一类广义 Schur 型多项式的不可约性

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摘 要：设 n 为一个正整数，$a_1,\cdots,a_n$ 均为整数. Schur 通过素理想分解证明：当 $a_n=\pm1$ 时，多项式 $1+a_1x+a_2\frac{x^2}{2!}+\cdots+a_{n-1}\frac{x^{n-1}}{(n-1)!}+a_n\frac{x^n}{n!}$ 是不可约多项式. 随后，Coleman 利用 p-adic Newton 多边形重新证明了 Schur 的结果. 本文研究了一类特殊的广义 Schur 型多项式 $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots+\frac{x^{p^a}}{p^a(p^a-1)}$ 的不可约性. 借助 p-adic Newton 多边形工具，本文基于局部整体原则证明：当 p 为素数，a 为正整数时该多项式在有理数域上不可约.

关键词：不可约；p-adic 牛顿多边形；局部整体原则
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On the irreducibility of a class of generalized Schur-type polynomials

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Abstract: Let n be a positive integer and $a_1,\cdots,a_n\in\mathbf{Z}$. Schur proved that the polynomial $1+a_1x+a_2\frac{x^2}{2!}+\cdots+a_{n-1}\frac{x^{n-1}}{(n-1)!}+a_n\frac{x^n}{n!}$ is irreducible over $\mathbf{Q}$ by using the factorization of prime ideal, where $a_n=\pm1$. Then Coleman reproved Schur's result by using the method of p-adic Newton polygon. In this paper, we study the irreducibility of the generalized Schur-type polynomial $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots+\frac{x^{p^a}}{p^a(p^a-1)}$. By using the tool of p-adic Newton polygon and applying the local-global principle, we prove the irreducibility of this polynomial, where p is a prime number and a is a positive integer.

Keywords: Irreducibility; p-adic Newton polygon; Local-global principle
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1 Introduction

Let $\mathbf{Z}$ and $\mathbf{Q}$ denote the ring of integers and the field of rational numbers respectively. The so-called Schur-type polynomial is a polynomial $f(x)$ of the following form:

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$$f(x) := 1 + a_1 x + a_2 \frac{x^2}{2!} + \cdots + a_{n-1} \frac{x^{n-1}}{(n-1)!} \pm \frac{x^n}{n!} \quad (1)$$

where $n \in \mathbf{Z}^+$ and $a_i \in \mathbf{Z}$. If $a_i = 1$ for all $1 \leqslant i \leqslant n-1$ and the positive sign is taken for the term $\frac{x^n}{n!}$, then (1) becomes the n-th truncated exponential Taylor polynomial $e_n(x) := \sum_{i=0}^{n} \frac{x^i}{i!}$. In 1929, Schur proved that any Schur-type polynomial is irreducible over **Q**. He also computed the Galois group of $e_n(x)$ over **Q**. Coleman[1] reproved Schur's result by the p-adic Newton polygon. We call the following polynomial a generalized Schur-type polynomial if it has the form

$$f(x) = 1 + a_1 x + a_2 \frac{x^2}{2!} + \cdots + a_{n-1} \frac{x^{n-1}}{(n-1)!} + a_n \frac{x^n}{n!} \quad (2)$$

where $a_i \in \mathbf{Z}$ for $1 \leqslant i \leqslant n$. On the irreducibility of (2), Filaseta[2] showed the following two results.

(i) If the leading coefficient of the generalized Schur-type polynomial (2) satisfies that $0 < |a_n| < n$, then $f(x)$ is irreducible over **Q** unless $a_n = \pm 5$ and $n = 6$ or $a_n = \pm 7$ and $n = 10$. In these cases, either $f(x)$ is irreducible or $f(x)$ equals to the product of two irreducible polynomials of the same degree.

(ii) If $|a_n| = n$, then either $f(x)$ is irreducible or $f(x)$ is $x \pm 1$ times an irreducible polynomial of degree $n-1$.

Meanwhile, Filaseta[3-5] also do some extension over the result of Schur.

Naturally, we may ask about the irreducibility of other kinds of generalized Schur-type polynomials. We may notice that given $a_i = i!$, $1 \leqslant i \leqslant n$, we have some new polynomials such as $f(x) = 1 + x + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$. In this case we simply know that $f(x)$ is irreducible over the field of rational numbers **Q** if and only if $n+1$ is a prime by the knowledge of cyclotomic field. Another example is that given $a_i = (i-1)!$, (2) recovers the n-th truncated polynomial of the Taylor expansion of $1 - \log(1-x)$ at the original point. Monsef and coworkers[6] proved that the polynomial $L(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}$ is irreducible over **Q** and further computed the Galois group of $L(x)$ for some special cases.

Motivated by these works, we in this paper consider a generalized Schur-type polynomial

$$f_n(x) := 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n(n-1)}$$

by setting $a_1 = 1$ and $a_i = (i-2)!$ for $2 \leqslant i \leqslant n$. Since $f'_n(x) = L_{n-1}(x)$, it is quite interesting to discuss the irreducibility of this polynomial. In fact, we obtain the following result.

Theorem 1.1 If n is a prime power, the polynomial $f_n(x)$ is irreducible over **Q**.

This paper is organized as follows. We present the definitions of p-adic valuation and p-adic Newton polygon, and introduce the main theorem of p-adic Newton polygon as well as some other preliminary lemmas in Section 2. In Section 3, we give the proof of Theorem 1.1. Finally, Section 4 is devoted to some concluding remarks.

2 Preliminaries

In this section we give some definitions and lemmas needed in the proof of Theorem 1.1.

Definition 2.1 The p-adic valuation of an integer m with respect to p, denoted by $v_p(m)$, is defined as

$$v_p(m) = \begin{cases} \max\{k : p^k \mid m\}, & m \neq 0, \\ \infty, & m = 0. \end{cases}$$

Clearly, we can extend Definition 2.1 to the rational field **Q** and the local field $\mathbf{Q}_p$.

We recall the definition of p-adic Newton polygons as follows.

Definition 2.2 The p-adic Newton polygon $NP_p(f)$ of a polynomial $f(x) = \sum_{j=0}^{n} c_j x^j \in Q[x]$ is the lower convex hull of the set of points $S_p(f) = \{(j, v_p(c_j)) \mid 0 \leqslant j \leqslant n\}$. It is the highest polygonal line passing on or below the points in $S_p(f)$. The vertices $(x_0, y_0), (x_1, y_1), \cdots, (x_r, y_r)$ where the slope of the Newton polygon changes are called the corners of $NP_p(f)$; their

x-coordinates ($0=x_0<x_1<\cdots<x_r=n$) are called the breaks of $NP_p(f)$; the lines connected two vertices are called the segments of $NP_p(f)$.

For a given polynomial, by the definition of lower convex hull, all points of $S_p(f)$ lies above $NP_p(f)$. In other words, although $S_p(f)$ contains all information of the coefficients of $f(x)$, $NP_p(f)$ reflects the arithmetic properties of all roots of $f(x)$ over the local field $\mathbf{Q}_p$.

We shall introduce the main theorem of the p-adic Newton polygon below. This theorem provides a rough factorizationof $f(x)$ over $\mathbf{Q}_p$.

Lemma 2.3[7] Let (x_0,y_0), (x_1,y_1), $\cdots$, (x_r,y_r) denote the successive vertices of $NP_p(f)$. Then there exist polynomials $f_1,\cdots,f_r$ in $\mathbf{Q}_p[x]$ such that

(i) $f(x)=f_1(x)f_2(x)\cdots f_r(x)$;

(ii) The degree of f_i is x_i-x_{i-1};

(iii) All the roots of f_i in $\overline{\mathbf{Q}_p}$ have p-adic valuations $-\dfrac{y_i-y_{i-1}}{x_i-x_{i-1}}$.

The following lemma is a generalization of the famous Eisenstein irreducibility criterion over $\mathbf{Q}_p$, which provides an upper bound for the number of irreducible factors of a polynomial over $\mathbf{Q}_p$ according to its p-adic Newton polygon. For the following lemma plays an important role in supporting of Theorem 1.1, we also give its proof in this paper.

Lemma 2.4[8] Let (x_{i-1},y_{i-1}) and (x_i,y_i) be two consecutive vertices of $NP_p(f)$, and let $d_i=\gcd(x_i-x_{i-1},y_i-y_{i-1})$. Then for each i, $f_i(x)$ has at most d_i irreducible factors in $\mathbf{Q}_p$ and the degree of the factors of $f_i(x)$ is a multiple of $\dfrac{x_i-x_{i-1}}{d_i}$. Particularly, if $d_i=1$, then $f_i(x)$ is irreducible over $\mathbf{Q}_p$.

Proof Let $x_i-x_{i-1}=u_i$ and $y_i-y_{i-1}=v_i$. By Lemma 2.3, we have $\deg f_i=u_i$ and all the roots of $f_i(x)$ in $\overline{\mathbf{Q}_p}$ have p-adic valuation $-\dfrac{v_i}{u_i}$. Let $h(x)\in\mathbf{Q}_p[x]$ with $\deg h(x)=t$ such that $h(x)\mid f_i(x)$, and $\alpha_1,\cdots,\alpha_t$ be roots of $h(x)$ in $\overline{\mathbf{Q}_p}$. Since $h(0)\in\mathbf{Q}_p$, we have

$$v_p\Big(\prod_{j=1}^{t}\alpha_j\Big)=v_p\big((-1)^t h(0)\big)\in\mathbf{Z}.$$

Noticing that for each i and j, we have $v_p(\alpha_i)=v_p(\alpha_j)$. Therefore, we derive that $\dfrac{-tv_i}{u_i}\in\mathbf{Z}$. Since $\gcd(u_i,v_i)=d_i$, one writes $u_i=u_i'd_i$, $v_i=v_i'd_i$, where $\gcd(u_i',v_i')=1$. It follows that $u_i'\mid t$, and one claims that the degree of every factor of $f_i(x)$ is a multiple of u_i'. Since $u_i=u_i'd_i$, it follows that $f_i(x)$ has at most d_i irreducible factors in $\mathbf{Q}_p$. This finishes the proof of Lemma 2.4.

We also need the following lemma which give a result of the existence of prime number between two real numbers.

Lemma 2.5[9] There exists a prime p satisfying $x<p<\dfrac{6}{5}x$ for real number $x\geqslant 25$.

Lemma 2.6 For any real number $x>6$, there exist distinct primes p_1 and p_2 satisfying that

$$\lfloor x\rfloor\leqslant x<x+1\leqslant p_1<p_2<2\lfloor x\rfloor<2x \qquad (*)$$

Proof The proof of Lemma 2.6 is divided into the following two cases.

Case 1 $x\geqslant 25$. By Lemma 2.5, there exist primes p_1 and p_2 satisfying that $x<p_1<\dfrac{6}{5}x<p_2<\dfrac{36}{25}x<2x$. The statement is true for this case.

Case 2 $6<x<25$. We can check that there exist two different prime numbers p_1 and p_2 satisfying that $\lfloor x\rfloor<p_1<p_2<2\lfloor x\rfloor$. Since the inequality $(*)$ holds, it follows that Lemma 2.6 also holds for real number x satisfying $6<x<25$. The statement is true for this case. This completes the proof.

3 Proof of Theorem 1.1

We first consider the cases that $n<12$. For the cases $n=2,3,5,7,11$, one can check the conclusion of the theorem via Eisenstein criterion directly.

It follows that $f_4(x)=F_1(x)F_2(x)$ in $\mathbf{Q}_3$, where $\deg F_1(x)=3$ and $\deg F_2(x)=1$, by Lemma 2.4, we have both $F_1(x)$ and $F_2(x)$ are irreducible over $\mathbf{Q}_3$. Then consider the 2-adic Newton polygon of $f_4(x)$, its vertices are $(0,0)$, (4, −2). Hence we have either $f_4(x)$ is irreducible over $\mathbf{Q}_2$ or $f_4(x)=G_1(x)G_2(x)$ over $\mathbf{Q}_2$ with $\deg G_1(x)=\deg G_2(x)=2$ by Lemma 2.4. If $f_4(x)$ is reducible over $\mathbf{Q}$, it leads a contradiction with the factorization of $f_4(x)$ over the local field $\mathbf{Q}_2$ and $\mathbf{Q}_3$ by local-global principle. It follows that Theorem 1.1 is true for $n=4$. Similarly, we take the 2-adic and 7-adic Newton polygon into account for $f_8(x)$. For $f_9(x)$, we consider the 3-adic and 7-adic Newton polygon. By the same argument as in the case $n=4$, we can always arrive at a contradiction by local-global principle. We omitted the tedious details here.

Now we may assume that $n\geqslant 12$. We first prove that if $f_n(x)$ is reducible over $\mathbf{Q}$, then one has $f_n(x)=(x+a)g(x)$, where a is a rational number. Since $n>12$, by Lemma 2.6, there exist distinct prime numbers p_1 and p_2 satisfying that $\frac{n}{2}<p_1<p_2<n$. Consequently, we consider the factorization of $f_n(x)$ in the local field $\mathbf{Q}_{p_1}$. The p_1-adic Newton polygon of $f_n(x)$ has vertices $(0,0),(p_1,-1),(p_1+1,-1),(n,0)$. By (i) of Lemma 2.3 and Lemma 2.4, we have $f_n(x)=(x+a_0)F_1(x)F_2(x)$ in $\mathbf{Q}_{p_1}$, where $F_1(x)$ and $F_2(x)$ are both irreducible over $\mathbf{Q}_{p_1}$ with

$$\deg F_1(x)=p_1,\ \deg F_2(x)=n-p_1-1.$$

Similarly, the vertices of the p_2-adic Newton polygon of $f_n(x)$ are given by $(0,0),(p_2,-1),(p_2+1,-1),(n,0)$. By (i) of Lemma 2.3 and Lemma 2.4 again, one has

$$f_n(x)=(x+a_1)G_1(x)G_2(x)$$

in $\mathbf{Q}_{p_2}$, where $G_1(x)$ and $G_2(x)$ are both irreducible over $\mathbf{Q}_{p_2}$ with $\deg G_1(x)=p_2$ and $\deg G_2(x)=n-p_2-1$. If $f_n(x)$ is reducible over $\mathbf{Q}$, the local-global principle implies that $f_n(x)$ has at most 3 factors in $\mathbf{Q}$. Clearly, $f_n(x)$ can't have exactly 3 factors in $\mathbf{Q}$, otherwise the factorization of $f_n(x)$ in the local field $\mathbf{Q}_{p_1}$ and $\mathbf{Q}_{p_2}$ can't coincide. Hence, we have the factorization $f_n(x)=g(x)h(x)$ in $\mathbf{Q}$, where both $g(x)$ and $h(x)$ are irreducible over $\mathbf{Q}$.

Without loss of generality, it is natural for us to assume that $\deg g(x)\leqslant n/2$. Noticing that $6\leqslant n/2<p_1<p_2<n$, we have $p_1+1<p_2$. By local-global principle, if $f_n(x)$ is reducible over $\mathbf{Q}$, it admits a factor whose degree is greater than or equal to p_2. Since $f_n(x)=(x+a_0)F_1(x)F_2(x)$ in $\mathbf{Q}_{p_1}$, $f_n(x)=g(x)h(x)$ in $\mathbf{Q}$ and

$$\deg F_2(x)<\deg F_1(x)=p_1<p_1+1<p_2,$$

by comparing the degree of the polynomials $f_n(x)$ in $\mathbf{Q}$ and $\mathbf{Q}_{p_1}$, it follows that $h(x)=F_1(x)F_2(x)$, which implies that $\deg g(x)=1$. This proves that $f_n(x)=(x+a)g(x)$ as desired.

In what follows, we prove that such linear factor doesn't exist. Since n is a prime power, we may let $n=p^f$, where p is a prime number and f is a positive integer. The p-adic Newton polygon of $f_n(x)$ has vertices

$$\begin{cases}(0,0),(p,-1),\ldots,(p^f,-f),\ p\neq 2,\\(0,0),(4,-2),\ldots,(2^f,-f),\ p=2.\end{cases}$$

If $p\neq 2$, then by (i) of Lemma 2.3 and Lemma 2.4, we have $f_n(x)=\prod_{i=1}^{f}g_i(x)$, where $g_i(x)$ are irreducible over $\mathbf{Q}_p$ with $\deg g_1(x)=p$ and

$$\deg g_i(x)=p^i-p^{i-1},\ i=2,\cdots,f.$$

It follows that $f_n(x)$ can't have a linear factor in $\mathbf{Q}_p$. Furtherly, by local-global principle $f_n(x)$ can't have a linear factor in $\mathbf{Q}$ either.

If $p=2$, by (i) of Lemma 2.3 and Lemma 2.4, we have $f_n(x)=\prod_{i=1}^{f}g_i(x)$, where $g_1(x)$ has at most two irreducible factors in $\mathbf{Q}_2$ and the degree of each factor of $g_1(x)$ is greater than or equal to 2. For $i=2,\cdots,f$, $g_i(x)$ are irreducible over $\mathbf{Q}_2$ and $\deg g_i(x)=2^i-2^{i-1}$. Thus $f_n(x)$ can't be with a linear factor in $\mathbf{Q}_2$. This finishes the proof of Theorem 1.1.

4 Conclusions

In this paper we have studied the irreducibility of a class of generalized Schur-type polynomial (2) with $a_1=1$ and $a_i=(i-2)!$ for $2\leqslant i\leqslant n=p^a$. By introducing the tool of p-adic Newton polygon and local-global principle, the irreducibility of the polynomial over $\mathbf{Q}$ was given. Here we point out that one can characterize the irreducibility and other properties of some more generalized

Schur-type polynomials by relaxing the restrictions on coefficients of the polynomial (2).

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