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一类带积分边值条件的非线性分数阶微分方程的正解

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摘　要：利用 Green 函数的性质和 Schauder 不动点定理，本文研究一类带积分边值条件的非线性 Caputo 型分数阶微分方程边值问题，得到该边值问题正解的存在性定理，推广了相关结果，并举例说明了主要结果的适用性.

关键词：分数阶微分方程；积分边值条件；正解；不动点定理
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Positive solutions for nonlinear fractional differential equations with integral boundary value conditions

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Abstract: We studied the existence of positive solutions for nonlinear Caputo fractional differential equations with integral boundary value conditions. The existence results of positive solutions for the boundary value problems are obtained by applying the properties of Green function and Schauder fixed point theorem, which partly extend the corresponding results of fractional differential equations. Two examples are also presented to illustrate the applications of the main results.

Keywords: Fractional differential equations; Integral boundary conditions; Positive solutions; Fixed point theorem
(2010 MSC 34B18)

1 Introduction

Fractional differential equations can be applied to describe many phenomena in various fields, for examples in physics, control theory, blood flow phenomena, regular variation in thermodynamics, chemistry, polymer rheology, biophysics, fluid dynamics, and so forth. There are many papers dealing with the positive solutions of boundary value problems for nonlinear differential equations of fractional order[1-9]. At the same time, boundary value problems with integral boundary value conditions of nonlinear fractional differential equations have aroused considerable attention. Boundary value problems with integral boundary value conditions have various applications in chemical engineer-

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ing, population dynamics, etc. For some recent development on the integral boundary conditions, we refer to the reader to the Refs. [5～14] and the references cited therein.

Recently, Cabada and Wang[5] used the known Guo-Krasnoselskii fixed point theorem to obtain the existence for the boundary value problem as

$$\begin{cases} {}_0^C D_t^\alpha u(t) + f(t,u(t)) = 0, 2<\alpha<3, 0<t<1, \\ u(0) = u''(0) = 0, u(1) = \\ \quad \lambda \int_0^1 u(s)\mathrm{d}s, 0 < \lambda < 2, \end{cases}$$

where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative of order α and $f \in C([0,1]\times[0,\infty),[0,\infty))$.

Motivated by Ref. [5], the main aim of our work is to establish the existence criteria for positive solutions to the following nonlinear Caputo fractional differential equations with integral boundary value conditions

$$\begin{cases} {}_0^C D_t^\alpha u(t) + f(t,u(t)) = 0, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n)}(0) = 0, \\ u(1) = \lambda \int_0^1 u(s)\mathrm{d}s \end{cases} \tag{1}$$

where $0 < t < 1, n < \alpha < n+1, n \geq 2 (n \in \mathbf{N}), 0 < \lambda < n, f: [0,\infty) \to \mathbf{R}$ is the standard Caputo fractional derivative of order α and $f \in C([0,1]\times[0,\infty),[0,\infty))$ is a given function. $C([0,1]\times[0,\infty),[0,\infty))$ represents the set of all continuous functions from $[0,1]\times[0,\infty)$ into $[0,\infty)$ in this article. The existence results of positive solutions for the boundary value problems are obtained by applying the properties of Green function and Schauder fixed point theorem, which partly extend the corresponding results of Refs. [5] and [6]. Two examples are also presented to illustrate the applications of our main results.

2 Preliminaries

For the convenience of the reader, we will recall some necessary definitions and theorems which can been founded in Refs. [9, 15～18].

Definition 2.1[15] For a function $f: [0,\infty) \to \mathbf{R}$, the Caputo fractional derivative of order $\alpha > 0$ is given as follows

$$ {}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\mathrm{d}s,$$

provided the integral exists, where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of α.

Definition 2.2[15] For a function $f: [0,\infty) \to \mathbf{R}$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is given as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s$$

provided the integral exists.

Lemma 2.3[16] Let $\alpha > 0$, then the equation ${}_0^C D_t^\alpha u(t) = 0$ has a unique solution given as

$$u(t) = \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{\Gamma(j+1)} t^j.$$

Lemma 2.4[16] Let $\alpha > 0$, then the above-mentioned integral and derivative have the property as

$$I^\alpha \{ {}^c D^\alpha u(t) \} = u(t) - \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{\Gamma(j+1)} t^j.$$

To get the solution of the boundary problem (1), we consider the following fractional differential equation with integral boundary value condition

$$\begin{cases} {}^c D^\alpha u(t) + f(t,u(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = \cdots \\ \quad = u^{(n-2)}(0) = u^{(n)}(0) = 0, \\ u(1) = \lambda \int_0^1 u(s)\mathrm{d}s \end{cases} \tag{2}$$

Then we have the following useful lemma

Lemma 2.5[9] For arbitrary $y(t) \in C[0,1], n < \alpha < n+1, n \geq 2 (n \in \mathbf{N}), 0 < \lambda < n$, then the problem (2) has a unique solution given as

$$u(t) = \int_0^1 G(t,s) y(s)\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \dfrac{nt^{n-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (n-\lambda)\alpha(t-s)^{\alpha-1}}{(n-\lambda)\Gamma(\alpha+1)}, 0 \leq s \leq t \leq 1, \\ \dfrac{nt^{n-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(n-\lambda)\Gamma(\alpha+1)}, 0 \leq t \leq s \leq 1 \end{cases} \tag{3}$$

Lemma 2.6[9]　Let $n<\alpha<n+1, n\geq 2(n\in \mathbf{N})$, then the Green's function $G(t,s)$ given by (3) satisfies the following conditions

(H_1) $G(t,s)>0$ for any $t,s\in(0,1)$ if and only if $\lambda\in(0,n)$;

(H_2) $G(t,s)$ and $\frac{G(t,s)}{t^{n-1}}$ are continuous functions for any $t,s\in(0,1),\lambda\neq n$;

(H_3) $G(t,s)\leq\frac{n}{(n-\lambda)\Gamma(\alpha)}$ for any $t,s\in[0,1],\lambda\in[0,n)$;

(H_4) $t^{n-1}G(1,s)\leq G(t,s)\leq\frac{n\alpha}{\lambda(\alpha-n)}G(1,s)$ for any $t,s\in[0,1],\lambda\in(0,n)$.

Proof　From the expression of the Green's function $G(t,s)$, (H_1)、(H_2) and (H_3) are trivial, here we omit them. Now, we have only to prove (H_4).

Set

$$\varphi(t,s)\equiv\frac{G(t,s)}{G(1,s)}.$$

Suppose in a first case that $0<t\leq s<1$, we have

$$\varphi(t,s)=\frac{nt^{n-1}(\alpha-\lambda(1-s))}{\lambda(\alpha-n+ns)},$$

from which, we can deduce that

$$t^{n-1}\leq\frac{nt^{n-1}}{\lambda}\leq\varphi(t,s)\leq\frac{nt^{n-1}\alpha}{\lambda(\alpha-n)}\leq\frac{n\alpha}{\lambda(\alpha-n)},\quad \forall\, 0<t\leq s<1.$$

On the other hand, if $0<s\leq t<1$, we have

$$\frac{nt^{n-1}(\alpha-\lambda(1-s))-\alpha(n-\lambda)(t-s)^{\alpha-1}(1-s)^{1-\alpha}}{\lambda(\alpha-n+ns)}.$$

Clearly, we can obtain

$$t^{n-1}\leq\varphi(t,s)\leq\frac{nt^{n-1}(\alpha-\lambda(1-s))}{\lambda(\alpha-n+ns)}\leq\frac{n\alpha}{\lambda(\alpha-n)},\quad \forall\, 0<s\leq t<1.$$

This concludes our desired result.

Lemma 2.7(Arzelà-Ascoli theorem[17])　Let $D\subset X$ be a compact set with a sequence $\{x_n\}\subset D$ being uniformly bounded and equicontinuous, then the sequence has a uniformly convergent subsequence.

Lemma 2.8(Schauder fixed point theorem[18]) Let X be a Banach space with $U\subset X$ being closed, convex and nonempty. Let $P:U\to U$ be a continuous mapping such that $P(U)$ is a relatively compact subset of X. Then P has at least one fixed point in U.

3　Main results

In this section, we show sufficient conditions of the existence results of solutions for boundary value Problem(1). First, we establish some notions. Let $E=C[0,1]$ represents the set of all continuous functions defined on $[0,1]$ with real values. Then E is a Banach space endowed with the norm given as

$$\|u\| = \max_{t\in[0,1]}|u(t)|.$$

For any $t\in[0,1]$, define the operator $F:E\to E$ by

$$(Fu)(t)=\int_0^1 G(t,s)f(s,u(s))\,ds \qquad (4)$$

It follows from Lemma 2.5 that the fixed points of the operator F coincide with the solutions of fractional differential equation (1). Now, we will show that the operator $F:E\to E$ is completely continuous.

Lemma 3.1　The operator $F:E\to E$ given by (4) is completely continuous.

Proof　By continuity of functions $f(t,u)$ and $G(t,s)$, the operator F is continuous. Let $\Omega\subset E$ be bounded. Then for any $t\in[0,1]$ and $u\in\Omega$, there exists a positive constant $K_1>0$ such that $|f(t,u)|\leq K_1$. From Lemma 2.6 and (4), we can deduce that

$$|(Fu)(t)|\leq\int_0^1|G(t,s)f(s,u(s))|\,ds\leq \int_0^1\frac{n}{(n-\lambda)\Gamma(\alpha)}|f(s,u(s))|\,ds\leq \frac{n}{(n-\lambda)\Gamma(\alpha)}K_1:=K_2,$$

which implies that $\|(Fu)(t)\|\leq K_2$. Analogously, for the derivative, we obtain that

$$|(Fu)'(t)|=\left|-\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s))\,ds+ \int_0^1\frac{n(n-1)t^{n-2}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(n-\lambda)\alpha\Gamma(\alpha)}f(s,u(s))\,ds\right|$$

$$\leq\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}|f(s,u(s))|\,ds+ \frac{n(n-1)t^{n-2}}{(n-\lambda)\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha-1}(\alpha- \lambda+\lambda s)|f(s,u(s))|\,ds\leq$$

$$\frac{K_1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\mathrm{d}s + \frac{n(n-1)\alpha K_1}{(n-\lambda)\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha-1}\mathrm{d}s \le \frac{K_1}{\Gamma(\alpha)} + \frac{n(n-1)K_1}{(n-\lambda)\Gamma(\alpha+1)} := K_3.$$

Therefore, for any $0 \le t_1 < t_2 \le 1$, we have

$$|(Fu)(t_2)-(Fu)(t_1)| = \left|\int_{t_1}^{t_2}(Fu)'(s)\mathrm{d}s\right| \le \int_{t_1}^{t_2}|(Fu)'(s)|\mathrm{d}s \le K_3(t_2-t_1),$$

which implies that the operator F is equicontinuous on $[0,1]$. Thus, by Lemma 2.7, the operator $F:E\to E$ given by (4) is completely continuous.

In the following, for convenience we denote

$$f_0 = \lim_{u\to 0^+}\left\{\max_{t\in[0,1]}\frac{f(t,u)}{u}\right\},$$

$$f_\infty = \lim_{u\to\infty}\left\{\max_{t\in[0,1]}\frac{f(t,u)}{u}\right\},$$

$$\rho = \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\mathrm{d}s\right\}.$$

Theorem 3.2 Assume that $f \in C([0,1]\times[0,+\infty),[0,+\infty))$ is a given function and exists real constant $K>0$ such that

$$f(t,u) \le \frac{K}{\rho}, \forall u \in (0,K], t\in[0,1] \quad (5)$$

Then the boundary value Problem (1) has at least one solution on $[0,1]$.

Proof Define $\Omega_K \subset E$ as

$$\Omega_K = \left\{u \mid u\in E, \frac{u(t)}{t^{n-1}}\in E, \|u\| \le K, u(t) \ge \frac{t^{n-1}\lambda(\alpha-n)}{n\alpha}\|u\|, \forall t\in[0,1]\right\}.$$

Since $n<\alpha<n+1, 0<\lambda<n, n\ge 2 (n\in\mathbf{N})$, then we have

$$0 < \frac{\lambda(\alpha-n)}{n\alpha} < \frac{\lambda}{n\alpha} < \frac{1}{n},$$

which implies that Ω_K is nonempty. Moreover, it is easy to see that Ω_K is a closed convex and bounded subset of E by the definition of Ω_K. When $u\in\Omega_K$, note that the non-negativeness and continuity of $f(t,u)$ and $G(t,s)$, in view of Lemma 2.6 we have $Fu \in E$ and

$$\frac{Fu(t)}{t^{n-1}} = \int_0^1 \frac{G(t,s)}{t^{n-1}} f(s,u(s))\mathrm{d}s \in E.$$

On the other hand, by (5) and the definition of ρ, we have

$$Fu(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \le \int_0^1 G(t,s)\frac{K}{\rho}\mathrm{d}s \le \frac{K}{\rho}\max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\mathrm{d}s\right\} = K,$$

which implies that

$$\|(Fu)(t)\| \le K, \forall u\in\Omega_K, t\in[0,1].$$

By virtue of Lemma 2.6, we deduce that

$$Fu(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \ge t^{n-1}\int_0^1 G(1,s)f(s,u(s))\mathrm{d}s \ge \frac{t^{n-1}\lambda(\alpha-n)}{n\alpha}\int_0^1 \max_{t\in[0,1]}\{G(t,s)\}f(s,u(s))\mathrm{d}s \ge \frac{t^{n-1}\lambda(\alpha-n)}{n\alpha}\max_{t\in[0,1]}\left\{\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s\right\} = \frac{t^{n-1}\lambda(\alpha-n)}{n\alpha}\|Fu\|.$$

Then we have

$$F(\Omega_K)\subset\Omega_K, \text{i.e. } F:\Omega_K\to\Omega_K.$$

By virtue of Lemma 3.1, we obtain that $F:\Omega_K\to\Omega_K$ is completely continuous. It follows from above that all conditions of Lemma 2.8 hold. Consequently, the boundary value Problem (1) has at least one positive solution on $[0,1]$.

Theorem 3.3 Assume that $f \in C([0,1]\times[0,+\infty),[0,+\infty))$ is a given function and exists real constant $L>0$ such that

$$f(t,u) \le \frac{L}{2\rho}, \forall u \in (L,\infty), t\in[0,1] \quad (6)$$

Then the boundary value Problem (1) has at least one solution on $[0,1]$.

Proof Take a real constant M such that

$$M \ge L+1+2\rho\max_{u\in[0,L]} f(t,u) \quad (7)$$

Define $\Omega_M\subset E$ as

$$\Omega_M = \left\{u \mid u\in E, \frac{u(t)}{t^{n-1}}\in E, \|u\| \le M, u(t) \ge \frac{t^{n-1}\lambda(\alpha-n)}{n\alpha}\|u\|, \forall t\in[0,1]\right\}.$$

Then, like the proof of Theorem 3.2, we get that Ω_M is a closed convex bounded and nonempty subset of E. In addition, one can also obtain that

$$Fu\in E, \frac{Fu(t)}{t^{n-1}}\in E,$$

$$(Fu)(t) \geq \frac{t^{n-1}\lambda(\alpha-n)}{n\alpha}\|Fu\|.$$

By (6), (7) and the definition of ρ, for any $u \in \Omega_M, t \in [0,1]$, we have

$$\begin{aligned} Fu(t) &= \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s = \\ &\int_{D_1} G(t,s)f(s,u(s))\mathrm{d}s + \\ &\int_{D_2} G(t,s)f(s,u(s))\mathrm{d}s \leq \\ &\int_0^1 G(t,s)\,\frac{L}{2\rho}\mathrm{d}s + \int_0^1 G(t,s)\mathrm{d}s \max_{u\in[0,L]} f(t,u) \leq \\ &\frac{L}{2\rho}\max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\mathrm{d}s\right\} + \rho \max_{u\in[0,L]} f(t,u) \leq \\ &\frac{L}{2} + \frac{M}{2} \leq M, \end{aligned}$$

where

$$D_1 = \{s \in [0,1], u(s) > L\},$$
$$D_2 = \{s \in [0,1], u(s) \leq L\},$$

which implies that $\|(Fu)(t)\| \leq M$ for any $u \in \Omega_M, t \in [0,1]$. By means of Lemma 3.1, we obtain that $F:\Omega_M \to \Omega_M$ is completely continuous. Thank to Lemma 2.8, the boundary value Problem (1) has at least one positive solution on [0, 1].

4 Examples

Example 4.1 Consider the following problem of fractional differential equations with integral boundary conditions

$$\begin{cases} {}^cD^{7/2}u(t) + f(t,u(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = u'''(0) = 0, \\ \quad u(1) = 2\int_0^1 u(s)\mathrm{d}s \end{cases} \tag{8}$$

where

$$3 < \alpha = \frac{7}{2} < 4, 0 < \lambda = 2 < 3,$$

$$f(t,u) = \frac{tu\sin^4 u}{\rho}, \rho = \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\mathrm{d}s\right\},$$

here $G(t,s)$ is defined by (3). It is easy to find that $f \in C([0,1]\times[0,+\infty),[0,+\infty))$. For any $u \in (0,K]$, one can get that

$$f(t,u) = \frac{tu\sin^4 u}{\rho} \leq \frac{K}{\rho}, \forall K > 0, t \in [0,].$$

Hence all the conditions of Theorem 3.2 are satisfied, the boundary value problem (8) has at least one positive solution on [0,1].

Example 4.2 Consider the following problem of fractional differential equations with integral boundary conditions

$$\begin{cases} {}^cD^{9/2}u(t) + f(t,u(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u^{(4)}(0) = 0, \\ \quad u(1) = \frac{11}{3}\int_0^1 u(s)\mathrm{d}s \end{cases} \tag{9}$$

where

$$4 < \alpha = \frac{9}{2} < 5, 0 < \lambda = \frac{11}{3} < 4,$$

$$f(t,u) = \frac{tu(t)}{(2+u(t))\rho}, \rho = \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\mathrm{d}s\right\},$$

here $G(t,s)$ is defined by (3). It is easy to find that $f \in C([0,1]\times[0,+\infty),[0,+\infty))$. For any $u \in (L,\infty)$, one can get that

$$f(t,u) = \frac{t\sin^2 u}{(2+u(t))\rho} \leq \frac{L}{(2+L)\rho} \leq \frac{L}{2\rho}, \forall L > 1, t \in [0,1].$$

Thus, by the use of Theorem 3.3, the boundary value problem (9) has at least one positive solution on [0,1].

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