doi：10.3969/j.issn.0490-6756.2019.02.017

彩虹黑弦的热力学性质

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摘　要：双重相对论(DSR)是描述平直时空中的量子引力的一个有效模型．当试图将该模型纳入到弯曲时空中考虑时，一种被称为“彩虹引力”的理论可帮助实现这一推广，此时背景时空依赖于探测粒子的能量．事实上受粒子能量影响的度规的形式依赖于正交系的选取．本文主要考虑了自由落体参考系下的彩虹静态柱对称黑洞(彩虹黑弦)．得到了取一阶近似的彩虹黑弦的霍金温度和熵，这种修正来源于考虑的彩虹引力效应．

关键词：黑洞热力学；黑弦；双重狭义相对论；彩虹引力
中图分类号：P145.8　　**文献标识码**：A　　**文章编号**：0490-6756(2019)02-0285-05

Thermodynamics of the rainbow black string

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Abstract: Doubly special relativity (DSR) is an effective model to describe quantum gravity in flat spacetime. One way to incorporate this formalism into curved spacetime is the “rainbow gravity”, where the background spacetime is dependent on the energy of the probing particles. The form the energy dependent metric actually depends on the choice of the orthonormal frame. In this paper, we consider the rainbow static cylindrical black hole, namely the rainbow black string, in the free-fall frame scenario. Specifically, we obtain the rainbow corrections to the Hawking temperature and entropy of the rainbow black string.

Keywords: Black hole thermodynamics; Black string; Doubly special relativity; Gravity’s rainbow

1　Introduction

After almost forty years since Hawking predicted that the black holes could emit radiation at temperature $T=\hbar\kappa/2\pi k_B$, where κ is the surface gravity [1,2], the thermodynamics of such objects is still of great interest to researchers in the direction of theoretical physics. At the very beginning, the Hawking radiation was calculated using quantum field theory in curved spacetime with the metric of the background spacetime being fixed. In other words, this formalism is semi-classical. However, there is a consensus among researchers that the framework of the smooth manifold and metric of general relativity is no longer applicable at very high energy scales. Instead, a nontrivial quantum gravity theory will play an important role. Although such a theory has yet been available, there are several candidates under study, including string theory[3], loop quantum gravity[4],

收稿日期：2017-09-14
基金项目：国家自然科学基金(11175039)
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and so forth.

Before we find the fundamental quantum description of gravity, it is worth exploring the effective models in terms of classical general relativity. Doubly Special Relativity (DSR), proposed by Amelino-Camelia[5, 6], is one of them. In this model, the Lorentz transformations are modified at very high energies. In fact, there is not only an observer-independent maximum velocity (the speed of light), but also an observer-independent minimum length scale (the Planck length). Consequently, the energy-momentum dispersion relation (MDR) for a particle of mass m is modified to

$$E^2 f^2(E/m_p) - p^2 g^2(E/m_p) = m^2, \qquad (1)$$

where m_p is the Planck mass, and the general functions $f(x)$ and $g(x)$ satisfy

$$\lim_{x\to 0} f(x) = 1,\ \lim_{x\to 0} g(x) = 1. \qquad (3)$$

To incorporate DSR into the framework of general relativity, Magueijo and Smolin proposed the "Gravity's rainbow"[7]. In this proposal, the spacetime is described by a one parameter family of metrics

$$g(E) = \eta^{ab} e_a(E) \otimes e_b(E), \qquad (3)$$

where the energy-dependent orthonormal frame fields are given by

$$e_0(E) = \frac{1}{f(E/m_p)}\tilde{e}_0,\ e_i(E) = \frac{1}{g(E/m_P)}\tilde{e}_i. \qquad (4)$$

Note that E is the energy of the probing particle. This is to say that particles with different energies move in different backgrounds. For more studies on the gravity's rainbow, please refer to Ref. [8, 9], *etc*.

In the following, we first consider the rainbow black string in free-fall frame, and then the effective temperature is obtained. After that, we take account of an example of modified dispersion relation

$$f(x) = 1 \text{ and } g(x) = \sqrt{1 - \eta x^n}, \qquad (5)$$

which is proposed by Amelino-Camelia *et al*. [10, 11]. Thermodynamics of the free-fall rainbow black string under this MDR is explored. We take Geometrized units $c = G = 1$ throughout this paper.

2 Free-fall frame black string in gravity's rainbow

The Einstein-Hilbert action in four dimensions, with a negative cosmological constant Λ is given by

$$S = \frac{1}{16\pi}\int d^4 x \sqrt{-g}(R - 2\Lambda), \qquad (6)$$

where g is the determinant of the metric, and R is the Ricci scalar. Assuming that the spacetime is cylindrically symmetric and time-independent, Lemos derived the solution of the static uncharged black string [12]. When we generalize the metric to the gravity's rainbow in the static orthonormal frame scenario, the line element can be written as

$$ds^2 = \frac{\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)}{f^2(E/m_p)}dt^2 - \frac{\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)^{-1}}{g^2(E/m_p)}dr^2 - \frac{r^2}{g^2(E/m_p)}(d\theta^2 + \alpha^2 dz^2), \qquad (7)$$

where

$$\alpha^2 = -\frac{\Lambda}{3},\ b = 4M,\ -\infty < t < \infty,$$
$$0 \le r < \infty,\ -\infty < z < \infty,\ 0 \le \theta \le 2\pi. \qquad (8)$$

Note that M is the mass density along the z line of the black string. This rainbow black string in static frame has been studied in Ref. [13]. However, there is another natural choice for the orthonormal frame, which is the one anchored to freely falling observers (FFO) along the radial direction. To describe FFO, it is better to use the Painleve-Gullstrand (PG) coordinate [14, 15]. The rainbow metric of Schwarzschild black hole in the free-fall frame scenario was obtained in Ref. [15]. Similarly, in the PG coordinate anchored to the FFO along the radial direction, the rainbow metric of black string in the free-fall frame scenario takes the form of

$$ds^2 = \frac{dt_p^2}{f^2(E/m_p)} - \frac{[dr - v(r)dt_p]^2}{g^2(E/m_p)} - \frac{r^2}{g^2(E/m_p)}(d\theta^2 + \alpha^2 dz^2), \qquad (9)$$

where $v(r)$ is the velocity of the FFO with respect to the static observer, and t_p is the proper time measured by the FFO. We assume that the observers is infalling toward the black string, *namely* $v<0$. The value of $v(r)$ is given by

$$v(r)=-\sqrt{1-\alpha^2r^2+\frac{b}{\alpha r}}. \tag{10}$$

For radial null curves with θ and z being constant, one has that $ds^2=0$ leads to

$$\frac{dt_p^2}{f^2(E/m_p)}=\frac{[dr-v(r)dt_p]^2}{g^2(E/m_p)}. \tag{11}$$

The condition that $dt_p/dr\to\pm\infty$ gives the position of the event horizon r_h, which reads

$$\alpha^2r_h^2-\frac{b}{\alpha r_h}=1-\frac{g^2(E/m_p)}{f^2(E/m_p)}. \tag{12}$$

One can see that in the freely falling frame the event horizon depends on the energy of the probing particle. Note this event horizon radius is different from that in the static frame scenario, which gives [13]

$$\alpha^2r_h^2-\frac{b}{\alpha r_h}=0. \tag{13}$$

We now apply the Hamilton-Jacobi method to calculate the Hawking temperature of the rainbow black string (9). This method was developed by Srinivasan *et al*. to investigate the tunneling process of Hawking radiation [16-18]. In the Hamilton-Jacobi method, the self-gravitation of emitted particles is ignored, and the action of the particles is assumed to satisfy the relativistic Hamilton-Jacobi equation. The tunneling probability for the classically forbidden trajectory from inside to outside the horizon is obtained by using the Hamilton-Jacobi equation to calculate the imaginary part of the action for the tunneling process. It has been shown in the metric $ds^2=g_{\mu\nu}(E)dx^\mu dx^\nu$ that the Hamilton-Jacobi equation for massless field was given by [13]

$$g^{\mu\nu}(E)\partial_\mu I\partial_\nu I=0, \tag{14}$$

where I is the action of the field. Taking Eq. (9) into consideration and making the ansatz for I which is $I=-Et_p+W(r)+\Theta(\theta,z)$, we have differential equations for $W(r)$ and $\Theta(\theta,z)$:

$$\partial_\theta\Theta\partial_\theta\Theta+\frac{1}{\alpha^2}\partial_z\Theta\partial_z\Theta=\lambda,$$

$$\partial_rW_\pm(r)=\frac{-r^2v(r)E\pm\sqrt{E^2\frac{g^2(E/m_p)}{f^2(E/m_p)}+\frac{\lambda}{r^2}\left[v^2(r)-\frac{g^2(E/m_p)}{f^2(E/m_p)}\right]\frac{g^2(E/m_p)}{f^2(E/m_p)}}}{g^2(E/m_p)/f^2(E/m_p)-v^2(r)}, \tag{15}$$

where +/− denotes the outgoing/ingoing solutions, and $\lambda=J_\theta^2+p_z^2/\alpha^2$ with J_θ being the angular momentum along *z*-axis. The periodicity condition gives $J_\theta=j$ and $p_z=2\pi l/a$ with $j,\ l\in\mathbb{Z}$. Here a is the length of the black string. It can be seen from Eq. (15) that $\partial_rW_\pm(r)$ is singular when $r=r_h$. Integrating $\partial_rW_\pm(r)$ along the semi-circle around the singularity, we get

$$\mathrm{Im}W_+(r)=\frac{2\pi E}{2\alpha^2r_h^2+b/(\alpha r_h^2)}\frac{g(E/m_p)}{f(E/m_p)},$$

$$\mathrm{Im}W_-(r)=0. \tag{16}$$

As shown in Ref. [19], the probability of a particle tunneling from inside to outside the horizon is

$$P_{emit}\sim\exp\left[-\frac{2}{\hbar}(\mathrm{Im}W_+-\mathrm{Im}W_-)\right]. \tag{17}$$

Taking $k_B=1$, we can get an effective Hawking temperature of the rainbow black string with the concept of Boltzmann factor:

$$T_{BS}=\frac{\hbar[2\alpha^2r_h^2+b/(\alpha r_h^2)]}{4\pi}\frac{f(E/m_p)}{g(E/m_p)}. \tag{18}$$

3 Thermodynamics of the rainbow black string

We now estimate the black string's temperature with the help of Heisenberg uncertainty principle. For a massless particle, the MDR in Eq. (1) becomes

$$\frac{E}{m_p}\frac{f(E/m_p)}{g(E/m_p)}=\frac{p}{m_p}. \tag{19}$$

The Heisenberg uncertainty principle gives a relation between the momentum p of an emitted particle and the event horizon radius r_h [20, 21]:

$$p/m_p\sim\delta p/m_p\sim\hbar/(m_p\delta x)\sim m_p/r_h. \tag{20}$$

Consequently, we have

$$\frac{x}{h(x)}=\frac{m_p}{r_h}, \tag{21}$$

where $x \equiv E/m_p$ and $h(x) \equiv g(x)/f(x) = \sqrt{1-\eta x^n}$. We need to solve Eq. (21) for x in terms of M, since our goal is to express the black string's temperature in terms of M. For convenience, we introduce a new variable y, which is defined as $y \equiv m_p/(4M)$. Solving Eq. (12) for r_h in terms of x and plugging it into Eq. (21) leads to

$$y=\frac{x^3}{h\alpha(-x^2+h^2x^2+h^2\alpha^2m_p^2)}, \tag{22}$$

which gives

$$x=\alpha(m_p^2y)^{\frac{1}{3}}-\eta\alpha^{n+1}(m_p^2y)^{\frac{n+1}{3}}\left[\frac{1}{2}+\frac{(m_p^2y)^{\frac{2}{3}}}{3m_p^2}\right]+O(\eta^2). \tag{23}$$

Thus we get the temperature of the black string in terms of M from Eq. (23)

$$T_{BS}=\frac{3\hbar\alpha(4M)^{\frac{1}{3}}}{4\pi}\left[1+\frac{\eta}{2}\alpha^n\left(\frac{m_p^3}{4M}\right)^{\frac{n}{3}}+O(\eta^2)\right] \tag{24}$$

Using the first law of black hole thermodynamics $dS_{BH}=dM/T_{BH}$, we find that the entropy per unit length of the z line of the rainbow black string (up to an irrelevant constant) is

$$S_{BS}=\int\frac{dM}{T_{BS}}=\begin{cases}\frac{\pi}{2\hbar\alpha}(4M)^{\frac{2}{3}}-\eta\frac{\pi(\alpha m_p)^n(4M)^{\frac{2-n}{3}}}{2(n-2)\hbar\alpha}+O(\eta^2),n\neq2\\ \frac{\pi}{2\hbar\alpha}(4M)^{\frac{2}{3}}-\eta\frac{\pi(\alpha m_p)^n}{6\hbar\alpha}\ln(4M)+O(\eta^2),n=2\end{cases} \tag{25}$$

In the standard gravity, we have (=0, and the event horizon $r_0=(4M)^{1/3}/\alpha$. Since the circumference of the standard black string is $A=2\pi\alpha r_0^2$, Eq. (25) becomes

$$S_{BS}=\begin{cases}\frac{A}{4\hbar}-\eta\frac{\pi(\alpha m_p)^n}{2(n-2)\hbar\alpha}\left(\frac{\alpha A}{2\pi}\right)^{\frac{2-n}{2}}+O(\eta^2),n\neq2\\ \frac{A}{4\hbar}-\eta\frac{\pi(\alpha m_p)^n}{4\hbar\alpha}\ln\left(\frac{\alpha A}{2\pi}\right)+O(\eta^2),n=2\end{cases} \tag{26}$$

The leading terms of Eq. (26) are the familiar Bekenstein-Hawking entropy. For $n=2$, we obtain the logarithmic term, in accordance with the result in Ref. [22-24].

4 Conclusions

In this paper, we considered the rainbow black string in free-fall frame scenario. Being different from the event horizon in static frame scenario, it shows that the position of the event horizon depends on the energy of the probing particles in free-fall frame scenario. By means of Hamilton-Jacobi method, we studied the effects of gravity's rainbow on the temperature and entropy of a free-fall frame black string. The temperature depends on the energy of the probing particles. Focusing on the AC dispersion relation, we obtained the effective temperature and entropy of the black string in terms of the mass density M or circumference A. For $n=2$, the leading correction on the entropy due to the effect of gravity's rainbow, has a logarithmic dependence on the circumference of the black string. This condition is the same as the one in Ref. [13, 15].

References:

[1] Hawking S W. Particle creation by black holes [J]. Commun Math Phys, 1975, 43: 199.

[2] Feng Z W, Jiang Q Q, Li G P, *et al*. General tortoise coordination transformation and radiation mechanism of dynamical Kerr-Newman black hole [J]. J Sichuan Univ: Nat Sci Ed, 2012, 49: 028.

[3] Förste S. Strings, branes and extra dimensions [J]. Fortschr Phys, 2002, 50: 221.

[4] Rovelli C. Loop quantum gravity [J]. Living Rev Relativ, 2008, 11: 5.

[5] Amelino-Camelia G. Testable scenario for relativity with minimum length [J]. Phys Lett B, 2001, 510: 255.

[6] Amelino-Camelia G. Relativity in spacetimes with short-distance structure governed by an observer-independent (Planckian) length scale [J]. Int J Mod Phys B, 2002, 11: 35.

[7] Magueijo J, Smolin L. Gravity's rainbow [J]. Classical Quant Grav, 2004, 21: 1725.

[8] Ling Y. Rainbow universe [J]. J Cosmol Astropart P, 2007, 2007: 017.

[9] Ling Y, Li X, Zhang H. Thermodynamics of modified black holes from gravity's rainbow [J]. Mod

Phys Lett A, 2007, 22: 2749.
[10] Amelino-Camelia G, Ellis J, Mavromatos N E, *et al*. Distance measurement and wave dispersion in a Liouville-string approach to quantum gravity [J]. Int J Mod Phys A, 1997, 12: 607.
[11] Amelino-Camelia G. Quantum-spacetime phenomenology [J]. Living Rev Relativ, 2013, 16: 5.
[12] Lemos J P S. Three dimensional black holes and cylindrical general relativity [J]. Phys Lett B, 1995, 353: 46.
[13] Mu B, Wang P, Yang H. Thermodynamics and luminosities of rainbow black holes [J]. J Cosmol Astropart P, 2015, 2015: 045.
[14] Corley S, Jacobson T. Hawking spectrum and high frequency dispersion [J]. Phys Rev D, 1996, 54: 1568.
[15] Tao J, Wang P, Yang H. Free-fall frame black hole in gravity's rainbow [J]. Phys Rev D, 2016, 94: 064068.
[16] Srinivasan K, Padmanabhan T. Particle production and complex path analysis [J]. Phys Rev D, 1999, 60: 024007.
[17] Angheben M, Nadalini M, Vanzo L, *et al*. Hawking radiation as tunneling for extremal and rotating black holes [J]. J High Energ Phys, 2005, 2005: 014.
[18] Kerner R, Mann R B. Tunnelling, temperature, and Taub-NUT black holes [J]. Phys Rev D, 2006, 73: 104010.
[19] Tao J, Wang P, Yang H. Black hole radiation with modified dispersion relation in tunneling paradigm: static frame [J]. Nucl Phys B, 2017, 922: 346.
[20] Bekenstein J D. Black holes and entropy [J]. Phys Rev D, 1973, 7: 2333.
[21] Adler R J, Chen P, Santiago D I. The generalized uncertainty principle and black hole remnants [J]. Gen Relat Gravit, 2001, 33: 2101.
[22] Kaul R K, Majumdar P. Logarithmic correction to the Bekenstein-Hawking entropy [J]. Phys Rev Lett, 2000, 84: 5255.
[23] Das S, Majumdar P, Bhaduri R K. General logarithmic corrections to black-hole entropy [J]. Classical Quant Grav, 2002, 19: 2355.
[24] Carlip S. Logarithmic corrections to black hole entropy, from the Cardy formula [J]. Classical Quant Grav, 2000, 17: 4175.

引用本文格式：
中　文：王鹏，魏欣. 彩虹黑弦的热力学性质 [J]. 四川大学学报：自然科学版，2019，56：285.
英　文：Wang P, Wei X. Thermodynamics of the rainbow black string [J]. J Sichuan Univ: Nat Sci Ed, 2019, 56: 285.