S$, $G_S(x)$ stands for the set of all greatest-type divisors of x in S. The concept of greatest-type divisor was introduced by Hong and played a key role in his solution Ref. [9] of the Bourque-Ligh conjecture Ref. [7]. By Ref. [8] we know that there are gcd-closed sets S with $\max_{x\in S}\{|G_S(x)|\}=2$ such that $(S)^{-1}[S]\notin M_n(\mathbf{Z})$. In 2008, Hong *et al.*[10] confirmed Conjecture 3.1 of Ref. [11] by constructing an integer matrix equal to the product $[S](S)^{-1}$ if S is a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\}=1$. The set S is called a divisor chain if there exists a permutation σ on $\{1,\ldots,n\}$ such that $x_{\sigma(1)}|\ldots|x_{\sigma(n)}$. Obviously, a divisor chain is gcd closed but the converse is not true.

Definition 1.1[12] Let T be a set of n distinct positive integers and $1 \leqslant r \leqslant n-1$ be an integer. We say that T is 0-fold gcd closed if T is gcd-closed. We say that T is *r-fold gcd closed* if there is a divisor chain $R \subset T$ with $|R|=r$ such that $\max(R) \mid \min(T \backslash R)$ and the set $T \backslash R$ is gcd closed.

Note that Definition 1. 1 is stated in a different way from that given in Ref. [12]. However, they are equivalent. It is easy to see that an r-fold gcd closed set is $(r-1)$-fold gcd closed, but the converse is not necessarily true. Hong[12] proved that the Bourque-Ligh conjecture is true when $n \leqslant 5$ and if $n \geqslant 6$ then the LCM matrix $[S]$ defined on any $(n-5)$-fold gcd-closed set S is nonsingular. In 2005, Zhou and Hong[13] considered the divisibility among power GCD and power LCM matrices for unique factorization domains. On the other hand, Hong[14] initiated the study of the divisibility properties among power GCD matrices and among power LCM matrices. Tan and Lin[15] studied the divisibility of determinants of power GCD matrices and power LCM matrices on finitely many quasi-coprime divisor chains.

In this paper, our main goal is to study the divisibility among the determinants of power matrices (S^a) and (S^b), among the determinants of power matrices $[S^a]$ and $[S^b]$ and among the determinants of power matrices (S^a) and $[S^b]$. The main result of this paper can be stated as follows.

Theorem 1. 2 Let a and b be positive integers with $a \mid b$ and let $n \geqslant 1$ be an integer.

(i) If $n \leqslant 3$, then for any gcd-closed set S of n distinct positive integers, one has $\det(S^a) \mid \det(S^b)$, $\det[S^a] \mid \det[S^b]$, $\det(S^a) \mid \det[S^b]$.

(ii) If $n \geqslant 4$, then for any $(n-3)$-fold gcd-closed set S of n distinct positive integers, one has $\det(S^a) \mid \det(S^b)$, $\det[S^a] \mid \det[S^b]$ and $\det(S^a) \mid [S^b]$.

(iii) For any integer $n \geqslant 4$, there is an $(n-4)$-fold gcd-closed set S of n distinct positive integers such that $\det(S) \nmid \det(S^2)$, $\det[S] \nmid \det[S^2]$, $\det(S) \nmid \det[S^2]$.

Evidently, Theorem 1. 2 extends Hong's theorem[16] obtained in 2003 and the theorems of Chen and Hong[17] gotten in 2020.

Throughout this paper, a and b stand for positive integers. We always assume that the set $S=\{x_1, \cdots, x_n\}$ satisfies that $x_1<\cdots<x_n$. This paper is organized as follows. In Section 2, we provide some lemmas needed in the proof of our main result. Section 3 is devoted to the proof of Theorem 1. 2.

2 Auxiliary results

In this section, we supply two lemmas that will be needed in the proof of Theorem 1. 2. We begin with a result due to Hong which gives the determinant formulas of a power GCD matrix and a power LCM matrix on a gcd-closed set.

Lemma 2. 1[18] Let $S=\{x_1, \ldots, x_n\}$ be a gcd-closed set. Then

$$\det(S^a)=\prod_{x \in S} \sum_{J \subseteq G_S(x)}(-1)^{|J|}(\gcd(J \cup\{x\}))^a$$

and

$$\det[S^a]=\prod_{x \in S} x^{2a} \sum_{J \subseteq G_S(x)} \frac{(-1)^{|J|}}{(\gcd(J \cup\{x\}))^a}.$$

We can now use Hong's formulae to deduce the formulae for $\det(S^a)$ and $\det[S^a]$ when S is a divisor chain.

Lemma 2. 2[17] Let $S=\{x_1, \ldots, x_n\}$ be a divisor chain such that $x_1 | \ldots | x_n$ and $n \geqslant 2$. Then

$$\det(S^a)=x_1^a \prod_{i=2}^n (x_i^a-x_{i-1}^a),$$

$$\det[S^a]=(-1)^{n-1} x_n^a \prod_{i=2}^n (x_i^a-x_{i-1}^a).$$

3 Proof of Theorem 1. 2

In this section, we use the lemmas presented in previous section to show Theorem 1. 2.

First, we prove part (i) as follows.

Let $n=1$. It is clear that the statement is true.

Let $n=2$. Since $S=\{x_1, x_2\}$ is gcd closed, we have $(x_1, x_2)=x_1$ and $x_1 \mid x_2$. It then follows that

$$\frac{\det(S^b)}{\det(S^a)}=\frac{\det\begin{pmatrix} x_1^b & x_1^b \\ x_1^b & x_2^b \end{pmatrix}}{\det\begin{pmatrix} x_1^a & x_1^a \\ x_1^a & x_2^a \end{pmatrix}}=x_1^{b-a} \cdot \frac{x_2^b-x_1^b}{x_2^a-x_1^a} \in \mathbf{Z},$$

$$\frac{\det[S^b]}{\det[S^a]}=\frac{\det\begin{pmatrix}x_1^b & x_2^b\\ x_2^b & x_2^b\end{pmatrix}}{\det\begin{pmatrix}x_1^a & x_2^a\\ x_2^a & x_2^a\end{pmatrix}}=x_2^{b-a}\cdot\frac{x_1^b-x_2^b}{x_1^a-x_2^a}\in\mathbf{Z}$$

and

$$\frac{\det[S^b]}{\det(S^a)}=\frac{\det\begin{pmatrix}x_1^b & x_2^b\\ x_2^b & x_2^b\end{pmatrix}}{\det\begin{pmatrix}x_1^a & x_1^a\\ x_1^a & x_2^a\end{pmatrix}}=$$

$$x_2^{b-a}\left(\frac{x_2}{x_1}\right)^a\cdot\frac{x_1^b-x_2^b}{x_2^a-x_1^a}\in\mathbf{Z}.$$

The statement is true for this case.

Let $n=3$. Since $S=\{x_1,x_2,x_3\}$ is gcd closed, we have $x_1|x_i(i=2,3)$ and $(x_2,x_3)=x_1$ or x_2. Consider the following two cases:

Case 1　$(x_2,x_3)=x_1$. Then

$$\frac{\det(S^b)}{\det(S^a)}=\frac{\det\begin{pmatrix}x_1^b & x_1^b & x_1^b\\ x_1^b & x_2^b & x_1^b\\ x_1^b & x_1^b & x_3^b\end{pmatrix}}{\det\begin{pmatrix}x_1^a & x_1^a & x_1^a\\ x_1^a & x_2^a & x_1^a\\ x_1^a & x_1^a & x_3^a\end{pmatrix}}=$$

$$x_1^{b-a}\cdot\frac{x_3^b-x_1^b}{x_3^a-x_1^a}\cdot\frac{x_2^b-x_1^b}{x_2^a-x_1^a}\in\mathbf{Z},$$

$$\frac{\det[S^b]}{\det(S^a)}=\frac{\det\begin{pmatrix}x_1^b & x_2^b & x_3^b\\ x_2^b & x_2^b & \frac{x_2^bx_3^b}{x_1^b}\\ x_3^b & \frac{x_2^bx_3^b}{x_1^b} & x_3^b\end{pmatrix}}{\det\begin{pmatrix}x_1^a & x_1^a & x_1^a\\ x_1^a & x_2^a & x_1^a\\ x_1^a & x_1^a & x_3^a\end{pmatrix}}=$$

$$x_3^{b-a}\cdot\left(\frac{x_2}{x_1}\right)^b\cdot\left(\frac{x_3}{x_1}\right)^a\cdot\frac{x_1^b-x_2^b}{x_1^a-x_2^a}\cdot\frac{x_1^b-x_3^b}{x_1^a-x_3^a}\in\mathbf{Z}$$

and

$$\frac{\det[S^b]}{\det[S^a]}=\frac{\det\begin{pmatrix}x_1^b & x_2^b & x_3^b\\ x_2^b & x_2^b & \frac{x_2^bx_3^b}{x_1^b}\\ x_3^b & \frac{x_2^bx_3^b}{x_1^b} & x_3^b\end{pmatrix}}{\det\begin{pmatrix}x_1^a & x_2^a & x_3^a\\ x_2^a & x_2^a & \frac{x_2^ax_3^a}{x_1^a}\\ x_3^a & \frac{x_2^ax_3^a}{x_1^a} & x_3^a\end{pmatrix}}=$$

$$x_3^{b-a}\cdot\left(\frac{x_2}{x_1}\right)^{b-a}\cdot\frac{x_1^b-x_2^b}{x_1^a-x_2^a}\cdot\frac{x_1^b-x_3^b}{x_1^a-x_3^a}\in\mathbf{Z}.$$

The statement is true for this case.

Case 2　$(x_2,x_3)=x_2$. Then $x_2|x_3$. It follows that

$$\frac{\det(S^b)}{\det(S^a)}=\frac{\det\begin{pmatrix}x_1^b & x_1^b & x_1^b\\ x_1^b & x_2^b & x_2^b\\ x_1^b & x_2^b & x_3^b\end{pmatrix}}{\det\begin{pmatrix}x_1^a & x_1^a & x_1^a\\ x_1^a & x_2^a & x_2^a\\ x_1^a & x_2^a & x_3^a\end{pmatrix}}=$$

$$x_1^{b-a}\cdot\frac{x_3^b-x_2^b}{x_3^a-x_2^a}\cdot\frac{x_2^b-x_1^b}{x_2^a-x_1^a}\in\mathbf{Z},$$

$$\frac{\det[S^b]}{\det[S^a]}=\frac{\det\begin{pmatrix}x_1^b & x_2^b & x_3^b\\ x_2^b & x_2^b & x_3^b\\ x_3^b & x_3^b & x_3^b\end{pmatrix}}{\det\begin{pmatrix}x_1^a & x_2^a & x_3^a\\ x_2^a & x_2^a & x_3^a\\ x_3^a & x_3^a & x_3^a\end{pmatrix}}=$$

$$x_3^{b-a}\cdot\frac{x_3^b-x_2^b}{x_3^a-x_2^a}\cdot\frac{x_2^b-x_1^b}{x_2^a-x_1^a}\in\mathbf{Z}$$

and

$$\frac{\det[S^b]}{\det(S^a)}=\frac{\det\begin{pmatrix}x_1^b & x_2^b & x_3^b\\ x_2^b & x_2^b & x_3^b\\ x_3^b & x_3^b & x_3^b\end{pmatrix}}{\det\begin{pmatrix}x_1^a & x_1^a & x_1^a\\ x_1^a & x_2^a & x_2^a\\ x_1^a & x_2^a & x_3^a\end{pmatrix}}=$$

$$x_3^{b-a}\left(\frac{x_3}{x_1}\right)^a\frac{x_3^b-x_2^b}{x_3^a-x_2^a}\cdot\frac{x_2^b-x_1^b}{x_2^a-x_1^a}\in\mathbf{Z}.$$

The statement is true for this case. Part (i) is proved.

Consequently, we prove part (ii). First of all, any $(n-3)$-fold gcd-closed set S must satisfy either $x_1|x_2|\cdots|x_{n-3}|x_{n-2}|x_{n-1}|x_n$, or $x_1|x_2|\cdots|x_{n-3}|x_{n-2}$ and $\gcd(x_n,x_{n-1})=x_{n-2}$.

Case a　S is a divisor chain. That is, $x_1|x_2|\cdots|x_{n-3}|x_{n-2}|x_{n-1}|x_n$. Then by Lemma 2.2, one deduces that

$$\frac{\det(S^b)}{\det(S^a)}=x_1^{b-a}\prod_{i=2}^{n}\left(\frac{x_i^b-x_{i-1}^b}{x_i^a-x_{i-1}^a}\right)\in\mathbf{Z},$$

$$\frac{\det[S^b]}{\det[S^a]}=x_n^{b-a}\prod_{i=2}^{n}\left(\frac{x_i^b-x_{i-1}^b}{x_i^a-x_{i-1}^a}\right)\in\mathbf{Z}$$

and

$$\frac{\det[S^b]}{\det(S^a)}=\frac{x_n^b}{x_1^a}\prod_{i=2}^{n}\left(\frac{x_i^b-x_{i-1}^b}{x_i^a-x_{i-1}^a}\right)\in \mathbf{Z}.$$

The statement is true for this case.

Case b $x_1 \mid x_2 \mid \cdots \mid x_{n-3} \mid x_{n-2}$ and $\gcd(x_n, x_{n-1})=x_{n-2}$. By Lemma 2.1, one has

$$\det(S^a)=\prod_{i=1}^{n}\sum_{J\subseteq G_S(x_i)}(-1)^{|J|}(\gcd(J\cup\{x_i\}))^a=x_1^a(x_n^a-x_{n-2}^a)(x_{n-1}^a-x_{n-2}^a)\prod_{i=2}^{n-2}(x_i^a-x_{i-1}^a)$$

and

$$\det[S^a]=\prod_{i=1}^{n}x_i^{2a}\sum_{J\subseteq G_S(x_i)}\frac{(-1)^{|J|}}{(\gcd(J\cup\{x_i\}))^a}=x_1^a x_{n-1}^{2a}\left(\frac{1}{x_{n-1}^a}-\frac{1}{x_{n-2}^a}\right)x_n^{2a}\left(\frac{1}{x_n^a}-\frac{1}{x_{n-2}^a}\right)\cdot\prod_{i=2}^{n-2}x_i^{2a}\left(\frac{1}{x_i^a}-\frac{1}{x_{i-1}^a}\right)=x_1^a\left(\frac{x_{n-1}}{x_{n-2}}\right)^a\cdot(x_{n-2}^a-x_{n-1}^a)\left(\frac{x_n}{x_{n-2}}\right)^a(x_{n-2}^a-x_n^a)\cdot\prod_{i=2}^{n-2}\left(\frac{x_i}{x_{i-1}}\right)^a(x_{i-1}^a-x_i^a).$$

Then

$$\frac{\det(S^b)}{\det(S^a)}=x_1^{b-a}\cdot\frac{x_n^b-x_{n-2}^b}{x_n^a-x_{n-2}^a}\cdot\frac{x_{n-1}^b-x_{n-2}^b}{x_{n-1}^a-x_{n-2}^a}\cdot\prod_{i=2}^{n-2}\left(\frac{x_i^b-x_{i-1}^b}{x_i^a-x_{i-1}^a}\right)\in\mathbf{Z},$$

$$\frac{\det[S^b]}{\det[S^a]}=x_1^{b-a}\left(\frac{x_{n-1}}{x_{n-2}}\right)^{b-a}\cdot\frac{x_{n-2}^b-x_{n-1}^b}{x_{n-2}^a-x_{n-1}^a}\left(\frac{x_n}{x_{n-2}}\right)^{b-a}\cdot\frac{x_{n-2}^b-x_n^b}{x_{n-2}^a-x_n^a}\cdot\prod_{i=2}^{n-2}\left(\frac{x_i}{x_{i-1}}\right)^{b-a}\left(\frac{x_{i-1}^b-x_i^b}{x_{i-1}^a-x_i^a}\right)\in\mathbf{Z}$$

and

$$\frac{\det[S^b]}{\det(S^a)}=x_1^{b-a}\left(\frac{x_{n-1}}{x_{n-2}}\right)^b\cdot\frac{x_{n-2}^b-x_{n-1}^b}{x_{n-1}^a-x_{n-2}^a}\cdot\left(\frac{x_n}{x_{n-2}}\right)^b\cdot\frac{x_{n-2}^b-x_n^b}{x_n^a-x_{n-2}^a}\cdot\prod_{i=2}^{n-2}\left(\frac{x_i}{x_{i-1}}\right)^b\left(\frac{x_{i-1}^b-x_i^b}{x_i^a-x_{i-1}^a}\right)\in\mathbf{Z}.$$

The statement is true for this case.

Finally, we prove part (iii). Let $n\geqslant 4$ be an integer, $a=1, b=2$ and

$$x_k=3^{k-1}, 1\leqslant k\leqslant n-3,$$

$$x_{n-2}=2\cdot 3^{n-4}, x_{n-1}=7\cdot 3^{n-4}, x_n=28\cdot 3^{n-4}.$$

By Definition 1.1, one knows that S is $(n-4)$-fold gcd closed. By Lemma 2.1, one has

$$\det(S^a)=x_1^a(x_{n-2}^a-x_{n-3}^a)(x_{n-1}^a-x_{n-3}^a)(x_n^a-x_{n-1}^a-x_{n-2}^a+x_{n-3}^a)\prod_{i=2}^{n-3}(x_i^a-x_{i-1}^a)=5\cdot 2^{n-1}\cdot 3^{\frac{n^2-3n-2}{2}},$$

$$\det(S^b)=61\cdot 2^{3n-6}\cdot 3^{n^2-3n-1},$$

$$\det[S^a]=x_1^a x_{n-1}^{2a}\left(\frac{1}{x_{n-1}^a}-\frac{1}{x_{n-3}^a}\right)x_{n-2}^{2a}\left(\frac{1}{x_{n-2}^a}-\frac{1}{x_{n-3}^a}\right)\cdot x_n^{2a}\left(\frac{1}{x_n^a}-\frac{1}{x_{n-1}^a}-\frac{1}{x_{n-2}^a}+\frac{1}{x_{n-3}^a}\right)\cdot\prod_{i=2}^{n-3}x_i^{2a}\left(\frac{1}{x_i^a}-\frac{1}{x_{i-1}^a}\right)=\left(\frac{x_{n-1}}{x_{n-3}}\right)^a\cdot(x_{n-3}^a-x_{n-1}^a)\left(\frac{x_{n-2}}{x_{n-3}}\right)^a(x_{n-3}^a-x_{n-2}^a)\cdot x_n^{2a}\left(\frac{1}{x_n^a}-\frac{1}{x_{n-1}^a}-\frac{1}{x_{n-2}^a}+\frac{1}{x_{n-3}^a}\right)\cdot\prod_{i=2}^{n-3}\left(\frac{x_i}{x_{i-1}}\right)^a(x_{i-1}^a-x_i^a)=(-1)^{n-4}\cdot 7^2\cdot 11\cdot 2^n\cdot 3^{\frac{n^2-n-10}{2}},$$

and

$$\det[S^b]=(-1)^{n-4}\cdot 191\cdot 7^4\cdot 2^{3n-2}\cdot 3^{n^2-n-9}.$$

Then we can compute and obtain that

$$\frac{\det(S^2)}{\det(S)}=61\cdot 2^{2n-5}\cdot 3^{\frac{n^2-3n}{2}}\cdot\frac{1}{5}\notin\mathbf{Z},$$

$$\frac{\det[S^2]}{\det[S]}=191\cdot 7^2\cdot 2^{2n-2}\cdot 3^{\frac{n^2-n-8}{2}}\cdot\frac{1}{11}\notin\mathbf{Z},$$

and

$$\frac{\det[S^2]}{\det(S)}=(-1)^{n-4}\cdot 191\cdot 7^4\cdot 2^{2n-1}\cdot 3^{\frac{n^2+n-16}{2}}\cdot\frac{1}{5}\notin\mathbf{Z}.$$

Part (iii) is proved. This finishes the proof of Theorem 1.2.

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