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非线性分数阶 Klein-Gordon 方程的新显式解

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摘 要：利用分数阶复变换技巧，本文将非线性分数阶 Klein-Gordon 方程转化为非线性常微分方程，然后应用扩展的(G'/G)-展开法构造了非线性分数阶 Klein-Gordon 方程的精确解，从而得到了一系列新显式解，包括双曲函数解，三角函数解和负幂次解.

关键词：显式解；非线性分数阶 Klein-Gordon 方程；扩展的(G'/G)-展开法
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New explicit solutions for nonlinear fractional Klein-Gordon equation

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Abstract: By using the fractional complex transformation, the nonlinear fractional Klein-Gordon equation is converted to a nonlinear ordinary differential equation. Then we apply the extended (G'/G)-expansion method to construct the exact solutions of the equation. Moreover, a series new explicit solutions are obtained, which include hyperbolic function, trigonometric and negative exponential solutions.

Keywords: Explicit solutions; Nonlinear fractional Klein-Gordon equation; Extended (G'/G)-expansion method
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1 Introduction

Fractional differential equations are generalizations of classical differential equations integer order. Many important phenomena in plasma, optical fibers, fluid dynamics, electromagnetic and acoustics are well described by fractional differential equations. The fractional differential equations have been investigated by many researchers[1-3]. In recent years, a variety of powerful methods have been introduced by some scientists to construct exact solutions. For example, the fractional exp-function method[4,5], the fractional first integral method[6,7] and the fractional sub-equation method[8].

Lately, Wang *et al.*[9] introduced a new auxiliary equation method, called (G'/G)-expansion method, which is a powerful method for seeking the solutions of nonlinear partial differential equations. In this method, the second order linear ordinary differential equation $G''+\lambda G'+\mu G=0$ is executed. Shehata[10] also presented an modified (G'/G)-expansion method, which add negative power exponent to seek more general traveling wave so-

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lutions. In this paper, we obtain abundant solutions method by using the improved (G'/G)-expansion method.

Jumarie[11] proposed a modified Riemann-Liouville derivative. Then, we give some definitions and properties of this modified Riemann-Liouville derivative. The Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))\mathrm{d}\xi, \\ 0<\alpha<1, \\ (f^n(t))^{(\alpha-n)}, n \le \alpha < n+1, n \ge 1 \end{cases} \tag{1}$$

Some important properties of Jumarie's derivatiive are:

$$D_t^\alpha f(t) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha} \tag{2}$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t) \tag{3}$$

$$D_t^\alpha f[g(t)] = f_g[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha \tag{4}$$

In other way, the Klein-Gordon equation[12,13] is an important nonlinear partial differential equation arising in relativistic quantum mechanics and quantum field theory, which is also used to model many types of phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles. Particularly, it is very interesting to study the nonlinear partial differential Klein-Gordon equation of fractional order[14-16]:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + \theta_1 u(x,t) + \theta_2 u^3(x,t), t>0, 0<\alpha\le 1 \tag{5}$$

where θ_1 and θ_2 are arbitrary constants and α is a parameter describing the order of the fractional time derivative. Taghizadel *et al*.[14] used the simplest auxiliary equation to solve (5) and obtained hyperbolic function solutions, Lu[15] investigated via first integral method. In this paper, we will apply improved (G'/G)-expansion method to construct exact solutions of (5) in the sense of modified Riemann-Liouville derivative.

2 The improved (G'/G)-expansion method

Let us consider a general fractional partial differential equation:

$$P(u, u_x, u_t, D_x^\alpha u, D_t^\alpha u, \cdots) = 0, 0<\alpha\le 1 \tag{6}$$

where $D_x^\alpha u$ and $D_t^\alpha u$ are Jumarie's modified Riemann-Liouville derivatives of $u(x,t)$, P is a polynomial in $u(x,t)$ and its various partial derivatives including fractional derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 1. Li and He[17] proposed a fractional complex transform to convert fractional differential equation into ordinary differential equation. Therefore, we can easily construct the solutions of fractional differential equation. With fractional complex transform, we can easily convert fractional partial differential equation to ordinary differential equation. By the fractional complex transformation

$$u(x,t) = u(\xi), \xi = lx - \frac{c}{\Gamma(1+\alpha)} t^\alpha \tag{7}$$

where l and c are non zero arbitrary constants with $c \neq 0$, we can reduce (6) to an ordinary differential equation of integer order in the form of

$$P(u, u', u'', u''', \cdots) = 0 \tag{8}$$

where the superscripts and the ordinary derivatives are taken with respect to ξ. If possible, we should integrate (8) term by term one or more times.

Step 2. Assume that the solution of (8) can be expressed as a polynomial of (G'/G) in the form of

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i + \sum_{i=1}^{m} b_i \left(\frac{G'}{G}\right)^{-i} \tag{9}$$

where $a_i (i=0,1,\cdots,m)$, $b_i (i=1,\cdots,m)$ are constants, $G=G(\xi)$ satisfies the following nonlinear ordinary differential equation:

$$G''G = B(G')^2 + CGG' + EG^2 \tag{10}$$

where the prime denotes derivative with respect to ξ, B, C and E are real parameters.

Step 3. To determine the positive integer m, taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in (8).

Step 4. Substituting (9) and (10) into (8), with the value of m obtained in Step 3, we obtain polynomials in $\left(\frac{G'}{G}\right)^m (m=0,1,\cdots)$ and $\left(\frac{G'}{G}\right)^{-m}$ (m

$=1,2,\cdots$). Then, collect each coefficient of the resulted polynomials to zero yields a set of algebraic equations for $a_i (i=0,1,2,\cdots,m)$, $b_i (i=1,2,\cdots,m)$, l and c.

Step 5. Suppose that the value of the constants $a_i (i=0,1,2,\cdots,m)$, $b_i (i=1,2,\cdots,m)$ can be found by solving the algebraic equations which are obtained in Step 4. Since the general solution of (10) is well known to us, substituting the values of $a_i (i=0,1,2,\cdots,m)$, $b_i (i=1,2,\cdots,m)$ into (9), we obtain more general type and new exact traveling wave solutions of the nonlinear partial differential (6).

3 Applications

Now, Substituting the wave transformation (7) into (5) we have

$$(l^2-c^2)u''+\theta_1 u+\theta_2 u^3=0 \tag{11}$$

By Balancing the order between the highest order derivative term and nonlinear term in (11). we can obtain $m=1$. So we have

$$u(\xi)=a_1(\frac{G'}{G})+a_0+b_1(\frac{G'}{G})^{-1} \tag{12}$$

By using (10) from (9) we have

$$u^3=a_1^3(\frac{G'}{G})3+3a_0a_1^2(\frac{G'}{G})^2+(3a_1^2b_1+3a_1a_0^2)(\frac{G'}{G})+(a_0^3+6a_0a_1b_1)+(3a_1b_1^2+3a_0^2b_1)(\frac{G'}{G})^{-1}+3a_0b_1^2(\frac{G'}{G})^{-2}+b_1^3(\frac{G'}{G})^{-3} \tag{13}$$

and

$$u''=2a_1(B-1)2(\frac{G'}{G})^3+3a_1(B-1)C(\frac{G'}{G})^2+[2a_1(B-1)E+a_1C^2]\frac{G'}{G}+[a_1CE+b_1C(B-1)]+[2b_1E(B-1)+b_1C^2](\frac{G'}{G})^{-1}+3b_1CE(\frac{G'}{G})^{-2}+2b_1E^2(\frac{G'}{G})^{-3} \tag{14}$$

Substituting (12)～(14) into (11), the left-hand side is converted into polynomials in $(\frac{G'}{G})^m$ $(m=0,1,2,\cdots)$ and $(\frac{G'}{G})^{-m}$ $(m=1,2,\cdots)$. Collecting each coefficient of these resulted polynomials to zero yields a set of simultaneous algebraic equations for a_0, a_1, b_1, l and c:

$$\begin{cases}2(l^2-c^2)(B-1)^2a_1+\theta_2a_1^3=0,\\3(l^2-c^2)(B-1)Ca_1+3\theta_2a_0a_1^2=0,\\(l^2-c^2)[2(B-1)E+C^2]a_1+\theta_1a_1+\\\quad\theta_2(3a_1^2b_1+3a_1a_0^2)=0,\\(l^2-c^2)[a_1CE+b_1C(B-1)]+\theta_1a_0+\\\quad\theta_2(a_0^3+6a_0a_1b_1)=0,\\(l^2-c^2)[2b_1E(B-1)+b_1C^2]+\theta_1b_1+\\\quad\theta_2(3a_1b_1^2+3a_0^2b_1)=0,\\(l^2-c^2)3CEb_1+3\theta_2a_0b_1^2=0,\\(l^2-c^2)2b_1E^2+\theta_2b_1^3=0\end{cases} \tag{15}$$

Solving these algebraic equations with the help of algebraic software Maple, we obtain

$$\begin{cases}a_0=\pm\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}\frac{B-1}{|B-1|}C,\\a_1=\pm2\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}|B-1|,\\b_1=0,c=\pm\sqrt{l^2-\frac{2\theta_1}{C^2+4E-4BE}}\end{cases} \tag{16}$$

$$\begin{cases}a_0=\pm\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}\frac{E}{|E|}C,\\a_1=0,b_1=\pm2\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}|E|,\\c=\pm\sqrt{l^2-\frac{2\theta_1}{C^2+4E-4BE}}\end{cases} \tag{17}$$

Substituting (16) into (12) yields

$$u(\xi)=\pm\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}\frac{B-1}{|B-1|}C\pm2\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}|B-1|(\frac{G'}{G}) \tag{18}$$

Substituting (17) into (12) yields

$$u(\xi)=\pm\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}\frac{E}{|E|}C\pm 2\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}\,|E|\,\left(\frac{G'}{G}\right)^{-1} \tag{19}$$

Family 1. When $C^2+4E-4BE>0, B\neq 1$, we obtain the hyperbolic function traveling wave solutions

$$u_1=\pm\sqrt{-\frac{\theta_1}{\theta_2}}\frac{|B-1|}{1-B}H_1 \tag{20}$$

$$u_2(\xi)=\pm\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}\frac{E}{|E|}C\pm 2\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{C^2+4E-4BE}}\,|E|\left[\frac{C}{2(1-B)}+\frac{\sqrt{C^2+4E-4BE}}{2(1-B)}H_1\right]^{-1} \tag{21}$$

where

$$H_1=\frac{C_1\sinh\frac{\sqrt{C^2+4E-4BE}}{2}\xi+C_2\cosh\frac{\sqrt{C^2+4E-4BE}}{2}\xi}{C_1\cosh\frac{\sqrt{C^2+4E-4BE}}{2}\xi+C_2\sinh\frac{\sqrt{C^2+4E-4BE}}{2}\xi},\ \xi=lx-\frac{c}{\Gamma(1+\alpha)}t^{\alpha},$$

C_1 and C_2 are arbitrary constants.

On the other hand, assuming $C_1=0, B>1$ and $C_2\neq 0$, the traveling wave solution of (20) can be written as

$$u_{1_1}(\xi)=\mp\sqrt{-\frac{\theta_1}{\theta_2}}\coth\frac{\sqrt{C^2+4E-4BE}}{2}\xi \tag{22}$$

Assuming $C_2=0, B>1$ and $C_1\neq 0$, we obtian

$$u_{1_2}(\xi)=\mp\sqrt{-\frac{\theta_1}{\theta_2}}\tanh\frac{\sqrt{C^2+4E-4BE}}{2}\xi \tag{23}$$

where $\xi=lx-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}$.

Family 2. When $C^2+4E-4BE<0, B\neq 1$, we obtain the trigonometric function traveling wave solutions

$$u_3(\xi)=\pm\sqrt{-\frac{\theta_1}{\theta_2}}\frac{|B-1|}{1-B}H_2 \tag{24}$$

$$u_4(\xi)=\pm\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{4BE-C^2-4E}}\frac{E}{|E|}C\pm 2\sqrt{-\frac{\theta_1}{\theta_2}\cdot\frac{1}{4BE-C^2-4E}}\,|E|\left[\frac{C}{2(1-B)}+\frac{\sqrt{4BE-C^2-4E}}{2(1-B)}H_2\right]^{-1} \tag{25}$$

where

$$H_2=\frac{-C_1\sin\frac{\sqrt{4BE-C^2-4E}}{2}\xi+C_2\cos\frac{\sqrt{4BE-C^2-4E}}{2}\xi}{C_1\cos\frac{\sqrt{4BE-C^2-4E}}{2}\xi+C_2\sin\frac{\sqrt{4BE-C^2-4E}}{2}\xi},$$

C_1 and C_2 are arbitrary constants.

On the other hand, assuming $C_1=0, B<1$ and $C_2\neq 0$, the traveling wave solution of (24) can be written as

$$u_{3_1}(\xi)=\pm\sqrt{-\frac{\theta_1}{\theta_2}}\cot\frac{\sqrt{4BE-C^2-4E}}{2}\xi \tag{26}$$

Assuming $C_2=0, B<1$ and $C_1\neq 0$, we obtain

$$u_{3_2}(\xi) = \mp\sqrt{-\frac{\theta_1}{\theta_2}}\tan\frac{\sqrt{4BE - C^2 - 4E}}{2}\xi \tag{27}$$

where $\xi = lx - \frac{c}{\Gamma(1+\alpha)}t^{\alpha}$.

In order to further understand these results, the solutions are plotted in Fig. 1~Fig. 4 with the help of software Maple in the case of suitable parameter.

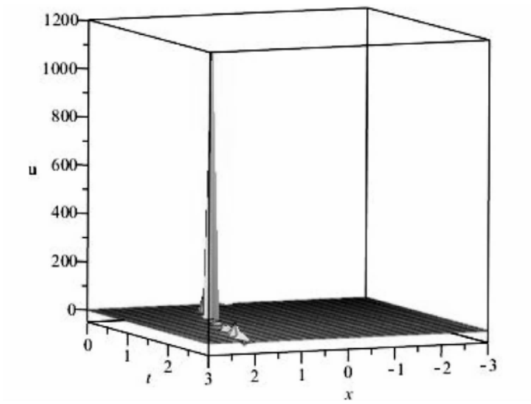

Fig. 1　The solitary solutions $u_{1_1}(x,t)$ with $B = 2, C = 3, E = 3, l = \sqrt{3}, \theta_1 = -\frac{3}{2}, \theta_2 = 6, \alpha = \frac{1}{2}, c = \frac{\sqrt{15}}{2}$

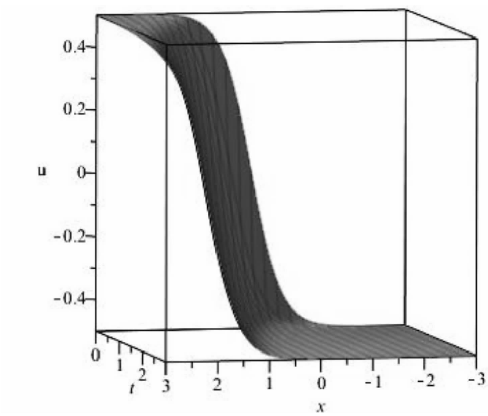

Fig. 2　The solitary solutions $u_{1_2}(x,t)$ with $B = 2, C = 3, E = 3, l = \sqrt{3}, \theta_1 = -\frac{3}{2}, \theta_2 = 6, \alpha = \frac{1}{2}, c = \frac{\sqrt{15}}{2}$

4　Conclusions

In this paper, we covert the fractional Klein-Gordon equation into a ordinary differential equation by the fractional complex transformation, then we use the extended (G'/G)-expansion method to obtain two family of solutions, which include hyperbolic function, trigonometric and negative exponential solutions. Comparing our results with the known results[15,16], we can find that the solution of the auxiliary equation (10) are are more richer than of the Bernoulli equation $\frac{dz}{d\xi} = az + bz^2$, In fact, the result are only the particularly case of our result.

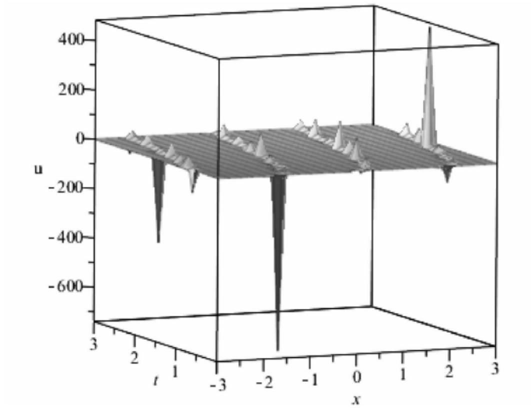

Fig. 3　The solitary solutions $u_{3_2}(x,t)$ with $B = 2, C = 4, E = 3, l = \sqrt{3}, \theta_1 = -\frac{3}{2}, \theta_2 = 6, \alpha = \frac{1}{2}, c = 2$

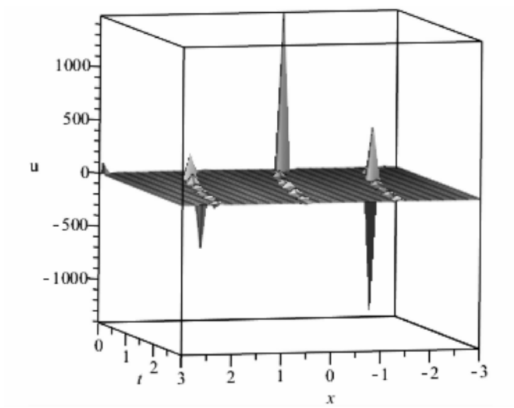

Fig. 4　The solitary solutions $u_{3_2}(x,t)$ with $B = 2, C = 4, E = 3, l = \sqrt{3}, \theta_1 = -\frac{3}{2}, \theta_2 = 6, \alpha = \frac{1}{2}, c = 2$

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