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含参原始向量混合拟均衡问题和含参对偶向量混合拟均衡问题解的 Hölder 连续性

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摘 要：本文首先将对偶法则应用于均衡问题，提出了含参对偶向量混合拟均衡问题，即混合 Minty-type 含参对偶向量混合拟均衡问题；其次在更一般的集合上研究含参原始向量混合拟均衡问题(PVMQEPi)(i=1,2)和含参对偶向量混合拟均衡问题(DVMQEPi)(i=1,2)解的 Hölder 连续性，并用适当的注和例子来逐一说明各个定理的结果；最后，将混合 Minty-type 含参对偶向量混合拟均衡问题应用于 Minty-type 变分不等式. 本文的结论是对其他作者的研究工作的推广和改进.

关键词：Hölder 连续性；含参原始向量混合拟均衡；含参对偶向量混合拟均衡

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Hölder continuity of solutions for parametric primal and dual vector mixed quasi-equilibrium problems

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Abstract: In this paper, following the dual rule for equilibrium problems, we introduce the corresponding parametric dual vector mixed quasi-equilibrium problem, which is mix Minty-type parametric dual vector quasi-equilibrium problem. We discuss the Hölder continuity of the solutions of parametric primal vector mixed quasi-equilibrium problem (PVMQEPi)(i=1,2) and parametric dual vector mixed quasi-equilibrium problem(DVMQEPi)(i=1,2) in general settings and also provide suitable remarks and examples to illustrate our results on by one. Finally, we apply mixed Minty-type dual vector mixed quasi-equilibrium problems to deal with mix Minty variational inequality. Our results improve and extend the corresponding results announced by many others.

Keywords: Hölder continuity; Parametric primal vector mixed quasi-equilibrium; Parametric dual vector mixed quasi-equilibrium
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1 Introduction

Vector equilibrium problem provides a unified model of several important problems, such as vector optimization problem, vector variational inequality, vector completeness problems, Nash equilibrium problems, etc. Recently, the sensitivity analysis[1-5], especially the upper semicontinuity and the Hölder continuity of the solution mappings to parametric vector equilibrium problems and parametric vector variational inequality have been intensively studied [7-15].

Hölder continuity of solutions plays an important role in theory of stability analysis for vector equilibrium, but there may be less works in the literature denoted to this property than to semicontinuity. The aim of this paper is to establish sufficient conditions for the Hölder continuity of the solutions to the two parametric primal and dual vector mixed quasi-equilibrium problems.

In this paper, firstly, we generalize results of Hölder continuity in Ref. [11] to more general settings and extend upper Hölder continuity[14] to Hölder continuity. Secondly, following the dual rules for an equilibrium problem, we introduce the corresponding parametric dual vector mixed quasi-equilibrium (DVMQEPi)(i=1,2), which is mix Minty-type parametric dual vector quasi-equilibrium (see Remark 2). We discuss the Hölder continuity of the solutions of (PVMQEPi)(i=1,2)and (DVMQEPi)(i=1,2) and provide suitable remarks and examples to illustrate our results on by one. Finally, we apply (DVMQEPi)(i=1,2) to deal with mix Minty variational inequality as an application.

2 Prelimaries

Throughout this paper, unless otherwise specified, let X,Λ,M and Y be metric linear spaces, and let $\Omega\subset Y$ be a closed subset with $\Omega\neq\varnothing$. Let $K:X\times\Lambda\to 2^X\varphi:X\times X\times M\to 2^Y$ and $\varphi:X\times M\to 2^Y$ be three set-valued mappings with nonempty values.

For the parameters $\lambda\in\Lambda$ and $\mu\in M$, we consider the following two parametric vector mixed quasi-equilibrium problems of finding $\bar{x}\in K(\bar{x},\lambda)$ such that

(PVMQEP 1) $[\varphi(\bar{x},y,\mu)+\psi(y,\mu)-\psi(\bar{x},\mu)]\cap\Omega\neq\varnothing,\ \forall y\in K(\bar{x},\lambda)$,

and of finding $\bar{x}\in K(\bar{x},\lambda)$, such that

(PVMQEP 2) $[\varphi(\bar{x},y,\mu)+\varphi(y,\mu)-\varphi(\bar{x},\mu)]\subset\Omega,\ \forall y\in K(\bar{x},\lambda)$.

Denote $E(\lambda)=\{x\in X\mid x\in K(x,\lambda)\}$ and

$F(x,y,\mu)=\varphi(x,y,\mu)+\psi(y,\mu)-\psi(x,\mu)$,

for each $(\lambda,\mu)\in\Lambda\times M$.

Let $S_i(\lambda,\mu)(i=1,2)$ be the solution set of (PVMQEPi)(i=1,2), i. e.

$S_1(\lambda,\mu)=\{x\in E(\lambda)\mid F(x,y,\mu)\cap\Omega\neq\varnothing,\ \forall y\in K(x,\lambda)\}$

and

$S_2(\lambda,\mu)=\{x\in E(\lambda)\mid F(x,y,\mu)\subset\Omega,\ \forall y\in K(x,\lambda)\}$.

***Remark* 1** The models (PVMQEPi)(i=1, 2) contain many problems as special cases, for example, we can replace Ω by setting $\Omega=Y\backslash-\mathrm{int}C(\mathrm{int}C\neq\varnothing)$, $\Omega=(Y\backslash-C)\cup l(C)(=Y\backslash(-C\backslash l(C)))$, $\Omega=C$, $\Omega=Y\backslash-C$respectively, where $\varnothing\neq C\subset Y$, int$C$-stands for the interior of C, and $l(C)=C\cap(-C)$ In particular, if C is a closed convex pointed cone in $Y(l(C))=\{0\}$ in this case) then above cases become the so-called weak and strong vector quasi-equilibrium problem considered by many authors.

Following the dual rules for an equilibrium problem proposed by Konnov and Schaible[15], whose schemes is an extension of the classical dual theory for variational inequality, we now introduce the corresponding parametric dual vector mixed quasi-equilibrium to (PVMQEPi) of finding $\bar{x}\in K(\bar{x},\lambda)$ such that

(DVMQEP 1) $F(y,\bar{x},\mu)\cap(-\Omega)\neq\phi$, $\forall y\in K(\bar{x},\lambda)$,

and of finding $\bar{x}\in K(\bar{x},\lambda)$ such that

(DVMQEP 2) $F(y,\bar{x},\mu)\subset-\Omega$, $\forall y\in K(\bar{x},\lambda)$.

Let $S_i^d(\lambda,m)$, $i=1,2$ be the solution set of (DVMQEPi)(i=1,2), i. e.

$S_1^d(\lambda,\mu)=\{x\in E(\lambda)\mid F(y,x,\mu)\cap(-\Omega)\neq\varnothing,\ \forall y\in K(x,\lambda)\}$, and

$S_1^d(\lambda,\mu)=\{x\in E(\lambda)\mid F(y,x,\mu)\subset(-\Omega)\neq\varnothing,$

$\forall y\in K(x,\lambda)\},$

***Remark* 2** Let $T:X\times\Lambda\to L(X,Y)$ be a vector-valued mapping and $C\subset Y$ be a closed cone with int$C\neq\varnothing$ where $L(X,Y)$ is a set of bounded linear mapping from X to Y. If $\varphi(x,y,\mu)=[T(x,\mu),y-x]$, $K(x,\lambda)=K(\lambda)$, and $\Omega=Y\backslash-(C\backslash\{0\})$, let $Y=R$, and $C=R_+$ then (DVMQEPi)(i=1,2) reduce to the mixed Minty vector variational inequality (MMVVI) involving parameters: finding $\bar{x}\in K(\lambda)$ such that $\psi(\bar{x},\mu)=[T(y,\mu),\bar{x}-y]+\psi(\bar{x},\mu)-\psi(y,\mu)\leqslant 0,\forall y\in K(\lambda)$.

Especially, when $\psi=0$ (DVMQEPi)(i=1,2) simultaneously collapse to the well-known parametric Minty variational inequality (MVI). Whence, we may view parametric dual vector mixed quasi-equilibrium with general settings proposed in this paper as Mix Minty type. To the best of our knowledge, there was nearly no result denote to Hölder continuity of mix Minty-type parametric dual vector quasi-equilibrium in the literature.

Throughout this paper, we always assume that $S_i(\lambda,\mu)\neq\varnothing$ (i=1,2) and for all (λ,μ) in a neighborhood of some point $(\bar{\lambda},\bar{\mu})\in\Lambda\times M$.

Now we recall some basic definitions which are needed in this paper. In the sequel, $B_X(\theta,\delta)$ denotes the closed ball with center $\theta\in X$ and radius $\delta>0$, B_X denotes the closed unit ball of X and $d(\cdot,\cdot)$ denotes the distance in metric spaces.

Definition 2.1 Let X and Y be two topological spaces, and $F:X\to 2^Y$ be a set-valued mapping.

(i) F is said to be upper semicontinuous at $x_0\in X$, if for every open set U with $F(x_0)\subset U$ there is a neighborhood $N(x_0)$ of x_0 in X such that $F(x)\subset U,\forall x\in \mathbf{N}(x_0)$.

(ii) F is said to be lower semicontinuous at $x_0\in X$, if for every open set U with $F(x_0)\cap U\neq\varnothing$ there is a neighborhood $N(x_0)$ of x_0 in X such that $F(x)\cap U\neq\varnothing,\forall x\in\mathbf{N}(x_0)$

Lemma 2.2 Let X and Y be two topological spaces, and $F:X\to 2^Y$ be a set-valued mapping.

(i) Let F is lower semicontinuous at $x_0\in X$, if and only if for any $y_0\in F(x_0)$ and for any net $\{x_\alpha\}$ satisfying $x_\alpha\to x_0$, there exists a net $\{y_\alpha\}$ such that $y_\alpha\in F(x_\alpha)$ and $y_\alpha\to y_0$.

(ii) Let F be compact-valued on X. Then F is upper semicontinuous at x_0 if and only if for any net $\{x_\alpha\}\subset X$ with $x_\alpha\to x_0$ and for every $y_\alpha\in F(x_\alpha)$, there exist $y_0\in F(x_0)$ and subset $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta\to y_0$.

Definition 2.3 A set-valued mapping $G:\Lambda\to 2^X$ is said to be $l.\alpha$-Hölder continuous in $M\subset\Lambda$, if $\forall\mu_1,\mu_2\in M$

$$G(\mu_1)\subset G(\mu_2)+lB_X(\theta,d^\alpha(\mu_1,\mu_2)),$$

where $l\geqslant 0$ and $\alpha\geqslant 0$.

Definition 2.4 A set-valued mapping $G:X\times\Lambda\to 2^X$ is said to be $(l_1.\alpha_1,l_2.\alpha_2)$- Hölder continuous in $M\subset X\times\Lambda$, if $\forall(x_1,\lambda_1),(x_2,\lambda_2)\in M$,

$$G(x_1,\lambda_1)\subset\{x\in X\mid\exists z\in G(x_2,\lambda_2),$$
$$d(x,z)\leqslant l_1d^\alpha(x_1,x_2)+l_2d^\alpha(\lambda_1,\lambda_2)\}.$$

where $l_1,l_2\geqslant 0$ and $\alpha_1,\alpha_2\geqslant 0$.

Definition 2.5 Let (X,d) be a metric space and H be a Hausdorff metric on the collection $CB(X)$ of all nonempty closed bounded subsets of X, which is defined as

$$H(A,B)=\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\},$$
$$\forall A,B\in CB(X),$$

where $d(a,B)=\inf_{b\in B}d(a,b)$ and $d(A,b)=\inf_{a\in A}d(a,b)$.

3 Hölder continuity of solutions to (PVMQEP1)

In this section, we discuss the Hölder continuity of solutions to (PVMQEP1).

Proposition 3.1 Assume that the solutions for the problem(PVMQEP1) exist in a neighborhood $N(\bar{\lambda})\times N(\bar{\mu})$ of the reference point $N(\bar{\lambda},\bar{\mu})\in\Lambda\times M$ Assume further that the following conditions hold:

(a) For each $\lambda\in N(\bar{\lambda})$, $E(\lambda)$ is compact and $K(\cdot,\lambda)$ is lower semicontinuous in $E(\lambda)$;

(b) For each ($(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu})$, and $\varphi(\cdot,\cdot,\mu)$ is upper semicontinuous with compact

values in $E(\lambda)\times K(E(\lambda),\lambda)\psi(\cdot,\mu)$ is upper continuous with compact values in $K(E(\lambda),\lambda)$.
Then, for any $(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu})S_1(\lambda,\mu)$ is a compact subset in $E(\lambda)$.

Proof It suffices to show that $S_1(\lambda,\mu)$ is closed in $E(\lambda)$, since $E(\lambda)$ is compact. Take any sequence $\{x_n\}\subset S_1(\lambda,\mu)$ with $x_n\to x_0$. It follows from $x_n\in E(\lambda)$ and the compactness of $E(\lambda)$ that $x_0\in E(\lambda)$. Suppose that $x_0\notin S_1(\lambda,\mu)$ Then, $\exists y_0\in K(x_0,\lambda)$, such that $F(x_0,y_0,\mu)\cap\Omega=\varnothing$. Since $K(\cdot,\lambda)$ is lower semicontinuity at x_0, it follows from (i) of Lemma 2.2 that, there exist $\bar{y}_n\in K(x_n,\lambda)$ such that $\bar{y}_n\to y_0$. As $x_n\in S_1(\lambda,\mu)$, there exist $z_n\in F(x_n,y_n,\mu)\cap\Omega$. From condition (b), we have $F(\cdot,\cdot,\mu)$ is upper semicontinuous with compact values, there exists $z_0\in F(x_0,y_0,\mu)$ such that $z_n\to z_0$. Noting the closeness of Ω, we have $z_0\in\Omega$ This leads to a contraction. Thus, $x_0\in S_1(\lambda,\mu)$ and $S_1(\lambda,\mu)$ is closed set.

Theorem 3.2 Assume that the solutions for the (PVMQEP1) exist in a neighborhood $N(\bar{\lambda})\times N(\bar{\mu})$ of the reference point $(\bar{\lambda},\bar{\mu})\in\Lambda\times M$ and the conditions of Proposition 3.1 are satisfied. Assume further that the following conditions hold:

(i) $K(\cdot,\cdot)$ is $(l_1.\alpha_1,l_2.\alpha_2)$-Hölder continuous in $E(N(\bar{\lambda}))\times N(\bar{\lambda})$;

(ii) There are constants $\alpha>0$ and $\beta>0$ such that $\forall(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu})$,

$\forall y\in E(N(\bar{\lambda}))\backslash S_1(\lambda,\mu)$, $\exists\hat{x}\in S_1(\lambda,\mu)$ satisfying

$$\alpha d^{\beta}(\hat{x},y)\leqslant\inf_{g\in F(y,\hat{x},\mu)}d(g,\Omega)+\inf_{f\in F(\hat{x},y,\mu)}d(f,\Omega);$$

(iii) $\forall\lambda\in N(\bar{\lambda})$, $\forall x,y\in E(\lambda)$, $F(x,y,\cdot)$ is $m.r$-Hölder continuous at $\bar{\mu}$;

(iv) $\forall(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu})$, $\forall x\in E(\lambda)$, $F(x,\cdot,\mu)$ is $n.\delta$-Hölder continuous in $K(E(N(\bar{\lambda})),N(\bar{\lambda}))$;

(v) $\beta=\alpha_1\delta$, $\alpha>2nl_1^{\delta}$.

Then for any $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N(\bar{\lambda})\times N(\bar{\mu})$,

$$H(S_1(\lambda_1,\mu_1),S_1(\lambda_2,\mu_2))\leqslant\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2)+\left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2)\tag{1}$$

Proof By Proposition 3.1 for each $(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu})S_1(\lambda,\mu)$ is a compact subset. Let $(\lambda_1,\mu_1)(\lambda_2,\mu_2)\in N(\bar{\lambda})\times N(\bar{\mu})$, we split the proof into three steps.

Step 1. We prove that

$$H(S_1(\lambda_1,\mu_1),S_1(\lambda_2,\mu_2))\leqslant\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2)\tag{2}$$

Obviously, if $S_1(\lambda_1,\mu_1)=S_1(\lambda_2,\mu_2)$ we have that (2) holds. So we suppose

$S_1(\lambda_1,\mu_1)\neq S_1(\lambda_2,\mu_2)$.

There are two cases to be considered.

Case 1. $S_1(\lambda_1,\mu_1)\not\subset S_1(\lambda_1,\mu_2)$, and $S_1(\lambda_1,\mu_1)\not\subset S_1(\lambda_1,\mu_1)$. For any $x(\lambda_1,\mu_1)\in S_1(\lambda_1,\mu_1)\backslash S_1(\lambda_1,\mu_2)$ by virtue of the assumption (ii), there exists $x(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)$ such that

$$\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant\inf_{g\in F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)}d(g,\Omega)+\inf_{f\in F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)}d(f,\Omega)\tag{3}$$

Since $x(\lambda_1,\mu_1)\in K(x(\lambda_1,\mu_1),\lambda_1)$, $x(\lambda_1,\mu_2)\in K(x(\lambda_1,\mu_2),\lambda_1)$, $K(\cdot,\cdot)$ is $(l_1.\alpha_1,l_2.\alpha_2)$-Hölder continuous in $E(N(\bar{\lambda}))\times N(\bar{\lambda})$, there exist $x_1\in K(x(\lambda_1,\mu_2),\lambda_1)$, $x_2\in K(x(\lambda_1,\mu_1),\lambda_1)$ such that

$$d(x(\lambda_1,\mu_1),x_1)\leqslant l_1d^{\alpha_1}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)),$$
$$d(x(\lambda_1,\mu_2),x_2)\leqslant l_1d^{\alpha_1}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\tag{4}$$

Because $x(\lambda_1,\mu_1)\in S_1(\lambda_1,\mu_1)$, $x(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)$, we have

$\exists z_1\in F(x(\lambda_1,\mu_1),x_2,\mu_1)\cap\Omega$,

$\exists z_2\in F(x(\lambda_1,\mu_2),x_1,\mu_2)\cap\Omega$,

From (3), we get

$$\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant\inf_{g\in F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)}d(g,z_1)+\inf_{f\in F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)}d(f,z_2)\leqslant$$
$$H(F(x(\lambda_1,\mu_1),x_2,\mu_1),F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_2),\mu_2))+$$
$$H(F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2),F(x(\lambda_1,\mu_2),x_1,\mu_2))\leqslant$$

$$H(F(x(\lambda_1,\mu_1),x_2,\mu_1),F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1))+$$
$$H(F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1),F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2))+$$
$$H(F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2),F(x(\lambda_1,\mu_2),x_1,\mu_2)).$$

From the Hölder continuity of F and (4), we get

$$\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant nd^{\delta}(x(\lambda_1,\mu_1),x_1)+ md^{\gamma}(\mu_1,\mu_2)+nd^{\delta}(x(\lambda_1,\mu_2),x_2)\leqslant 2nl_1^{\delta}d^{\alpha_1\delta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))+ md^{\gamma}(\mu_1,\mu_2) \quad (5)$$

By assumption (v), we can obtain

$$d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant \left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2).$$

Noting the arbitrariness of $x(\lambda_1,\mu_1)\in S_1(\lambda_1,\mu_1)\backslash S_1(\lambda_1,\mu_2)$ we have

$$\sup_{x(\lambda_1,\mu_1)\in S_1(\lambda_1,\mu_1)\backslash S_1(\lambda_1,\mu_2)}\ \inf_{x(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)} d(x(\lambda_1,\mu_1), x(\lambda_1,\mu_2))\leqslant\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2).$$

By the definition of the distance $d(\cdot,\cdot)$, we have

$$\sup_{x(\lambda_1,\mu_1)\in S_1(\lambda_1,\mu_1)} d(x(\lambda_1,\mu_1),S_1(\lambda_1,\mu_2))= \sup_{x(\lambda_1,\mu_1)\in S_1(\lambda_1,\mu_1)\backslash S_1(\lambda_1,\mu_2)} d(x(\lambda_1,\mu_1),S_1(\lambda_1,\mu_2))\leqslant \left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2) \quad (6)$$

Similarly, we can deduce

$$\sup_{x(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)} d(x(\lambda_1,\mu_2),S_1(\lambda_1,\mu_1))\leqslant \left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2) \quad (7)$$

From (6) and (7), we know (2) holds.

Case 2. $S_1(\lambda_1,\mu_1)\subset S_1(\lambda_1,\mu_2)$ or $S_1(\lambda_1,\mu_2)\subset S_1(\lambda_1,\mu_1)$. Without loss of generality, we can assume that $S_1(\lambda_1,\mu_1)\subset S_1(\lambda_1,\mu_2)$ From the definition of the distance $d(\cdot,\cdot)$ we have

$$\sup_{x(\lambda_1,\mu_1)\in S_1(\lambda_1,\mu_1)} d(x(\lambda_1,\mu_1),S_1(\lambda_1,\mu_2))=0 \quad (8)$$

By using the same argument as in Case 1, we also have that (7) holds in Case 2. It follows from (7), (8) that (2) also holds.

Step. 2. Now we show that

$$H(S_1(\lambda_1,\mu_2),S_1(\lambda_2,\mu_2))\leqslant \left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\alpha_2\gamma}{\beta}}(\lambda_1,\lambda_2) \quad (9)$$

Obviously, if $S_1(\lambda_1,\mu_2) = S_1(\lambda_2,\mu_2)$, we have that (9) holds. Similar to Step 1, we suppose $S_1(\lambda_1,\mu_2)\neq S_1(\lambda_2,\mu_2)$. There are two cases to be considered.

Case 1. $S_1(\lambda_1,\mu_2)\not\subset S_1(\lambda_1,\mu_2)$ and $S_1(\lambda_2,\mu_2)\not\subset S_1(\lambda_1,\mu_2)$. For any $x(\lambda_2,\mu_2)\in S_1(\lambda_1,\mu_2)\backslash S_1(\lambda_1,\mu_2)$ by virtue of the assumption (ii), there exists $\bar{x}(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)$, such that

$$\alpha d^{\beta}(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2))\leqslant \inf\nolimits_{g\in F(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2),\mu_2)}d(g,\Omega)+ \inf\nolimits_{f\in F(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2)}d(f,\Omega) \quad (10)$$

Since $x(\lambda_2,\mu_2)\in K(x(\lambda_2,\mu_2),\lambda_2)$, $\bar{x}(\lambda_1,\mu_2)\in K(\bar{x}(\lambda_1,\mu_2),\lambda_1)$, $K(\cdot,\cdot)$ is Hölder continuous, there exist $\bar{x}_1\in K(\bar{x}(\lambda_2,\mu_2),\lambda_1)$, $\bar{x}_2\in K(\bar{x}(\lambda_1,\mu_2),\lambda_2)$ such that

$$d(x(\lambda_2,\mu_2),\bar{x}_1)\leqslant l_2 d^{\alpha_2}(\lambda_1,\lambda_2),$$
$$d(\bar{x}(\lambda_1,\mu_2),\bar{x}_2)\leqslant l_2 d^{\alpha_2}(\lambda_1,\lambda_2) \quad (11)$$

By the Hölder continuity of $K(\cdot,\cdot)$ again, there exist $\bar{x}_1'\in K(\bar{x}(\lambda_1,\mu_2),\lambda_1)$, $\bar{x}_2'(\lambda_1,\mu_2)\in K(x(\lambda_2,\mu_2),\lambda_2)$
such that

$$d(\bar{x}_1,\bar{x}_1')\leqslant l_1 d^{\alpha_1}(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2)),$$
$$d(\bar{x}_2,\bar{x}_2')\leqslant l_1 d^{\alpha_1}(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) \quad (12)$$

Because $x(\lambda_2,\mu_2)\in S_1(\lambda_2,\mu_2)$, $\bar{x}(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)$ we have

$$\exists z_1'\in F(x(\lambda_2,\mu_2),\bar{x}_2',\mu_2)\cap\Omega,$$
$$\exists z_2'\in F(x(\lambda_1,\mu_2),\bar{x}_1',\mu_2)\cap\Omega.$$

From (10), we get

$$\alpha d^{\beta}(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2))\leqslant \inf\nolimits_{g\in F(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2),\mu_2)}d(g,z_1')+\inf\nolimits_{f\in F(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2)}d(f,z_2')\leqslant$$
$$H(F(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2),F(\bar{x}(\lambda_1,\mu_2),\bar{x}_1',\mu_2))+$$
$$H(F(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2),\mu_2),F(x(\lambda_2,\mu_2),\bar{x}_2',\mu_2))\leqslant$$
$$H(F(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2),F(\bar{x}(\lambda_1,\mu_2),\bar{x}_1,\mu_2))+$$

$$H(F(\bar{x}(\lambda_1,\mu_2),\bar{x}_1,\mu_2),F(\bar{x}(\lambda_1,\mu_2),\bar{x}'_1,\mu_2))+$$
$$H(F(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2),\mu_2),F(x(\lambda_2,\mu_2),\bar{x}_2,\mu_2))+$$
$$H(F(x(\lambda_2,\mu_2),\bar{x}_2,\mu_2),F(x(\lambda_2,\mu_2),\bar{x}'_2,\mu_2)).$$

From (11), (12) and the Hölder continuity of F, we have

$$\alpha d^{\beta}(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2))\leqslant nd^{\delta}(x(\lambda_2,\mu_2),\bar{x}_1)+nd^{\delta}(\bar{x}_1,\bar{x}'_1)+nd^{\delta}(\bar{x}(\lambda_1,\mu_2),\bar{x}_2)+nd^{\delta}(\bar{x}_2,\bar{x}'_2)\leqslant 2nl_1^{\delta}d^{\alpha_1\beta}(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2))+2nl_2^{\delta}d^{\alpha_2\beta}(\lambda_1,\lambda_2).$$

The condition (v) yields that

$$d(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2))\leqslant\left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2).$$

By the arbitrariness of $x(\lambda_2,\mu_2)\in S_1(\lambda_2,\mu_2)\backslash S_1(\lambda_1,\mu_2)$ and the definition of $d(\cdot,\cdot)$, we have

$$\sup_{x(\lambda_2,\mu_2)\in S_1(\lambda_2,\mu_2)}d(S_1(\lambda_1,\mu_1),x(\lambda_2,\mu_2))\leqslant\left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2)\tag{13}$$

Similarly, we get

$$\sup_{\bar{x}(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)}d(\bar{x}(\lambda_1,\mu_2),S_1(\lambda_2,\mu_2))\leqslant\left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2)\tag{14}$$

From (13) and (14), we have that (9) holds.

Case 2. $S_1(\lambda_1,\mu_2)\subset S_1(\lambda_2,\mu_2)$ or $S_1(\lambda_2,\mu_2)\subset S_1(\lambda_1,\mu_2)$. Without loss of generality, we assume that $S_1(\lambda_1,\mu_2)\subset S_1(\lambda_2,\mu_2)$. From the definition of the distance $d(\cdot,\cdot)$ we have

$$\sup_{x(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)}d(x(\lambda_1,\mu_2),S_1(\lambda_2,\mu_2))=0.$$

By use the same argument as in Case 1, we also have that (9) holds.

Step 3. Since

$$H(S_1(\lambda_1,\mu_1),S_1(\lambda_2,\mu_2))\leqslant H(S_1(\lambda_1,\mu_1),S_1(\lambda_1,\mu_2))+H(S_1(\lambda_1,\mu_2),S_1(\lambda_2,\mu_2)).$$

It follows from (2) and (9) that (1) holds.

Now we give an example to show that Theorem 3.2 is applicable when the solution mapping is set-valued.

Example 3.3 Let $X=Y=R,\Lambda=M=[0,1],\Omega=\mathbf{R}_+$,

$$K(x,\lambda)=\left[\frac{(1+\lambda)^2+x}{16},2\right],$$
$$\psi(x,\lambda)=[(1+\lambda)x,2],$$
$$\varphi(x,y,\lambda)=[(1+\lambda)x-2(1+\lambda)y+xy+2,(1+\lambda)x+6]$$

then

$$E(\lambda)=\left[\frac{(1+\lambda)^2}{15},2\right].$$

Consider that $\bar{\lambda}=0.5$ and $N(\bar{\lambda})=\Lambda$. Direct computation show that

$$E(\lambda)=E(N(\bar{\lambda}))=\left[\frac{1}{15},2\right],$$
$$S_1(\lambda)=[1+\lambda,2]$$

and

$$E(\Lambda)\backslash S_1(\lambda)=\left[\frac{1}{15},1+\lambda\right],\forall\lambda\in\Lambda.$$

Obviously, for all $\lambda\in\Lambda,E(\Lambda)$ is compact. $K(\cdot,\lambda)$ is continuous and $F(\cdot,\cdot,\lambda)$ is continuous with compact value; $K(\cdot,\cdot)$ is $\left(\frac{1}{16}.1,\frac{1}{4}.1\right)$-Hölder continuous; For all $x,y\in E(\Lambda),F(x,y,\cdot)$ is 4.1-Hölder continuous. For all $x\in E(\Lambda),\lambda\in\Lambda,F(x,\cdot,\lambda)$ is 4.1-Hölder continuous. Here $l_1=\frac{1}{16},l_2=\frac{1}{4},m=4$, $n=4,\alpha_1=\alpha_2=\delta=\gamma=1$. Taking $\beta=1,\alpha=1$. For any $\lambda\in\Lambda$ and $y\in\left[\frac{1}{15},1+\lambda\right)$, taking $\bar{x}=1+\lambda\in S_1(\lambda)$, we have

$$\inf_{g\in F(y,\bar{x},\lambda)}d(g,\mathbf{R}_+)+\inf_{f\in F(\bar{x},y,\lambda)}d(f,\mathbf{R}_+)=(1+\lambda)|y-(1+\lambda)|\geqslant|y-(1+\lambda)|=\alpha d^{\beta}(\bar{x},y).$$

It holds that $\beta=\alpha_1\delta=1,\alpha>2nl_1^{\delta}=\frac{1}{2}$. Hence, all assumptions of Theorem 3.2 hold and thus it is valid.

4 Hölder continuity of solutions to (PVMQEP2)

In this section, we discuss the Hölder continuity of solutions to (PVMQEP2).

Proposition 4.1 Assume that the solutions for the problem(PVMQEP2) exist in a neighborhood $N(\bar{\lambda}) \times N(\bar{\mu})$ of the reference point $(\bar{\lambda},\bar{\mu}) \in \Lambda \times M$, the condition (a) in Proposition 3.1 holds, and condition (b) in Proposition 3.1 is replaced by

(b′) For each $(\lambda,\mu) \in N(\bar{\lambda}) \times N(\bar{\mu})$ and $\varphi(\cdot,\cdot,\mu)$ is lower semicontinuous in $E(\lambda) \times K(E(\lambda),\lambda)$, $\psi(\cdot,\mu)$ is lower semicontinuous in $K(E(\lambda),\lambda)$,

Then for any $(\lambda,\mu) \in N(\bar{\lambda}) \times N(\bar{\mu}) S_2(\lambda,\mu)$ is a compact subset in $E(\lambda)$.

Proof It suffices to show that $S_2(\lambda,\mu)$ is closed in $E(\lambda)$ since $E(\lambda)$ is compact. Take any sequence $\{x_n\} \subset S_2(\lambda,\mu)$ with $x_n \to x_0$ It follows from $x_n \in E(\lambda)$ and the compactness of $E(\lambda)$ that $x_0 \in E(\lambda)$. Suppose that $x_0 \notin S_2(\lambda,\mu)$. Then, $\exists y_0 \in K(x_0,\lambda)$, such that $F(x_0,y_0,\mu) \not\subset \Omega$. Taking $z_0 \in F(x_0,y_0,\mu) \cap (Y\backslash\Omega)$. Since $K(\cdot,\lambda)$, is lower semicontinuous at x_0, for y_0 and $\{x_n\}$, there exist $\bar{y}_n \in K(x_n,\lambda)$, such that $\bar{y}_n \to y_0$. As $z_0 \in F(x_0,y_0,\mu)$ from condition (b′), there exists $\bar{z}_n \in F(x_n,\bar{y}_n,\mu)$ such that $\bar{z}_n \to z_0$. Because $z_0 \in Y\backslash\Omega$, when n is large enough, $\bar{z}_n \in Y\backslash\Omega$. It follows from $x_n \notin S_2(\lambda,\mu)$ that

$$\bar{z}_n \in F(x_n,\bar{y}_n,\mu) \subseteq \Omega.$$

This leads to a contraction. Thus $x_0 \in S_2(\lambda,\mu)$ and $S_2(\lambda,\mu)$ is closed.

Theorem 4.2 Assume that the solutions for the (PVMQEP2) exist in a neighborhood $N(\bar{\lambda}) \times N(\bar{\mu})$ of the reference point $(\lambda,\mu) \in \Lambda \times M$ and the conditions of Proposition 3.1 are satisfied. Assume further that the conditions (i), (iii), (iv) and (v) in Theorem 3.2 hold and (ii) in Theorem 3.2 is replaced by the following condition:

(ii′) There are constants $\alpha > 0, \beta > 0$ such that $\forall (\lambda,\mu) \in N(\bar{\lambda}) \times N(\bar{\mu}), \forall y \in E(N(\bar{\lambda}))\backslash S_2(\lambda,\mu)$, $\exists \hat{x} \in S_2(\lambda,\mu)$ satisfying

$$\alpha d^{\beta}(\hat{x},y) \leqslant \sup_{g \in F(y,\hat{x},\mu)} d(g,\Omega) + \sup_{f \in F(\hat{x},y,\mu)} d(f,\Omega).$$

Then $S_2(\cdot,\cdot)$ satisfies the Hölder continuous condition that for any (λ_1,μ_1) and (λ_2,μ_2) in $N(\bar{\lambda}) \times N(\bar{\mu})$,

$$H(S_2(\lambda_1,\mu_1),S_2(\lambda_2,\mu_2)) \leqslant \left(\frac{m}{\alpha - 2nl_1^{\delta}}\right)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2) + \left(\frac{2nl_2^{\delta}}{\alpha - 2nl_1^{\delta}}\right)^{\frac{1}{\beta}} d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2) \tag{15}$$

Proof The proof is similar as the Theorem 3.2 with suitable modifications. For Case 1 of Step 1 in the proof of Theorem 3.2, we consider that $S_2(\lambda_1,\mu_1) \not\subset S_2(\lambda_1,\mu_2)$ and $S_2(\lambda_1,\mu_2) \not\subset S_2(\lambda_1,\mu_1)$. For any $x(\lambda_1,\mu_1) \in S_2(\lambda_1,\mu_1)\backslash S_2(\lambda_1,\mu_2)$, by virtue of the assumption(ii′), there exists $x(\lambda_1,\mu_2) \in S_2(\lambda_1,\mu_2)$, such that

$$\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)) \leqslant \sup_{g \in F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)} d(g,\Omega) + \sup_{f \in F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)} d(f,\Omega) \tag{16}$$

Since $x(\lambda_1,\mu_1) \in K(x(\lambda_1,\mu_1),\lambda_1), x(\lambda_1,\mu_2) \in K(x(\lambda_1,\mu_2),\lambda_1), K(\cdot,\cdot)$ is Hölder continuous, there exist $x_1 \in K(x(\lambda_1,\mu_2),\lambda_1), x_2 \in K(x(\lambda_1,\mu_1),\lambda_1)$ such that (4) holds. Because $x(\lambda_1,\mu_1) \in S_2(\lambda_1,\mu_1), x(\lambda_1,\mu_2) \in S_2(\lambda_1,\mu_2)$ we have

$$F(x(\lambda_1,\mu_1),x_2,\mu_1) \subset \Omega,$$
$$F(x(\lambda_1,\mu_2),x_1,\mu_2) \subset \Omega.$$

From (16), we have

$$\begin{aligned}&\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)) \leqslant \\ &\sup_{g \in F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)} d(g,F(x(\lambda_1,\mu_1),x_2,\mu_1)) + \\ &\sup_{f \in F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)} d(f,F(x(\lambda_1,\mu_2),x_1,\mu_2)) \leqslant \\ &H(F(x(\lambda_1,\mu_1),x_2,\mu_1),F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)) + \\ &H(F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2),F(x(\lambda_1,\mu_2),x_1,\mu_2)).\end{aligned}$$

Then the proof follows by similar argument in Theorem 3.2.

Now we give an example to show that Theorem 4.2 is applicable when the solution mapping is set-valued.

Example 4.3 Let $X = Y = R, \Lambda = M = [0,1]$, $\Omega = \mathbf{R}_+$,

$$K(x,\lambda) = \left[\frac{(1+\lambda)^2 + x}{16}, 2\right],$$
$$\psi(x,\lambda) = [(1+\lambda)x, 2],$$
$$\varphi(x,y,\lambda) = [xy - 2(1+\lambda)y + 2, +\infty)$$

then $E(\lambda) = \left[\frac{(1+\lambda)^2}{15}, 2\right]$.

Consider that $\bar{\lambda} = 0.5$ and $N(\bar{\lambda}) = \Lambda$, Direct computation show that

$$E(\lambda) = E(N(\bar{\lambda})) = \left[\frac{1}{15}, 2\right],$$
$$S_2(\lambda) = [1+\lambda, 2]$$

and

$$E(\Lambda)\backslash S_2(\lambda)=\left[\frac{1}{15},1+\lambda\right],\forall\lambda\in\Lambda.$$

Obviously, for all $\lambda\in\Lambda,E(\Lambda)$ is compact set. $K(\cdot,\lambda)$ is continuous and $F(\cdot,\cdot,\lambda)$ is continuous with compact value; $K(\cdot,\cdot)$ is $\left(\frac{1}{16}.1,\frac{1}{4}.1\right)$-Hölder continuous. For all $x,y\in E(\Lambda),F(x,y,\cdot)$ is 2. 1-Hölder continuous, For all $F(x,\cdot,\lambda)$ is 4. 1-Hölder. Continuous Here $l_1=\frac{1}{16},l_2=\frac{1}{4},m=2,n=4,\alpha_1=\alpha_2=\delta=\gamma=1$. Taking $\beta=1,\alpha=1$. For any $\lambda\in\Lambda$ and $y\in\left[\frac{1}{15},1+\lambda\right)$, taking $\bar{x}=1+\lambda\in S_2(\lambda)$, we have

$$\sup_{g\in F(y,\bar{x},\lambda)}d(g,R_+)+\sup_{f\in F(\bar{x},y,\lambda)}d(f,R_+)=(1+\lambda)\,|y-(1+\lambda)|\geqslant|y-(1+\lambda)|=\alpha d^{\beta}(\bar{x},y).$$

It holds that $\beta=\alpha_1\delta=1,\alpha>2nl_1^{\delta}=\frac{1}{4}$. Hence all assumptions of Theorem 3. 2 hold and thus it is valid.

5 Hölder continuity of solutions to (DVMQEP1)

In this section, we discuss the Hölder continuity of solutions to (DVMQEP1).

Proposition 5. 1 Assume that the solutions for the problem(DVMQEP1) exist in a neighborhood $N(\bar{\lambda})\times N(\bar{\mu})$ of the reference point $(\bar{\lambda},\bar{\mu})\in\Lambda\times M$. Assume further that the conditions in Proposition 3. 1 hold. Then for any $(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu}),S_1^d(\lambda,\mu)$ is a compact subset in $E(\lambda)$.

Proof It suffices to show that $S_1^d(\lambda,\mu)$ is closed in $E(\lambda)$, since $E(\lambda)$ is compact. Take any sequence $\{x_n\}\in S_1^d(\lambda,\mu)$ with $x_n\to x_0$. It follows from $x_n\in E(\lambda)$ and the compactness of $E(\lambda)$, suppose that $x_0\notin S_1^d(\lambda,\mu)$. Then, $\exists y_0\in K(x_0,\lambda)$, such that $F(x_0,y_0,\mu)\cap(-\Omega)=\varnothing$. Since $K(\cdot,\lambda)$ is lower semicontinuous at x_0, for y_0 and $\{x_n\}$, there exist $\bar{y}_n$ such that $\bar{y}_n\to y_0$. As $x_n\in S_1^d(\lambda,\mu)$, there exist $z_n\in F(y_n,x_n,\mu)\cap(-\Omega)$. From condition (b), we have $F(\cdot,\cdot,\mu)$ is upper semicontinuity with compact values, there exists $z_0\in F(y_0,x_0,\mu)\,z_n\to z_0$. Noting the closeness of Ω, we have $z_0\in(-\Omega)$. This leads to a contraction. Thus, $x_0\in S_1^d(\lambda,\mu)$ and $S_1^d(\lambda,\mu)$ is closed.

Theorem 5. 2 Assume that the solutions for the (DVMQEP1) exist in a neighborhood $N(\bar{\lambda})\times N(\bar{\mu})$ of the reference point $(\bar{\lambda},\bar{\mu})\in\Lambda\times M$ and the conditions of Proposition 5. 1 are satisfied. Assume further that the conditions (i), (iii), (iv) in Theorem 3. 2 hold and (ii), (iv) in Theorem 3. 2 replaced by :

(ii)^d There are constants $\alpha>0$ and $\beta>0$, such that $\forall(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu})$ $\forall y\in E(N(\bar{\lambda}))\backslash S_1^d(\lambda,\mu)$, $\exists\hat{x}\in S_1^d(\lambda,\mu)$ satisfying

$$\alpha d^{\beta}(\hat{x},y)\leqslant\inf_{g\in F(y,\hat{x},\mu)}d(g,-\Omega)+\inf_{f\in F(\hat{x},y,\mu)}d(f,-\Omega).$$

(iv)^d $\forall(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu}),\forall x\in E(\lambda),F(\cdot,x,\mu)n.\delta$-Hölder continuous in $K(E(N(\bar{\lambda})),N(\bar{\lambda}))$,

Then $S_1^d(\cdot,\cdot)$ satisfies the Hölder continuous condition that for any

$$(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N(\bar{\lambda})\times N(\bar{\mu}),$$

$$H(S_1^d(\lambda_1,\mu_1),S_1^d(\lambda_2,\mu_2))\leqslant\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2)+\left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2)\tag{17}$$

Proof By Proposition 5. 1 for each $(\lambda,\mu)\in N(\bar{\lambda})\times N(\bar{\mu}),S_1^d(\lambda,\mu)$ is compact subset. Let $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N(\bar{\lambda})\times N(\bar{\mu})$. We prove Theorem 5. 2 by the following three steps.

Step1. We prove that

$$H(S_1^d(\lambda_1,\mu_1),S_1^d(\lambda_1,\mu_2))\leqslant\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu,\mu_2)\tag{18}$$

Obviously, if $S_1^d(\lambda_1,\mu_1)=S_1^d(\lambda_1,\mu_2)$, we have that (18) holds. So we suppose

$$S_1^d(\lambda_1,\mu_1)\neq S_1^d(\lambda_1,\mu_2).$$

There are two cases to be considered.

Case1. $S_1^d(\lambda_1,\mu_1)\not\subset S_1^d(\lambda_1,\mu_2)$ and $S_1^d(\lambda_1,\mu_2)\not\subset S_1^d(\lambda_1,\mu_1)$. For any $x(\lambda_1,\mu_1)\in S_1^d(\lambda_1,\mu_1)\backslash S_1^d(\lambda_1,\mu_2)$, by virtue of the assumption (ii)^d, there exists $x(\lambda_1,\mu_2)\in S_1^d(\lambda_1,\mu_2)$, such that

$$\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant\inf_{g\in F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)}d(g,-\Omega)+\inf_{f\in F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)}d(f,-\Omega)\tag{19}$$

Since $x(\lambda_1,\mu_1)\in K(x(\lambda_1,\mu_1),\lambda_1), x(\lambda_1,\mu_2)\in K(x(\lambda_1,\mu_2),\lambda_1)$, $K(\cdot,\cdot)$ is Hölder continuous, there exists $x_1\in K(x(\lambda_1,\mu_2),\lambda_1)$, $x_2\in K(x(\lambda_1,\mu_1),\lambda_1)$ such that

$$d(x(\lambda_1,\mu_1),x_1)\leqslant l_1 d^{\alpha_1}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\,d(x(\lambda_1,\mu_2),x_2)\leqslant l_1 d^{\alpha_1}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)) \quad (20)$$

Because $x(\lambda_1,\mu_1)\in S_1^d(\lambda_1,\mu_1)$, $x(\lambda_1,\mu_2)\in S_1^d(\lambda_1,\mu_2)$, we have

$$\exists z_1\in F(x_2,x(\lambda_1,\mu_1),\mu_1)\cap(-\Omega),$$
$$\exists z_2\in F(x_1,x(\lambda_1,\mu_2),\mu_2)\cap(-\Omega).$$

From (19), we have

$$\begin{aligned}\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant&\inf_{g\in F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)}d(g,z_1)+\inf_{f\in F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)}d(f,z_2)\leqslant\\ &H(F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2),F(x(\lambda_1,\mu_2),x_2,\mu_1))+\\ &H(F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2),F(x(\lambda_1,\mu_2),x_1,\mu_2))\leqslant\\ &H(F(x(\lambda_1,\mu_1),x_2,\mu_1),F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1))+\\ &H(F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1),F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2))+\\ &H(F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2),F(x(\lambda_1,\mu_2),x_1,\mu_2)).\end{aligned}$$

From the Hölder continuity of F and (20), we get

$$\begin{aligned}\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant&\ n d^{\delta}(x(\lambda_1,\mu_1),x_1)+m d^{\gamma}(\mu_1,\mu_2)+\\ &n d^{\delta}(x(\lambda_1,\mu_2),x_2)\leqslant\\ &2nl_1^{\delta}d^{\alpha_1\delta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))+\\ &m d^{\gamma}(\mu_1,\mu_2)\end{aligned}\quad (21)$$

By assumption (v) we can obtain

$$d(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2).$$

Noting the arbitrariness of $x(\lambda_1,\mu_1)\in S_1^d(\lambda_1,\mu_1)\backslash S_1^d(\lambda_1,\mu_2)$ we have

$$\sup_{x(\lambda_1,\mu_1)\in S_1^d(\lambda_1,\mu_1)\backslash S_1^d(\lambda_1,\mu_2)}\ \inf_{x(\lambda_1,\mu_2)\in S_1^d(\lambda_1,\mu_2)} d(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2))\leqslant\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2).$$

By the definition of the distance $d(\cdot,\cdot)$, we have

$$\begin{aligned}\sup_{x(\lambda_1,\mu_1)\in S_1^d(\lambda_1,\mu_1)} d(x(\lambda_1,\mu_1),S_1^d(\lambda_1,\mu_2))=&\sup_{x(\lambda_1,\mu_1)\in S_1^d(\lambda_1,\mu_1)\backslash S_1^d(\lambda_1,\mu_2)} d(x(\lambda_1,\mu_1),S_1^d(\lambda_1,\mu_2))\leqslant\\ &\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2)\end{aligned}\quad (22)$$

Similarity, we can deduce

$$\sup_{x(\lambda_1,\mu_2)\in S_1^d(\lambda_1,\mu_2)} d(x(\lambda_1,\mu_2),S_1^d(\lambda_1,\mu_1))\leqslant\left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2)\quad (23)$$

From (22) and (23), we know (18) holds.

Case2. $S_1^d(\lambda_1,\mu_1)\subset S_1^d(\lambda_1,\mu_2)$ or $S_1^d(\lambda_1,\mu_2)\subset S_1^d(\lambda_1,\mu_1)$. Without loss of generality, we can assume that $S_1^d(\lambda_1,\mu_1)\subset S_1^d(\lambda_1,\mu_2)$. From the definition of the distance $d(\cdot,\cdot)$ we have

$$\sup_{x(\lambda_1,\mu_1)\in S_1^d(\lambda_1,\mu_1)} d(x(\lambda_1,\mu_1),S_1^d(\lambda_1,\mu_2))=0\quad (24)$$

By use the same argument as in Case 1, we also have that (23) holds in Case 2. It follows from (23),(24) that (18) also holds.

Step2. Now we show that

$$H(S_1^d(\lambda_1,\mu_2),S_1^d(\lambda_2,\mu_2))\leqslant\left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}}d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2)\quad (25)$$

Obviously, if $S_1^d(\lambda_1,\mu_2)=S_1^d(\lambda_2,\mu_2)$, we have that (25) holds. Similar to Step 1, we suppose $S_1^d(\lambda_1,\mu_2)\neq S_1^d(\lambda_2,\mu_2)$, there are two cases to be considered.

Case 1. $S_1^d(\lambda_1,\mu_2)\not\subset S_1^d(\lambda_1,\mu_2)$ and $S_1^d(\lambda_2,\mu_2)\not\subset S_1^d(\lambda_1,\mu_2)$.

for any $x(\lambda_2,\mu_2)\in S_1^d(\lambda_2,\mu_2)\backslash S_1^d(\lambda_1,\mu_2)$, by virtue of the assumption (ii), there exists $\bar{x}(\lambda_1,\mu_2)\in S_1^d(\lambda_1,\mu_2)$ such that

$$\begin{aligned}\alpha d^{\beta}(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2))\leqslant&\inf_{g\in F(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2),\mu_2)}d(g,-\Omega)+\\ &\inf_{f\in F(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2)}d(f,-\Omega)\end{aligned}\quad (26)$$

Since $x(\lambda_2,\mu_2)\in K(x(\lambda_2,\mu_2),\lambda_2)$, $\bar{x}(\lambda_1,\mu_2)\in K(\bar{x}(\lambda_1,\mu_2),\lambda_1)$, $K(\cdot,\cdot)$ is Hölder continuous, there exist $\bar{x}_1\in K(\bar{x}(\lambda_2,\mu_2),\lambda_1)$, $\bar{x}_2\in K(\bar{x}(\lambda_1,$

$\mu_2),\lambda_2)$ such that

$$d(x(\lambda_2,\mu_2),\bar{x}_1)\leqslant l_2 d^{\alpha_2}(\lambda_1,\lambda_2),$$
$$d(\bar{x}(\lambda_1,\mu_2),\bar{x}_2)\leqslant l_2 d^{\alpha_2}(\lambda_1,\lambda_2) \quad (27)$$

By the Hölder continuity of $K(\cdot,\cdot)$ again, there exist $\bar{x}_1' \in K(\bar{x}(\lambda_1,\mu_2),\lambda_1)$, $\bar{x}_2' \in K(x(\lambda_2,\mu_2),\lambda_2)$ such that

$$d(\bar{x}_1,\bar{x}_1')\leqslant l_1 d^{\alpha_1}(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2)),$$
$$d(\bar{x}_2,\bar{x}_2')\leqslant l_1 d^{\alpha_1}(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2)) \quad (28)$$

Because $x(\lambda_2,\mu_2)\in S_1(\lambda_2,\mu_2)$, $\bar{x}(\lambda_1,\mu_2)\in S_1(\lambda_1,\mu_2)$ we have

$$\exists z_1' \in F(\bar{x}_2',x(\lambda_2,\mu_2),\mu_2)\cap(-\Omega),$$
$$\exists z_2' \in F(\bar{x}_1',\bar{x}(\lambda_1,\mu_2),\mu_2)\cap(-\Omega).$$

From (26), we get

$$\begin{aligned}\alpha d^{\beta}(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2))\leqslant{}& \inf_{g\in F(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2),\mu_2)} d(g,z_1')+\inf_{f\in F(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2)} d(f,z_2')\leqslant\\ & H(F(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2),F(\bar{x}(\lambda_1,\mu_2),\bar{x}_1',\mu_2))+\\ & H(F(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2),\mu_2),F(x(\lambda_2,\mu_2),\bar{x}_2',\mu_2))\leqslant\\ & H(F(\bar{x}(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2),F(\bar{x}(\lambda_1,\mu_2),\bar{x}_1,\mu_2))+\\ & H(F(\bar{x}(\lambda_1,\mu_2),\bar{x}_1,\mu_2),F(\bar{x}(\lambda_1,\mu_2),\bar{x}_1',\mu_2))+\\ & H(F(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2),\mu_2),F(x(\lambda_2,\mu_2),\bar{x}_2,\mu_2))+\\ & H(F(x(\lambda_2,\mu_2),\bar{x}_2,\mu_2),F(x(\lambda_2,\mu_2),\bar{x}_2',\mu_2)).\end{aligned}$$

From (24),(28) and the Hölder continuity of F, we have

$$\begin{aligned}&\alpha d^{\beta}(x(\lambda_1,\mu_1),\bar{x}(\lambda_1,\mu_2))\leqslant\\ &\quad nd^{\delta}(x(\lambda_2,\mu_2),\bar{x}_1)+nd^{\delta}(\bar{x}_1,\bar{x}_1')+\\ &\quad nd^{\delta}(\bar{x}(\lambda_2,\mu_2),\bar{x}_2)+nd^{\delta}(\bar{x}_2',\bar{x}_2')\leqslant\\ &\quad 2nl_1^{\delta}d^{\alpha_1\delta}(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2))+\\ &\quad 2nl_2^{\delta}d^{\alpha_2\beta}d^{r}(\lambda_1,\lambda_2)\end{aligned}$$

The condition (v) yields that

$$d(x(\lambda_2,\mu_2),\bar{x}(\lambda_1,\mu_2))\leqslant \left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}} d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2).$$

By the arbitrariness of $x(\lambda_2,\mu_2)\in S_1^d(\lambda_2,\mu_2)\backslash S_1^d(\lambda_1,\mu_2)$ and the definition of $d(\cdot,\cdot)$, we have

$$\sup_{x(\lambda_2,\mu_2)\in S_1^d(\lambda_2,\mu_2)} d(S_1^d(\lambda_1,\mu_2),x(\lambda_2,\mu_2))\leqslant \left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}} d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2) \quad (29)$$

Similarly, we get

$$\sup_{\bar{x}(\lambda_1,\mu_2)\in S_1^d(\lambda_1,\mu_2)} d(\bar{x}(\lambda_1,\mu_2),S_1^d(\lambda_2,\mu_2))\leqslant \left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}} d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2) \quad (30)$$

From (29) and (30), we have that (25) holds.

Case 2. $S_1^d(\lambda_1,\mu_2)\subset S_1^d(\lambda_2,\mu_2)$ or $S_1^d(\lambda_2,\mu_2)\subset S_1^d(\lambda_1,\mu_2)$. Without loss of generality, we assume that $S_1^d(\lambda_1,\mu_2)\subset S_1^d(\lambda_2,\mu_2)$. From the definition of the distance $d(\cdot,\cdot)$, we have

$$\sup_{\bar{x}(\lambda_1,\mu_2)\in S_1^d(\lambda_1,\mu_2)} d(\bar{x}(\lambda_1,\mu_2),S_1^d(\lambda_2,\mu_2))=0.$$

By use the same argument as in Case 1, we also have that (25) holds.

Step 3. Finally, since

$$\begin{aligned}&H(S_1^d(\lambda_1,\mu_1),S_1^d(\lambda_2,\mu_2))\leqslant\\ &\quad H(S_1^d(\lambda_1,\mu_1),S_1^d(\lambda_1,\mu_2))+\\ &\quad H(S_1^d(\lambda_1,\mu_2),S_1^d(\lambda_2,\mu_2)).\end{aligned}$$

It follows from (18) and (25) that (17) holds.

Example 5.3 Let $X=Y=R$, $\Lambda=M=[0,1]$, $\Omega=\mathbf{R}_+$,

$$K(x,\lambda)=\left[\frac{\lambda^2+x}{2},1\right],$$
$$\psi(x,\lambda)=\left[\frac{\lambda x}{3},2\right],$$
$$\varphi(x,y,\lambda)=\left[-\frac{\lambda y}{3}+2+(\lambda-y)\left(\frac{\lambda x}{3}+1\right),\frac{\lambda x}{3}+6\right].$$

Then

$$E(\lambda)=\left[\frac{(1+\lambda)^2}{15},2\right].$$

Consider that $\bar{\lambda}=0.5$ and $N(\bar{\lambda})=\Lambda$. Then $F(x,y,\lambda)\neq F(y,x,\lambda)$. Direct computation show that $E(\lambda)=[\lambda^2,1]$, $S_1(\lambda)=[\lambda^2,1]$ and $S_1^d(\lambda)=[\lambda,1]$, $\forall\lambda\in\Lambda$. Obviously, for all $\lambda\in\Lambda$, $E(\Lambda)$ is compact set. $K(\cdot,\lambda)$ is continuous and $F(\cdot,\cdot,\lambda)$ is continuous compact value; $K(\cdot,\cdot)$ is $\left(\frac{1}{2}.1,1.1\right)$-Hölder continuous. For all $x,y\in$

$E(\Lambda)$, $F(x,y,\cdot)$ is $\frac{4}{3}$. 1- Hölder continuous. For all $x \in E(\Lambda)$, $\lambda \in \Lambda$, $F(x,\cdot,\lambda)$ is 2. 1-Hölder continuous. Here $l_1 = \frac{1}{2}$, $l_2 = 1$, $m = \frac{4}{3}$, $n = \frac{1}{3}$, $\alpha_1 = \alpha_2 = \delta = \gamma = 1$ and $\beta = 1$, $\alpha = 1$. For all $\lambda \in \Lambda$ and $y \in E(\lambda) \setminus S_1^d(\bar{\lambda})$, taking $\bar{x} = \frac{1}{2} \in S_1^d(\bar{\lambda})$, we have

$$\inf_{g\in F(y,\bar{x},\lambda)} d(g,\mathbf{R}_-) + \inf_{f\in F(\bar{x},y,\lambda)} d(f,\mathbf{R}_-) = \inf_{g\in[\frac{7}{6}(\frac{1}{2}-y),8]} d(g,\mathbf{R}_-) + \inf_{f\in[0,8]} d(f,\mathbf{R}_-) = \frac{7}{6}\left|y-\frac{1}{2}\right| \geqslant \alpha d^{\beta}(\bar{x},y).$$

It holds that $\beta = \alpha_1\delta = 1$, $\alpha > 2nl_1^{\delta} = \frac{1}{3}$. Hence all assumptions of Theorem 5. 2 hold and thus it is valid.

6 Hölder continuity of solutions to (DVMQEP2)

In this section, we discuss the Hölder continuity of solutions to (DVMQEP2).

Proposition 6. 1 Assume that the solutions for the problem(DVMQEP2) exist in a neighborhood $N(\bar{\lambda}) \times N(\bar{\mu})$ of the reference point $(\bar{\lambda},\bar{\mu}) \in \Lambda \times M$. Assume further that the conditions in Proposition 3. 1 hold, then for any $(\lambda,\mu) \in N(\bar{\lambda}) \times N(\bar{\mu})$, $S_2^d(\lambda,\mu)$ is compact subset in $E(\lambda)$.

Proof It suffices to show that $S_2^d(\lambda,\mu)$ is closed in $E(\lambda)$. Since $E(\lambda)$ is compact. Take any sequence $\{x_n\} \in S_2^d(\lambda,\mu)$ with $x_n \to x_0$. It follows from $x_n \in E(\lambda)$ and the compactness of $E(\lambda)$. Suppose that $x_0 \notin S_2^d(\lambda,\mu)$. Then $\exists y_0 \in K(x_0,\lambda)$, such that $F(x_0,y_0,\mu) \cap \Omega = \varnothing$. Since $K(\cdot,\lambda)$ is lower semicontinuous at x_0 for y_0 and $\{x_n\}$, there exist $\bar{y}_n \to y_0$. As $x_n \in S_2^d(\lambda,\mu)$, there exist $z_n \in F(x_n,y_n,\mu) \cap \Omega$. From condition (b), we have $F(\cdot,\cdot,\mu)$ is upper semicontinuity with compact values, there exists $z_0 \in F(x_0,y_0,\mu)$ such that $z_n \to z_0$. Noting the closeness of Ω, we have $z_0 \in \Omega$. This leads to a contraction. Thus, $x_0 \in S_2^d(\lambda,\mu)$ and $S_2^d(\lambda,\mu)$ is closed set.

Theorem 6. 2 Assume that the solutions for the (DVMQEP2) exist in a neighborhood $N(\bar{\lambda}) \times N(\bar{\mu})$ of the reference point $(\lambda,\mu) \in \Lambda \times M$ and the conditions of Proposition 6. 1 are satisfied. Assume further that the conditions (i), (iii), (iv)^d and (v) in Theorem 5. 2 hold and the condition (ii) in Theorem 5. 2 is replaced by

(ii$'^d$) There are constants $\alpha > 0$, $\beta > 0$ such that $\forall (\lambda,\mu) \in N(\bar{\lambda}) \times N(\bar{\mu})$, satisfying

$$\alpha d^{\beta}(\hat{x},y) \leqslant \sup_{g\in F(y,\hat{x},\mu)} d(g,-\Omega) + \sup_{f\in F(\hat{x},y,\mu)} d(f,-\Omega).$$

Then the Hölder continuous condition satisfies for any (λ_1,μ_1) and (λ_2,μ_2) in $N(\bar{\lambda}) \times N(\bar{\mu})$,

$$H(S_2^d(\lambda_1,\mu_1),S_2^d(\lambda_2,\mu_2)) \leqslant \left(\frac{m}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta}}(\mu_1,\mu_2) + \left(\frac{2nl_2^{\delta}}{\alpha-2nl_1^{\delta}}\right)^{\frac{1}{\beta}} d^{\frac{\alpha_2\delta}{\beta}}(\lambda_1,\lambda_2) \tag{31}$$

Proof The proof is similar as the Theorem 5. 2 with suitable modifications. For Case 1 of Step 1 in the proof of Theorem 5. 2, we consider that $S_2^d(\lambda_1,\mu_1) \not\subset S_2^d(\lambda_1,\mu_2)$ and $S_2^d(\lambda_1,\mu_2) \not\subset S_2^d(\lambda_1,\mu_1)$.

For any $x(\lambda_1\mu_1) \in S_2^d(\lambda_1\mu_1) \setminus S_2^d(\lambda_1\mu_2)$, by virtue of the assumption (ii$'^d$), there exists such that

$$\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)) \leqslant \sup_{g\in F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)} d(g,-\Omega) + \sup_{f\in F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)} d(f,-\Omega). \tag{32}$$

Since $x(\lambda_1,\mu_1) \in K(x(\lambda_1,\mu_1),\lambda_1)$, $x(\lambda_1,\mu_2) \in K(x(\lambda_1,\mu_2),\lambda_1)$, $K(\cdot,\cdot)$ is Hölder continuous, there exist $x_1 \in K(x(\lambda_1,\mu_2),\lambda_1)$, $x_2 \in K(x(\lambda_1,\mu_1),\lambda_1)$ such that (20) holds. Because $x(\lambda_1,\mu_1) \in S_2^d(\lambda_1,\mu_1)$, $x(\lambda_1,\mu_2) \in S_2^d(\lambda_1,\mu_2)$, we have

$$F(x_2,x(\lambda_1,\mu_1),\mu_1) \subset (-\Omega),$$
$$F(x_1,x(\lambda_1,\mu_2),\mu_2) \subset (-\Omega).$$

From (32), we have

$$\alpha d^{\beta}(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)) \leqslant \sup_{g\in F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2)} d(g,F(x_2,x(\lambda_1,\mu_1),\mu_1)) + \sup_{f\in F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)} d(f,F(x_1,x(\lambda_1,\mu_2),\mu_2)) \leqslant H(F(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_2),F(x_2,x(\lambda_1,\mu_2),\mu_1)) + H(F(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2),$$

$F(x_1, x(\lambda_1, \mu_2), \mu_2))$.

Then the proof follows by similar argument in Theorem 5.2.

Now we give an example to show that Theorem 6.2 is applicable when the solution mapping is set-valued.

Example 6.3 Let $X = Y = \mathbf{R}, \Lambda = M = [0,1], \Omega = \mathbf{R}_+$,

$$K(x,\lambda) = \left[\frac{\lambda^2 + x}{2}, 1\right],$$

$$\psi(x,\lambda) = \left[\frac{\lambda^3}{3}, \frac{\lambda x}{3}\right], \varphi(x,y,\lambda) = \left(-\infty, -\frac{\lambda y}{3} + \frac{\lambda^3}{3}(\lambda - y)\left(\frac{\lambda x}{3} + 1\right)\right],$$

then

$$E(\lambda) = \left[\frac{(1+\lambda)^2}{15}, 2\right].$$

Consider that $\bar{\lambda} = 0.5$ and $N(\bar{\lambda}) = \Lambda$. Then $F(x,y,\lambda) \neq F(y,x,\lambda)$. Direct computation show that

$$E(\lambda) = [\lambda^2, 1], S_1(\lambda) = [\lambda^2, 1]$$

and $S_2^d(\lambda) = [\lambda, 1], \forall \lambda \in \Lambda$. Obviously, for all $\lambda \in \Lambda$, $E(\Lambda)$ is compact set. $K(\cdot,\lambda)$ is continuous and $F(\cdot, \cdot, \lambda)$ is continuous compact value; $K(\cdot,\cdot)$ is $\left(\frac{1}{2}.1, 1.1\right)$-Hölder continuous. For all $x, y \in E(\Lambda)$, $F(x,y,\cdot)$ is $\frac{4}{3}.1$-Hölder continuous. For all $x \in E(\Lambda), \lambda \in \Lambda$, $F(x,\cdot,\lambda)$ is 2.1-Hölder continuous. Here $l_1 = \frac{1}{2}, l_2 = 1, m = \frac{4}{3}, n = \frac{2}{3}, \alpha_1 = \alpha_2 = \delta = \gamma = 1$ and $\beta = 1, \alpha = 1$. For all $\lambda \in \Lambda$ and $y \in E(\Lambda) \backslash S_2^d(\bar{\lambda})$ taking $\bar{x} = \frac{1}{2} \in S_2^d(\bar{\lambda})$, we have

$$\inf_{g \in F(y,\bar{x},\lambda)} d(g, \mathbf{R}_-) + \inf_{f \in F(\bar{x},y,\lambda)} d(f, \mathbf{R}_-) =$$
$$\inf_{g \in \left[\frac{7}{6}\left(\frac{1}{2}-y\right), 8\right]} d(g, \mathbf{R}_-) + \inf_{f \in [0,8]} d(f, \mathbf{R}_-) =$$
$$\frac{7}{6}\left|y - \frac{1}{2}\right| \geqslant \alpha d^\beta(\bar{x}, y).$$

It holds that $\beta = \alpha_1 \delta = 1, \alpha > 2nl_1^\delta = \frac{1}{3}$. Hence all assumptions of Theorem 6.2 hold and thus it is valid.

Remark 3 Although the structure between Theorem 3.2, 4.2 and Theorem 5.1, 6.1 are symmetric, the applications of Theorem 3.2, 4.2 and Theorem 5.2, 6.2 are independent each other whenever the solution set of primal and dual problems do not coincide.

7 Application

In this section, we apply mixed Minty-type dual vector mixed quasi-equilibrium problems to mixed Minty variational inequality (MMVI) (see Remark 2).

Let X be a normed space, M, Λ be a nonempty subset of normed $Y = \mathbf{R}$ and $C = \mathbf{R}_+$. Let $K(x,\lambda) = K(\lambda), K: \Lambda \to 2^X$ be a set-valued mapping with nonempty values. We use $S^d(\cdot,\cdot)$ to denote the solution set of MMVI.

Corollary 7.1 Assume the solution for (MMVI) exist in neighborhood $N(\bar{\lambda}) \times N(\bar{\mu})$ of the reference point (λ, μ) and the conditions of Proposition 4.1 are satisfied. Assume further that the following conditions hold:

(i) There are constants $\alpha > 0$ and $\beta > 0$ such that $\forall (\lambda,\mu) \in N(\bar{\lambda}) \times N(\bar{\mu})$, $\forall y \in E(N(\bar{\lambda})) \backslash S^d(\lambda,\mu)$, $\exists \hat{x} \in S^d(\lambda,\mu)$ satisfying

$$\alpha d^\beta(\hat{x}, y) \leqslant d([T(\hat{x},\mu), y - \hat{x}] + \psi(y,\mu) - \psi(\hat{x},\mu), \mathbf{R}_-) + d([T(y,\mu), \hat{x} - y] + \psi(\hat{x},\mu) - \psi(y,\mu), \mathbf{R}_-);$$

(ii) $K(\cdot,\cdot)$ is $l.\alpha$- Hölder in $N(\bar{\lambda})$;

(iii) $\forall \lambda \in N(\bar{\lambda}), \forall x \in K(\lambda), T(x,\cdot) m_1.r$-Hölder continuous at $\bar{\mu}$ and $\psi(x,\cdot)$ is $n_2.1$-Hölder continuous in $K(N(\bar{\lambda}))$;

(iv) $\forall (\lambda,\mu) \in N(\bar{\lambda}) \times N(\bar{\mu}), \forall x \in K(\lambda)$, $T(\cdot,\mu)$ is $n_1.1$-Höldercontinuous in $K(N(\bar{\lambda}))$ and $\psi(\cdot,\mu)$ is $n_2.1$-Hölder continuous in $K(N(\bar{\lambda}))$;

(v) $K(N(\bar{\lambda}))$ is bounded and $\forall x \in K(N(\bar{\lambda})), T(x,\mu) \leqslant \gamma > 0$ for some $\upsilon > 0$.

Then for any (λ_1, μ_1) and (λ_2, μ_2) in $N(\bar{\lambda}) \times N(\bar{\mu})$,

$$H(S^d(\lambda_1,\mu_1), S^d(\lambda_2,\mu_2)) \leqslant \left(\frac{m_1\tau + 2m_2}{\alpha}\right)^{\frac{1}{\beta}} \mu_1 - \mu_2^{\frac{\gamma}{\beta}} + \left(\frac{2(n_1\tau + \upsilon + n_2)l}{\alpha}\right)^{\frac{1}{\beta}} \lambda_1 - \lambda d^{\frac{\alpha}{\beta}}.$$

Proof To apply Theorem 5.1, we just need to verify the Hölder continuity of F. From Remark 2, we know $F(x,y,\mu)=[T(x,\mu),y-x]+\psi(y,\mu)-\psi(x,\mu)$.

Firstly, $\forall\lambda\in N(\bar{\lambda}), \forall x,y\in K(\lambda), \forall \mu_1,\mu_2 \in N(\bar{\mu})$,

$$|F(x,y,\mu_1)-F(x,y,\mu_2)|=$$
$$|[T(x,\mu_1),y-x]+\psi(y,\mu_1)-\psi(x,\mu_1)-$$
$$[T(x,\mu_2),y-x]-\psi(y,\mu_2)-\psi(x,\mu_2)|\leqslant$$
$$|[T(x,\mu_1)-T(x,\mu_2),y-x]|+$$
$$|\psi(y,\mu_1)-\psi(y,\mu_2)|+|\psi(x,\mu_1)-\psi(x,\mu_2)|\leqslant$$
$$T(x,\mu_1)-T(x,\mu_2)y-x+2m_2\mu_1-\mu_2{}^{\gamma}\leqslant$$
$$m_1\mu_1-\mu_2{}^{\gamma}y-x+2m_2\mu_1-\mu_2{}^{\gamma}.$$

Notice the boundedness of $K(N(\bar{\lambda}))$,

$$|F(x,y,\mu_1)-F(x,y,\mu_2)|\leqslant$$
$$(m_1\tau+2m_2)\mu_1-\mu_2{}^{\gamma},$$

where $\tau:=\sup_{x,y\in K(N(\bar{\lambda}))}x-y<+\infty$.

Secondly, $\forall x \in K(N(\bar{\lambda})), \forall y_1,y_2 \in K(N(\bar{\lambda}))$,

$$|F(y_1,x,\mu)-F(y_2,x,\mu)|=$$
$$|[T(y_1,\mu),x-y_1]+\psi(x,\mu)-\psi(y_1,\mu)-$$
$$[T(y_2,\mu),x-y_2]-\psi(x,\mu)+\psi(y_2,\mu)|\leqslant$$
$$|[T(y_1,\mu),x-y_1]-[T(y_2,\mu),x-y_2]|+$$
$$|\psi(y_1,\mu)-\psi(y_2,\mu_2)|\leqslant$$
$$|[T(y_1,\mu),x-y_1]-[T(y_2,\mu),x-y_1]|+$$
$$|[T(y_2,\mu),x-y_1]-[T(y_2,\mu),x-y_2]|+$$
$$n_2y_1-y_2\leqslant n_1y_1-y_2x-y_1+$$
$$T(y_2,\mu)y_1-y_2+n_2y_1-y_2\leqslant$$
$$(n_1\tau+v+n_2)y_1-y_2.$$

Hence, the Hölder constant of Theorem 5.1 are fullfilled with $l_1=0$, α_1 is arbitrary,

$$l_2=l, \alpha_2=\alpha m=m_1\tau+2m_2,$$
$$n=n_1\tau+v+n_2 \text{ and } \delta=1.$$

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