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非线性三阶差分方程三点特征值问题的正解

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摘　要：利用锥不动点定理得到离散非线性三阶三点特征值问题的正解

$$\begin{cases}\Delta^3 u(t-1) = \lambda a(t) f(t,u(t)),\ t \in [1,T]_Z \\ u(0) = \Delta u(\eta) = \Delta^2 u(T) = 0,\end{cases}$$

这里 $\eta \in [[\frac{T^2+T}{3T+2}]+1,T]_Z, \lambda > 0$ 是一个参数.

关键词：非线性三阶差分方程；三点特征值问题；正解；锥
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Positive Solutions of three-point eigenvalue problem for nonlinear third-Order difference equation

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Abstract: In this paper, by using the fixed point theorem in cone, we obtain existence of positive solutions of the discrete nonlinear third-order three-point eigenvalue problem

$$\begin{cases}\Delta^3 u(t-1) = \lambda a(t) f(t,u(t)),\ t \in [1,T]_Z \\ u(0) = \Delta u(\eta) = \Delta^2 u(T) = 0,\end{cases}$$

where $\eta \in [[\frac{T^2+T}{3T+2}]+1,T]_Z, \lambda > 0$ is a parameter.

Keywords: Nonlinear third-order difference equation; Three-point eigenvalue problem; Positive solution; Cone
(2010 MSC 39A10, 39A12)

1 Introduction

Let a, b be two integers with $b>a$. Let us employ $[a, b]$ denotes the integer set $\{a, a+1, \cdots, b\}$. For any real number c, $[c]$ is the integer part of c. In this paper, we consider the existence of positive solutions of the discrete nonlinear third-order three-point eigenvalue problem

$$\begin{cases}\Delta^3 u(t-1) = \lambda a(t) f(t,u(t)),\ t \in [1,T]_Z \\ u(0) = \Delta u(\eta) = \Delta^2 u(T) = 0\end{cases} \tag{1}$$

where $T>2$ is an integer, $\lambda > 0$ is a parameter, $\eta \in [[\frac{T^2+T}{3T+2}]+1,T]_Z, f \in ([1,T]_Z \times [0,\infty),$

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$[0,\infty))$ is continuous and $a\in[1,T]_Z\rightarrow(0,\infty)$.

Difference equations appear in many mathematical models in diverse fields, such as economy, biology, physics and finance[1-5]. In recent years, the existence and multiplicity of positive solutions of discrete boundary value problems have received much attention from many authors and a great deal of work has been done by using classical methods such as fixed point theory in cone[6-12], lower and upper solutions method[13], bifurcation theory[14,15] and critical point theory[16-18], etc. Specially, the boundary value problems of third-order difference equations have been considered by several authors. For instance, by using the Guo-Krasnosel′skiis fixed point theorem, Agarwal and Henderson[10] considered the existence of positive solutions of the discrete third-order boundary value problems. Later, using the same theorem, Yang and Weng[19] considered the existence of at least one positive solution and two positive solutions of the discrete third-order nonlinear difference equation with several kinds of boundary conditions. In 2007, Karaca[20] studied the existence of positive solutions of the discrete third-order three-point eigenvalue problem.

Inspired by the work of above papers, the aim of the present paper is to establish some criteria for the existence of positive solutions of (1) in an explicit interval for λ. All results are based on the following fixed point theorem of cone expansion-compression type due to Guo-Krasnosel′skiis[21-22].

Theorem 1.1 Let E be a Banach space and K is a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0\in\Omega_1,\overline{\Omega_1}\subset\Omega_2$, and $A:K\cap(\overline{\Omega_2}\setminus\Omega_1)\rightarrow K$ is a completely continuous operator such that either

(i) $\|Au\|\leqslant\|u\|$ for $u\in K\cap\partial\Omega_1$ and $\|Au\|\geqslant\|u\|$ for $u\in K\cap\partial\Omega_2$, or

(ii) $\|Au\|\geqslant\|u\|$ for $u\in\cap K\cap\partial\Omega_1$ and $\|Au\|\leqslant\|u\|$ for $u\in K\cap\partial\Omega_2$.

Then A has fixed point in $K\cap(\overline{\Omega_2}\setminus\Omega_1)$.

The rest of this paper is arranged as follows. In section 2, we present some lemmas that will be used to prove our main results. In section 3, we discuss the existence of positive solutions of (1). We would like to point out that the existence results on infinitely many positive solutions of (1) we established has been given rather less attention in the existed literature. For every result, an open interval of eigenvalues is determined in an explicitly way.

2 Preliminaries

Let $E=\{u:[0,T+2]_Z\rightarrow\mathbf{R}\mid u(0)=\Delta u(\eta)=\Delta^2u(T)=0\}$. Then E is a Banach space under the norm $\|u\|=\max_{t\in[0,T+2]_Z}|u(t)|$ $E^+=\{u:[0,T+2]_Z\rightarrow[0,+\infty)\mid u(0)=\Delta u(\eta)=\Delta^2u(T)=0\}$ is the nonnegative functions in E.

At first, let us convert the following linear problem

$$\begin{cases}\Delta^3u(t-1)=y(t),t\in[1,T]_Z,\\ u(0)=\Delta u(\eta)=\Delta^2u(T)=0\end{cases}\tag{2}$$

to the equivalent summation equation. To get it, let us define the Green′s function $G(t;s)$ as follows.

$$G(t,s)=\begin{cases}t\min\{\eta,s\}+\dfrac{t-t^2}{2},t-2<s,\\[2ex] t\min\{\eta,s\}-st+\dfrac{s^2+s}{2},s\leqslant t-2\end{cases}\tag{3}$$

Let a, b are two integers with $b>a$. We define $\sum_{s=b}^{a}y(s)=0$. Then we get the following Lemma.

Lemma 2.1 The problem (2) has a unique solution

$$u(t)=\sum_{s=1}^{T}G(t,s)y(s)\tag{4}$$

where $G(t,s)$ is defined as in (3).

Proof Summing from $s=1$ to $s=t-1$ at both sides of the equation in (2), then we get

$$\Delta^2u(t-1)=\Delta^2u(0)+\sum_{s=1}^{t-1}y(s).$$

Repeating the above process, we obtain

$$\Delta u(t-1)=\Delta u(0)+(t-1)\Delta^2u(0)+\sum_{s=1}^{t-1}(t-s-1)y(s).$$

Summing from $s=1$ to $s=t$ at both sides of the above equation, we have

$$u(t) = t\,\Delta u(0) + \frac{t(t-1)}{2}\Delta^2 u(0) + \sum_{s=1}^{t-2}\left(\frac{(t-s)(t-s-1)}{2}\right)y(s).$$

By using the boundary condition $\Delta u(\eta) = \Delta^2 u(T) = 0$, we get that

$$\begin{cases} \Delta^2 u(0) + \sum_{s=1}^{T} y(s) = 0, \\ \Delta u(0) = \sum_{s=1}^{T} \eta y(s) - \sum_{s=1}^{\eta-1}(\eta - s)y(s) \end{cases}$$

Therefore,

$$u(t) = \sum_{s=1}^{T}\frac{2t\eta - t^2 + t}{2}y(s) - \sum_{s=1}^{\eta-1}t(\eta-s)y(s) + \sum_{s=1}^{t-2}\frac{(t-s)(t-s-1)}{2}y(s) \quad (5)$$

which implies (4) holds.

Lemma 2.2　For all $t\in[0,T+2]_Z$, $s\in[1,T]_Z$.

$$\theta^* G(\eta,s) \leqslant G(t,s) \leqslant G(\eta,s) \quad (6)$$

where

$$\theta^* = \min\left\{\frac{t}{\eta+1}, \frac{T+2-t}{T+1-\eta}\right\} \quad (7)$$

Proof　A direct computation shows that $0 < G(t,s) \leqslant G(\eta,s)$ for all $t\in[0,T+2]_Z$, $s\in[1,T]_Z$. For the lower bound, we proceed by case on the branches of the Green's function (3).

(i) If $t-2 < s \leqslant \eta - 1$: Then $G(t,s) = \frac{2st - t^2 + t}{2}$, $G(\eta,s) = \frac{2s\eta - \eta^2 + \eta}{2}$. Simple algebra yields

$$\frac{t}{\eta}G(\eta,s) \leqslant G(t,s).$$

(ii) $s\leqslant t-2 \leqslant \eta-1$: Since $G(t,s) = \frac{s^2+s}{2} = G(\eta,s)$, it follows that

$$\frac{T}{\eta+1}G(\eta,s) \leqslant G(t,s).$$

(iii) $s \leqslant \eta-1 < t-2$. As in case (2), $G(t,s) = \frac{s^2+s}{2} = G(\eta,s)$, thus

$$\frac{T+2-t}{T+1-\eta}G(\eta,s) \leqslant G(t,s).$$

(iv) $t-2 \leqslant \eta-1 < s$: $G(t,s) = \frac{2t\eta - t^2 + t}{2}$, and $G(\eta,s) = \frac{\eta^2+\eta}{2}$. Simple algebra yields

$$\frac{t}{\eta+1}G(\eta,s) \leqslant G(t,s).$$

(5) $\eta - 1 \leqslant t-2 < s$. As in case (4), $G(t,s) = \frac{2t\eta - t^2 + t}{2}$ and $G(\eta,s) = \frac{\eta^2+\eta}{2}$. Define

$$\omega(t) = G(t,s) - \frac{1}{2}\left(\frac{T+2-t}{T+1-\eta}\right)G(\eta,s) \quad (8)$$

Now $\omega(\eta+1) = 0$. $\omega(T+2) = G(T+2,s) > 0$, and $\Delta^2\omega(t) < 0$, i. e., ω is convex, then $\omega(t) \geqslant 0$ on $[\eta+1, T+2]_Z$; hence

$$\left(\frac{T+2-t}{T+1-\eta}\right)G(\eta,s) \leqslant G(t,s).$$

(6) $\eta - 1 < s \leqslant t-2 \leqslant T$. Note that $G(\eta,s) = \frac{\eta^2+\eta}{2}$, while

$$G(t,s) = \frac{2t\eta - t^2 + t}{2} + \frac{(t-s)(t-s-1)}{2} \geqslant \frac{2t\eta - t^2 + t}{2}.$$

consequently, the employment of as in (8) yields

$$\left(\frac{T+2-t}{T+1-\eta}\right)G(\eta,s) \leqslant G(t,s).$$

Since $\frac{t}{\eta} > \frac{t}{\eta+1}$, then $\theta^* = \min\left\{\frac{t}{\eta+1}, \frac{T+2-t}{T+1-\eta}\right\}$.

3　Existence results

In this section, we are concerned on the existence of at least one positive solution of (1). To get it, we assume that

(H_1) $f:[1,T]_Z\times[0,\infty)\to[0,+\infty)$ is continuous and the function $f(t,u)$ is increasing for each $(t,u)\in[1,T]_z\times[0,\infty)$;

(H_2) $a:[1,T]_Z\to[0,+\infty)$ is increasing.

Define the cone K by

$$K = \{u\in E^+ \mid u(t) \geqslant \theta^* \|u\|,\ t\in[0,T+2]_z\}.$$

where $\theta^* = \min\left\{\frac{t}{\eta+1}, \frac{T+2-t}{T+1-\eta}\right\}$.

Define the integral operator $T_\lambda: K\to K$ by

$$T_\lambda u(t) = \lambda\sum_{s=1}^{T}G(t,s)a(s)f(s,u(s)).$$

Obviously, if u is fixed point of T_λ in K, then u is positive solution of (1). Because E is finite dimensional, then we know that $T_\lambda: K\to K$ is completely continuous. Set

$$A = \max_{t\in[0,T+2]_Z} \sum_{s=1}^{T} G(\eta,s)a(s),$$

$$B = \max_{t\in[0,T+2]_Z} \sum_{s=1}^{T} \theta^* G(\eta,s)a(s),$$

$$f_0 = \lim_{u\to 0^+} \min_{t\in[0,T+2]_Z} \frac{f(t,u)}{u},$$

$$f_\infty = \lim_{u\to\infty} \min_{t\in[0,T+2]_Z} \frac{f(t,u)}{u},$$

$$f^0 = \lim_{u\to 0^+} \max_{t\in[0,T+2]_Z} \frac{f(t,u)}{u},$$

$$f^\infty = \lim_{u\to\infty} \max_{t\in[0,T+2]_Z} \frac{f(t,u)}{u}.$$

Theorem 3.1 Suppose that (H_1) and (H_2) hold. In addition, assume that there exist two positive constants r and R with $r \neq R$ such that

(A_1) $f(t,u)\leqslant\frac{r}{\lambda A}, \forall (t,u)\in[1,T]_z\times[0,r]$,

(A_2) $f(t,u)\geqslant\frac{r}{\lambda B}$, *for* $\forall (t,u)\in[1,T]_z\times[\theta^* R,R]$,

then (1) has at least one positive solution $u^*\in K$ with $\min\{r,R\}\leqslant\|u^*\|\leqslant\max\{r,R\}$.

Proof We only deal with the case $r<R$, the case $r<R$ is similar to it, so we omit the details. Let $\Omega_1=\{u\in E: \|u\|<r\}$ and $\Omega_2=\{u\in E: \|u\|<R\}$. It follows from ($A_1$) that, for any $u\in K\cap\partial\Omega_1$

$$\|T_\lambda u\| = \max_{t\in[0,T+2]_Z} \lambda\sum_{s=1}^{T} G(t,s)(a,s)f(s,u(s)) \leqslant \lambda \max_{t\in[0,T+2]_Z}\sum_{s=1}^{T} G(\eta,s)(a,s)\frac{r}{\lambda A} = r = \|u\|.$$

Therefore,

$$\|T_\lambda u\| \leqslant \|u\|, \text{ for } u\in K\cap\partial\Omega_1 \tag{9}$$

On the other hand, for any $u\in K\cap\partial\Omega_2$, $\theta^* R\leqslant|u(s)|\leqslant R$, for $s\in[1,T]_z$. It follows from (A_2) that for any $u\in K\cap\partial\Omega_2$,

$$\|T_\lambda u\| = \max_{t\in[0,T+2]_Z} \lambda\sum_{s=1}^{T} G(t,s)(a,s)f(s,u(s)) \geqslant \lambda \max_{t\in[0,T+2]_Z}\sum_{s=1}^{T} \theta^* G(\eta,s)(a,s)\frac{R}{\lambda B} = R = \|u\|.$$

Therefore,

$$\|T_\lambda u\| \geqslant \|u\|, \text{ for } u\in K\cap\partial\Omega_2 \tag{10}$$

Applying Theorem 1.1 (i) to (9) and (10) yields that T_λ has a fixed point $u^*\in K\cap(\overline{\Omega_2}\backslash\Omega_1)$, and then u^* is a positive solution of (1) with $r\leqslant u^*\leqslant R$.

Theorem 3.2 Suppose that (H_1) and (H_2) hold. In addition, assume that

(A_3) $f^0=0, f_\infty=\infty$, or

(A_4) $f_0=\infty, f^\infty=0$.

Then for any $\lambda\in(0,\infty)$, (1) has at least one positive solution.

Proof First we consider the case of (A_3) holds.

For any $\lambda\in(0,\infty)$, since $f^0=0$, for $\frac{1}{\lambda A}>0$, there exists $R_1>0$ such that $\frac{f(t,u)}{u}\leqslant\frac{1}{\lambda A}$ for $(t,u)\in[1,T]_z\times[0,R_1]$. Therefore,

$$f(t,u)\leqslant\frac{u}{\lambda A}\leqslant\frac{R_1}{\lambda A}, \text{ for } (t,u)\in[1,T]_z\times[0,R_1].$$

On the other hand, since $f_\infty=\infty$, for $\frac{1}{\theta^*\lambda B}>0$, there exists $R_2>R_1$ such that $\frac{f(t,u)}{u}\leqslant\frac{1}{\theta^*\lambda B}$, for $(t,u)\in[1,T]_z\times[\theta^* R_2,\infty)]$, which implies

$$f(t,u)\geqslant\frac{u}{\theta^*\lambda B}\geqslant\frac{\theta^* R_2}{\theta^*\lambda B}=\frac{R_2}{\lambda B}, \text{ for } (t,u)\in[1,T]_z\times[\theta^* R_2,\infty)].$$

Therefore, by using Theorem 3.1. T_λ has a fixed point $u^*\in K$.

Next, let (A_4) holds. In view of $f_0=\infty$, for $\frac{1}{\theta^*\lambda B}>0$, there exists $R_1>0$ such that

$$f(t,u)\geqslant\frac{u}{\theta^*\lambda B}, \text{ for } (t,u)\in[1,T]_z\times[0,R_1] \tag{11}$$

Set $\Omega_1=\{u\in E: \|u\|<R_1\}$. For $u\in K\cap\Omega_1$,

$$u(s)\geqslant\theta^*\|u\|=\theta^* R_1.$$

Thus from Lemma (3) and (11) one can conclude that

$$\|T_\lambda u\| = \max_{t\in[0,T+2]_z} \lambda\sum_{s=1}^{T} G(t,s)a(s)f(s,u(s)) \geqslant \frac{1}{\theta^* B}\max_{t\in[0,T+2]_z}\lambda\sum_{s=1}^{T}\theta^* G(\eta,s)a(s)\theta^*\|u\| = \|u\|,$$

which implies

$$\|T_\lambda u\| \geqslant \|u\|, \text{ for } u \in K \cap \partial\Omega_1 \quad (12)$$

Again, since $f^\infty = 0$, for $\frac{1}{\lambda A} > 0$, there exists $R_0 > 0$ such that

$$f(t,u) \leqslant \frac{u}{\lambda A}, \text{ for } (t,u) \in [1,T]_Z \times [R_0,\infty) \quad (13)$$

We consider two cases: f is bounded and f is unbounded.

Case 1. Supposed f is bounded, say $f \leqslant M$. Let $R_2 = \max\{2R_1, \lambda MA\}$. If $u \in K$ with $\|u\| = R_2$,

$$\|T_\lambda u\| \leqslant \lambda \max_{t\in[0,T+2]_z} \sum_{s=1}^{T} G(\eta,s)a(s)M = \lambda MA \leqslant R_2 = \|u\|.$$

Consequently, $\|T_\lambda u\| \leqslant \|u\|$. If we set $\Omega_2 = \{u \in K: \|u\| < R_2\}$, then

$$\|T_\lambda u\| \leqslant \|u\|, \text{ for } u \in K \cap \partial\Omega_2 \quad (14)$$

Case 2. Supposed f is unbounded, we let $R_2 > \max\{2R_1, R_0\}$ such that $f(t,u) \leqslant f(t,R_2)$, for $(t,u) \in [1,T]_Z \times [0,R_2)$. For $u \in K$ with $\|u\| = R_2$ by Lemma (3) and (13).

$$\|T_\lambda u\| = \max_{t\in[0,T+2]_z} \lambda \sum_{s=1}^{T} G(t,s)a(s)f(s,u(s)) \leqslant \lambda \max_{t\in[0,T+2]_z} \sum_{s=1}^{T} G(\eta,s)a(s)\frac{R_2}{\lambda A} \leqslant R_2 = \|u\|.$$

If we set $\Omega_2 = \{u \in K: \|u\| < R_2\}$, then

$$\|T_\lambda u\| \leqslant \|u\|, \text{ for } u \in K \cap \partial\Omega_2 \quad (15)$$

Applying Theorem 1.1 (ii) to (13) and (14) yields that T_λ has a fixed point $u^* \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$. Also applying Theorem 1.1 (ii) to (12) and (15) yields that T_λ has a fixed point $u^* \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$. Then u^* has a positive solution of (1).

Theorem 3.3 Suppose that (H_1) and (H_2) hold. In addition, assume that $0 < Af^0 < \theta^* Bf_\infty < \infty$. Then for each $\lambda \in (\frac{1}{\theta^* Bf_\infty}, \frac{1}{Af^0})$, (1) has one positive solution.

Proof We construct the set Ω_1 and Ω_2. In order to apply Theorem 1.1, let $\lambda \in (\frac{1}{\theta^* Bf_\infty}, \frac{1}{Af^0})$ and choose $\varepsilon > 0$ such that

$$\frac{1}{\theta^* B(f_\infty - \varepsilon)} \leqslant \lambda \leqslant \frac{1}{A(f^0 + \varepsilon)}.$$

By the definition of f^0, there exists $R_1 > 0$ such that $f(t,u) \leqslant (f^0 + \varepsilon)u$, for $(t,u) \in [1,T]_Z \times [0, R_1]$. Let $u \in K$ with $\|u\| = R_1$. Then

$$\|T_\lambda u\| = \max_{t\in[0,T+2]_z} \lambda \sum_{s=1}^{T} G(t,s)a(s)f(s,u(s)) \leqslant \lambda \max_{t\in[0,T+2]_z} \sum_{s=1}^{T} G(\eta,s)a(s)(f^0 + \varepsilon)\|u\| = \lambda A(f^0 + \varepsilon)\|u\| \leqslant \|u\|.$$

Consequently, $\|T_\lambda u\| \leqslant \|u\|$. If we set $\Omega_2 = \{u \in K: \|u\| < R_1\}$, then

$$\|T_\lambda u\| \leqslant \|u\|, \text{ for } u \in K \cap \partial\Omega_1 \quad (16)$$

Next we construct the set Ω_2. By the definition of f_∞, there exists $\overline{R_2}$ such that $f(t,u) \geqslant (f_\infty + \varepsilon)u$ for $(t,u) \in [1,T]_Z \times [0,R_1]$. Let $R_2 = \max\{2R_1, \frac{R_2}{\theta^*}\}$ and $\Omega_2 = \{u \in K: \|u\| < R_2\}$.

If $u \in K$ with $\|u\| = R_1$, then $u(s) \geqslant \theta^* \|u\| \geqslant \overline{R_2}$. Therefore

$$\|T_\lambda u\| = \max_{t\in[0,T+2]_z} \lambda \sum_{s=1}^{T} G(t,s)a(s)f(s,u(s)) \geqslant \lambda \max_{t\in[0,T+2]_z} \sum_{s=1}^{T} \theta^* G(\eta,s)a(s)(f_\infty - \varepsilon)\theta^* \|u\| = \lambda\theta^* B(f_\infty - \varepsilon)\|u\| \geqslant \|u\|.$$

Hence

$$\|T_\lambda u\| \leqslant \|u\|, \text{ for } u \in K \cap \partial\Omega_2 \quad (17)$$

Applying Theorem 1.1 (i) to (16) and (17) yields that T_λ has a fixed point $u^* \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, which is a positive solution of (1).

Theorem 3.4 Suppose that (H_1) and (H_2) hold. In addition, assume that $0 < Af^\infty < \theta^* Bf_0 < \infty$. Then for each $\lambda \in (\frac{1}{\theta^* Bf_0}, \frac{1}{Af^\infty})$, (1) has one positive solution.

Proof We construct the set Ω_1 and Ω_2. In order to apply Theorem 1.1, let $\lambda \in (\frac{1}{\theta^* Bf_0}, \frac{1}{Af^\infty})$ and choose $\varepsilon > 0$ such that

$$\frac{1}{\theta^* B(f_0 - \varepsilon)} \leqslant \lambda \leqslant \frac{1}{A(f^\infty + \varepsilon)}.$$

By the definition of f_0, there exists $R_1 > 0$ such that $f(t,u) \geqslant (f_0 - \varepsilon)u$, for $(t,u) \in [1,T]_Z \times [0,R_1]$. So if $u \in K$ with $\|u\| = R_1$, then

$u(s) \geqslant \theta^* \|u\| = \theta^* R_1$.

Thus, in the same way of above, we have

$$\|T_\lambda u\| = \max_{t\in[0,T+2]_z} \lambda \sum_{s=1}^{T} G(t,s)a(s)f(s,u(s)) \geqslant \lambda\theta^* B(f_0-\varepsilon)\|u\| \geqslant \|u\|.$$

Consequently, $\|T_\lambda u\| \geqslant \|u\|$. So if we set $\Omega_2 = \{u \in K: \|u\| < R_1\}$ then

$$\|T_\lambda u\| \geqslant \|u\|, \text{ for } u \in K \cap \partial\Omega_1 \quad (18)$$

Next we construct the set Ω_2. By the definition of f^∞, there exists $\overline{R_2}$ such that $f(t,u) \leqslant (f^\infty + \varepsilon)u$, for $(t,u) \in [1,T]_Z \times [\overline{R_2},\infty)$. We consider two cases: f is bounded and f is unbounded. If f is bounded by $M>0$, set $R_2 = \max\{2R_1, \lambda MA\}$. Then

If $u \in K$ with $\|u\| = R_2$,

$$\|T_\lambda u\| = \max_{t\in[0,T+2]_z} \lambda \sum_{s=1}^{T} G(t,s)a(s)f(s,u(s)) \leqslant \lambda \max_{t\in[0,T+2]_z} \sum_{s=1}^{T} G(\eta,s)a(s)M = \lambda AM \leqslant R_2 = \|u\|.$$

Consequently, $\|T_\lambda u\| \leqslant \|u\|$. So if we set $\Omega_2 = \{u \in K: \|u\| < R_2\}$ then

$$\|T_\lambda u\| \leqslant \|u\|, \text{ for } u \in K \cap \partial\Omega_2 \quad (19)$$

When f is unbounded, we let $R_2 \geqslant \max\{2R_1, \overline{R_2}\}$ be such that $f(t,u) \leqslant f(t,R_2)$, for $(t,u) \in [0,T]_z \times [0,R_2]$. If $\{u \in K: \|u\| = R_2\}$, by using method to get (3.10), we get that

$$\|T_\lambda u\| \leqslant \|u\|.$$

Hence if we set $\Omega_2 = \{u \in K: \|u\| < R_2\}$.

$$\|T_\lambda u\| \leqslant \|u\|, \text{ for } u \in K \cap \partial\Omega_2. \quad (20)$$

Applying Theorem 1.1 (ii) to (18) and (19) yields T_λ has a fixed point $u^* \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$. Also applying Theorem 1.1 (ii) to (18) and (19) yields T_λ has a fixed point $u^* \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$. Then u^* is a positive solution of (1). The proof is compete.

We state the following result similar to Theorem 3.3 and 3.4 without proof.

Theorem 3.5 Suppose (H_1) and (H_2) hold.

(1) If $f_\infty = \infty, 0 < f^0 < \infty$ then for each $\lambda \in (0, \frac{1}{Af^0})$, (1) has at least one positive solution.

(2) If $f_0 = \infty, 0 < f^\infty < \infty$ then for each $\lambda \in (0, \frac{1}{Af^\infty})$, (1) has at least one positive solution.

(3) If $f^0 = \infty, 0 < f_\infty < \infty$ then for each $\lambda \in (\frac{1}{\theta^* Bf_\infty}, \infty)$, (1) has at least one positive solution.

(4) If $f^\infty = \infty, 0 < f_0 < \infty$ then for each $\lambda \in (\frac{1}{\theta^* Bf_0}, \infty)$, (1) has at least one positive solution.

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