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单位球上多重调和 Bergman 空间上的 k-拟齐次 Toeplitz 算子

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摘　要：本文研究单位球上的多重调和 Bergman 空间 b_α^2 上的 k-拟齐次 Toeplitz 算子的基本性质，得到了该类算子所构成的交换子及半交换子的两个对称性质. 此外，本文还得到了 b_α^2 上的两个单项式型 Toeplitz 算子所构成的交换子和半交换子具有有限秩的充分必要条件.

关键词：Toeplitz 算子；多重调和 Bergman 空间；k-拟齐次函数
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k-quasi-homogeneous Toeplitz operators on pluriharmonic Bergman space of the unit ball

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Abstract: In this paper, we study some basic properties of the k-quasi-homogeneous Toeplitz operators on pluriharmonic Bergman space b_α^2 of the unit ball, and obtain two symmetric properties of the commutator and semi-commutator consisting of two such operators on b_α^2. Additionally, we obtain the necessary and sufficient conditions for the finite rank of commutator and semi-commutator of two monomial-type Toeplitz operators on b_α^2.

Keywords: Toeplitz operator; Pluriharmonic Bergman space; k-quasi-homogeneous function
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1　Introduction

Let B_n be the open unit ball in $\mathbf{C}^n$ and dv denote the standard volume measure on B_n. Throughout this paper we fix a parameter $\alpha > -1$ and consider the following weighted volume measure

$$dv_\alpha(z) = \frac{\Gamma(n+\alpha+1)}{\pi^n \Gamma(\alpha+1)} (1-|z|^2)^\alpha dv(z),$$

which is a probability measure on B_n.

Pluriharmonic Bergman space b_α^2 is the closed subspace of $L^2(B_n, dv_\alpha)$, consisting of all pluriharmonic functions on B_n. Given a function $f \in L^\infty(B_n, dv_\alpha)$, we define the Toeplitz operator T_f: $b_\alpha^2 \to b_\alpha^2$ by

$$T_f g = Q(fg), g \in b_\alpha^2,$$

where Q is the orthogonal projection from $L^2(B_n, dv_\alpha)$ onto b_α^2. For two Toeplitz operators T_{f_1} and T_{f_2} on b_α^2, we define their commutator and the semi-commutator, respectively, by $[T_{f_1}, T_{f_2}] = T_{f_1}T_{f_2} - T_{f_2}T_{f_1}$ and $(T_{f_1}, T_{f_2}] = T_{f_1}T_{f_2} - T_{f_1 f_2}$.

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Several classical operators on analytic function spaces have been widely studied in the past few decades, including multiplication operators, weighted composition operators, Toeplitz operators, Hankel operators, and so on, see Refs. [1-7] and references there. Particularly, the problem of determining when the commutator or semi-commutator of two Toeplitz operators on various classical function spaces has finite rank was considered in Refs. [8-12].

In the setting of pluriharmonic Bergman spaces of the unit ball, the problem is known to be much more delicate and more challenging. Just recently, the second author and Zhu in Ref. [13] completely characterized when the commutators and semi-commutators of two monomial Toeplitz operators on b_α^2 have finite rank. In this paper, we consider more general symbols, namely, the monomial-type symbols. Recall that the monomial-type symbol is the function $\varphi: B_n \to \mathbf{C}$ given by

$$\varphi(z)=r^l \xi^p \bar{\xi}^q, z=r\xi,$$

for $p,q\in\mathbf{N}^n$, $l\in\mathbf{R}_+$. Here $\mathbf{R}_+$ denotes the set of all nonnegative real numbers. In this case, the corresponding Toeplitz operator T_φ is called a monomial-type Toeplitz operator.

The paper is organized as follows. Motivated by Ref. [13], we first obtain some interesting symmetric properties of k-quasi-homogeneous Toeplitz operators on b_α^2 in Section 2. Then we give a complete characterization of when the commutators and semi-commutators of two monomial-type Toeplitz operators on b_α^2 have finite rank in Section 3.

2 Preliminaries

The concept of quasi-homogeneous functions on the unit disk was first defined in Ref. [14]. Then, Refs. [15] and [16] introduced the notions "quasi-homogeneous functions" and "separately quasi-homogeneous functions" on B_n. These two classes of functions together were referred to as k-quasi-homogeneous functions in Ref. [17]. The purpose of this section is to prove some preliminary results about k-quasi-homogeneous Toeplitz operators on b_α^2.

More specifically, if $k=(k_1,\cdots,k_m)$ is a tuple of positive integers with $|k|=k_1+\cdots+k_m=n$, and if we write $\mathbf{C}^n=\mathbf{C}^{k_1}\times\cdots\times\mathbf{C}^{k_m}$, then every point $z\in\mathbf{C}^n$ can be written as $z=(z_{(1)},\cdots,z_{(m)})$, where $z_{(j)}=(z_{j,1},\cdots,z_{j,k_j})\in\mathbf{C}^{k_j}$ for $j\in\{1,\cdots,m\}$. For each j we write $z_{(j)}\in B_{k_j}$ in the form $z_{(j)}=r_j\xi_{(j)}$ with $r_j=|z_{(j)}|$ and $\xi_{(j)}\in S_{k_j}$. Now, for $p,q\in\mathbf{N}^n$, a bounded function $f(z)$ on B_n is called k-quasi-homogeneous if it has the form

$$f(z_{(1)},\cdots,z_{(m)})=\xi^p\bar{\xi}^q\varphi(r_1,\cdots,r_m)=$$
$$\xi_{(1)}^{p_{(1)}}\cdots\xi_{(m)}^{p_{(m)}}\bar{\xi}_{(1)}^{q_{(1)}}\cdots\bar{\xi}_{(m)}^{q_{(m)}}\varphi(r_1,\cdots,r_m),$$

where $\varphi=\varphi(r_1,\cdots,r_m)$ is a function of the m nonnegative real variables $r_1,\cdots,r_m$. Furthermore, if $p\perp q$ in $\mathbf{C}^n$, that is, $p_1q_1+\cdots+p_nq_n=0$, then ξ^σ, for $\sigma=p-q\in\mathbf{Z}^n$, is always understood as $\xi^\sigma=\xi^p\bar{\xi}^q$, and the tuple (p,q) is called the k-quasi-homogeneous degree of $f(z)$. In this case the associated Toeplitz operator T_f is also called a k-quasi-homogeneous Toeplitz operator.

Take the monomial $z^p\bar{z}^q$ for example. Obviously, $z^p\bar{z}^q$ is a special k-quasi-homogeneous function with $k=(n)$ and $\varphi(r)=r^{|p|+|q|}$. In addition, we can also consider $z^p\bar{z}^q$ as k-quasi-homogeneous function with $k=(1,\cdots,1)$ and

$$\varphi(r_1,\cdots,r_n)=r_1^{p_1+q_1}\cdots r_n^{p_n+q_n}.$$

See Ref. [17] for more information about k-quasi-homogeneous functions.

Denote by $\tau(B_m)$ the base of B_m, considered as a Reinhard domain, namely,

$$\tau(B_m)=\{(r_1,\cdots,r_m)=(|z_{(1)}|,\cdots,|z_{(m)}|):$$
$$z=(z_{(1)},\cdots,z_{(m)})\in B_n\}.$$

For each $\beta=(\beta_{(1)},\cdots,\beta_{(m)})\in\mathbf{N}^n$, we write $\kappa(\beta)=(|\beta_{(1)}|,\cdots,|\beta_{(m)}|)$ and then define

$$\widehat{\varphi}\circ\kappa(\beta)=\int_{\tau(B_m)}\varphi(r_1,\cdots,r_m)(1-|r|^2)^\alpha\cdot$$
$$\prod_{j=1}^m r_j^{|\beta_{(j)}|+2k_j-1}\,\mathrm{d}r_j \quad (1)$$

This is a natural generalization of the Mellin transform to higher dimensions[3].

Lemma 2.1 Let $p,q\in\mathbf{N}^n$ and let $\xi^p\bar{\xi}^q\varphi(r_1,\cdots,r_m)$ be a bounded k-quasi-homogeneous function on B_n. Then on b_α^2, for each $\beta\in\mathbf{N}^n$, we have

$$T_{\xi^p\xi^q\varphi}(z^\beta)=\begin{cases}\dfrac{2^m\Gamma(n+|\beta|+|p|-|q|+\alpha+1)(\beta+p)!\ \widehat{\varphi}\circ\kappa(2\beta+p-q)}{\Gamma(\alpha+1)\prod_{j=1}^m(k_j-1+|\beta_{\langle j\rangle}+p_{\langle j\rangle}|)!\ (\beta+p-q)!}z^{\beta+p-q}, & \beta+p\geqslant q,\\ \dfrac{2^m\Gamma(n-|\beta|-|p|+|q|+\alpha+1)q!\ \widehat{\varphi}\circ\kappa(q-p)}{\Gamma(\alpha+1)\prod_{j=1}^m(k_j-1+|q_{\langle j\rangle}|)!\ (-\beta-p+q)!}\bar z^{q-\beta-p}, & \beta+p\leqslant q,\\ 0, & \text{otherwise}\end{cases}$$

and

$$T_{\xi^p\xi^q\varphi}(\bar z^\beta)=\begin{cases}\dfrac{2^m\Gamma(n+|\beta|+|q|-|p|+\alpha+1)(\beta+q)!\ \widehat{\varphi}\circ\kappa(2\beta+q-p)}{\Gamma(\alpha+1)\prod_{j=1}^m(k_j-1+|\beta_{\langle j\rangle}+q_{\langle j\rangle}|)!\ (\beta-p+q)!}\bar z^{\beta+q-p}, & \beta+q\geqslant p,\\ \dfrac{2^m\Gamma(n-|\beta|-|q|+|p|+\alpha+1)p!\ \widehat{\varphi}\circ\kappa(p-q)}{\Gamma(\alpha+1)\prod_{j=1}^m(k_j-1+|p_{\langle j\rangle}|)!\ (-\beta+p-q)!}z^{p-\beta-q}, & \beta+q\leqslant p,\\ 0, & \text{otherwise.}\end{cases}$$

Proof　First we assume $\beta+p\geqslant q$. Then for each $\lambda\in\mathbf{N}^n$ we have

$$\langle T_{\xi^p\xi^q\varphi}z^\beta,z^\lambda\rangle=\langle\xi^p\xi^q\varphi z^\beta,z^\lambda\rangle=$$
$$C_\alpha\int_{B_n}\varphi(r_1,\cdots,r_m)\xi^p\xi^q z^\beta\bar z^\lambda(1-|z|^2)^\alpha\mathrm{d}\upsilon(z),$$

where $C_\alpha=\dfrac{\Gamma(n+\alpha+1)}{\pi^n\Gamma(\alpha+1)}$. Changing the variables $z_{(j)}=r_j\xi_{(j)}$ and letting $\mathrm{d}S$ be the surface measure on S_{k_j} before normalization, we have

$$\langle T_{\xi^p\xi^q\varphi}(z^\beta),z^\lambda\rangle=$$
$$C_\alpha\int_{\tau(B_m)}\varphi(r_1,\cdots,r_m)(1-|r|^2)^\alpha\cdot\prod_{j=1}^m r_j^{|\beta_{(j)}+\lambda_{(j)}|+2k_j-1}\mathrm{d}r_j\cdot\prod_{j=1}^m\int_{S_{k_j}}\xi_{(j)}^{\beta_{(j)}+p_{(j)}}\bar\xi_{(j)}^{\lambda_{(j)}+q_{(j)}}\mathrm{d}S(\xi_{(j)}).$$

Using the notation from (1) and applying (1.22) and (1.23) of Ref. [18], we obtain

$$\langle T_{\xi^p\xi^q\varphi}(z^\beta),z^\lambda\rangle=\begin{cases}0, & \lambda\neq\beta+p-q,\\ \dfrac{2^m\Gamma(n+\alpha+1)(\beta+p)!\ \widehat{\varphi}\circ\kappa(2\beta+p-q)}{\Gamma(\alpha+1)\prod_{j=1}^m(k_j-1+|\beta_{\langle j\rangle}+p_{\langle j\rangle}|)!}, & \lambda=\beta+p-q,\end{cases}$$
$$=\frac{2^m\Gamma(n+|\beta|+|p|-|q|+\alpha+1)(\beta+p)!\ \widehat{\varphi}\circ\kappa(2\beta+p-q)}{\Gamma(\alpha+1)\prod_{j=1}^m(k_j-1+|\beta_{\langle j\rangle}+p_{\langle j\rangle}|)!\ (\beta+p-q)!}\langle z^{\beta+p-q},z^\lambda\rangle.$$

Moreover, for any nonzero $\lambda\in\mathbf{N}^n$, we have $\langle T_{\xi^p\xi^q\varphi}(z^\beta),\bar z^\lambda\rangle=\langle z^{\beta+p-q},\bar z^\lambda\rangle=0$ as $\beta+p\geqslant q$. Therefore,

$$T_{\xi^p\xi^q\varphi}(z^\beta)=\frac{2^m\Gamma(n+|\beta|+|p|-|q|+\alpha+1)(\beta+p)!\ \widehat{\varphi}\circ\kappa(2\beta+p-q)}{\Gamma(\alpha+1)\prod_{j=1}^m(k_j-1+|\beta_{\langle j\rangle}+p_{\langle j\rangle}|)!\ (\beta+p-q)!}z^{\beta+p-q}.$$

Next we assume $\beta+p\leqslant q$. Just like the previous case, we can show that

$$T_{\xi^p\xi^q\varphi}(z^\beta)=\frac{2^m\Gamma(n-|\beta|-|p|+|q|+\alpha+1)q!\ \widehat{\varphi}\circ\kappa(q-p)}{\Gamma(\alpha+1)\prod_{j=1}^m(k_j-1+|q_{\langle j\rangle}|)!\ (q-\beta-p)!}\bar z^{q-\beta-p}.$$

Finally, we assumethat $\beta+p\ngeqq q$ and $\beta+p\nleqq q$. Then $\beta_j+p_j<q_j$ and $\beta_i+p_i>q_i$ for some $i,j\in\{1,\cdots,n\}$, which implies $\beta+p-\lambda\neq q$ and $\beta+p+\lambda\neq q$ for any $\lambda\in\mathbf{N}^n$. Consequently,

$$\langle T_{\xi^p\xi^q\varphi}(z^\beta),z^\lambda\rangle=\langle T_{\xi^p\xi^q\varphi}(z^\beta),\bar z^\lambda\rangle=0,$$

which shows that $T_{\xi^p\xi^q\varphi}(z^\beta)=0$.

The computation for $T_{\xi^p\xi^q\varphi}(\bar z^\beta)$ is similar and we leave the details to the interested reader.

For convenience we will write $|l!|=|l_1|!\cdots|l_n|!$ for any multi-index $l=(l_1,\cdots,l_n)\in\mathbf{N}^n\cup(-\mathbf{N})^n$.

Lemma 2.2　Let $p,q\in\mathbf{N}^n$, $l\in\mathbf{N}^n\cup(-\mathbf{N})^n$, and let $\xi^p\xi^q\varphi(r_1,\cdots,r_m)$ be a bounded k-quasi-homogeneous function on B_n. Then

$$T_{\xi^p\xi^q\varphi}(r^{|l|}\xi^l)=$$

$$\begin{cases} C_{p,q,l}r^{|l+p-q|}\xi^{l+p-q}, & l+p-q\in\mathbf{N}^n\cup(-\mathbf{N})^n,\\ 0, & l+p-q\notin\mathbf{N}^n\cup(-\mathbf{N})^n\end{cases}$$

for some constant $C_{p,q,l}$. Moreover, if $l+p-q\in\mathbf{N}^n\cup(-\mathbf{N})^n$, then

$$\frac{|(l+p-q)!|}{\Gamma(n+|l+p-q|+\alpha+1)}C_{p,q,l}=\frac{|l!|}{\Gamma(n+|l|+\alpha+1)}C_{p,q,-l-p+q} \qquad (2)$$

Proof It follows from Lemma 2.1 that, if $l+p-q\notin\mathbf{N}^n\cup(-\mathbf{N})^n$, then $T_{\xi^p\bar\xi^q\varphi}(r^{|l|}\xi^l)=0$, and if $l+p-q\in\mathbf{N}^n\cup(-\mathbf{N})^n$ then

$$T_{\xi^p\bar\xi^q\varphi}(r^{|l|}\xi^l)=C_{p,q,l}r^{|l+p-q|}\xi^{l+p-q},$$

where the constant $C_{p,q,l}$ is given by $\frac{2m}{\Gamma(\alpha+1)}$ times

$$\begin{cases} \frac{\Gamma(n+|l|+|p|-|q|+\alpha+1)(l+p)!\ \hat{\varphi}\circ\kappa(2l+p-q)}{\prod_{j=1}^m(k_j-1+|l_{\langle j\rangle}+p_{\langle j\rangle}|)!\ (l+p-q)!}, & l\geqslant0, l+p\geqslant q,\\ \frac{\Gamma(n-|l|-|p|+|q|+\alpha+1)q!\ \hat{\varphi}\circ\kappa(q-p)}{\prod_{j=1}^m(k_j-1+|q_{\langle j\rangle}|)!\ (-l-p+q)!}, & l\geqslant0, l+p\leqslant q,\\ \frac{\Gamma(n+|-l|+|q|-|p|+\alpha+1)(-l+q)!\ \hat{\varphi}\circ\kappa(-2l-p+q)}{\prod_{j=1}^m(k_j-1+|-l_{\langle j\rangle}+q_{\langle j\rangle}|)!\ (-l-p+q)!}, & l\leqslant0, -l+q\geqslant p,\\ \frac{\Gamma(n-|-l|-|q|+|p|+\alpha+1)p!\ \hat{\varphi}\circ\kappa(p-q)}{\prod_{j=1}^m(k_j-1+|p_{\langle j\rangle}|)!\ (l+p-q)!}, & l\leqslant0, -l+q\leqslant p\end{cases} \qquad (3)$$

Observe that

- $l\geqslant0$ and $l+p\geqslant q\Leftrightarrow -l-p+q\leqslant0$ and $-(-l-p+q)+q\geqslant p$;
- $l\geqslant0$ and $l+p\leqslant q\Leftrightarrow -l-p+q\geqslant0$ and $(-l-p+q)+p\leqslant q$;
- $l\leqslant0$ and $-l+q\geqslant p\Leftrightarrow -l-p+q\geqslant0$ and $(-l-p+q)+p\geqslant q$;
- $l\leqslant0$ and $-l+q\leqslant p\Leftrightarrow -l-p+q\leqslant0$ and $-(-l-p+q)+q\leqslant p$.

So, by (3), the constant $C_{p,q,-l-p+q}$ is given by $\frac{2m}{\Gamma(\alpha+1)}$ times

$$\begin{cases} \frac{\Gamma(n+|l|+\alpha+1)(l+p)!\ \hat{\varphi}\circ\kappa(2l+p-q)}{\prod_{j=1}^m(k_j-1+|l_{\langle j\rangle}+p_{\langle j\rangle}|)!\ l!}, & l\geqslant0, l+p\geqslant q,\\ \frac{\Gamma(n+|l|+\alpha+1)q!\ \hat{\varphi}\circ\kappa(q-p)}{\prod_{j=1}^m(k_j-1+|q_{\langle j\rangle}|)!\ l!}, & l\geqslant0, l+p\leqslant q,\\ \frac{\Gamma(n+|-l|+\alpha+1)(-l+q)!\ \hat{\varphi}\circ\kappa(-2l-p+q)}{\prod_{j=1}^m(k_j-1+|-l_{\langle j\rangle}+q_{\langle j\rangle}|)!\ (-l)!}, & l\leqslant0, -l+q\geqslant p,\\ \frac{\Gamma(n+|-l|+\alpha+1)p!\ \hat{\varphi}\circ\kappa(p-q)}{\prod_{j=1}^m(k_j-1+|p_{\langle j\rangle}|)!\ (-l)!}, & l\leqslant0, -l+q\leqslant p\end{cases} \qquad (4)$$

A comparison of (3) with (4) shows that (2) holds. The proof is complete.

The next two propositions will be essential for our arguments in Section 3.

Proposition 2.3 Let $p,q,s,t\in\mathbf{N}^n$. Suppose $\xi^p\bar\xi^q\varphi_1(r_1,\cdots,r_m)$ and $\xi^s\bar\xi^t\varphi_2(r_1,\cdots,r_m)$ are two bounded k-quasi-homogeneous functions on B_n. Then for any $l\in\mathbf{N}^n\cup(-\mathbf{N})^n$ the following statements hold:

(a) If $-l-(p-q)-(s-t)\notin\mathbf{N}^n\cup(-\mathbf{N})^n$, then $[T_{\xi^p\bar\xi^q\varphi_1},T_{\xi^s\bar\xi^t\varphi_2}](r^{|l|}\xi^l)=0$;

(b) If $-l-(p-q)-(s-t)\in\mathbf{N}^n\cup(-\mathbf{N})^n$, then $[T_{\xi^p\bar\xi^q\varphi_1},T_{\xi^s\bar\xi^t\varphi_2}](r^{|l|}\xi^l)=0$ if and only if

$$[T_{\xi^p\bar\xi^q\varphi_1},T_{\xi^s\bar\xi^t\varphi_2}](r^{|-l-(p-q)-(s-t)|}\xi^{-l-(p-q)-(s-t)})=0.$$

Moreover, if neither $l+p-q$ nor $l+s-t$ is in $\mathbf{N}^n\cup(-\mathbf{N})^n$, then

$$[T_{\xi^p\bar\xi^q\varphi_1},T_{\xi^s\bar\xi^t\varphi_2}](r^{|l|}\xi^l)=[T_{\xi^p\bar\xi^q\varphi_1},T_{\xi^s\bar\xi^t\varphi_2}](r^{|-l-(p-q)-(s-t)|}\xi^{-l-(p-q)-(s-t)})=0.$$

(c) If $[-(p-q)-(s-t)]/2\in\mathbf{N}^n\cup(-\mathbf{N})^n$, then

$$[T_{\xi^p\bar\xi^q\varphi_1},T_{\xi^s\bar\xi^t\varphi_2}](r^{\left|\frac{-(p-q)-(s-t)}{2}\right|}\xi^{\frac{-(p-q)-(s-t)}{2}})=0.$$

Proof Since equation (2) is the same as

Ref. [13, Equation (12)], the proof is similar to that of Ref. [13, Proposition 10]. We omit the details here.

It follows from Proposition 2.3 that if $r^{|d|}\xi^d$ belongs to the range of $[T_{\xi^p\bar\xi^q\varphi_1}, T_{\xi^s\bar\xi^t\varphi_2}]$ for some $d\in\mathbf{N}^n\cup(-\mathbf{N})^n$ then $-d+(p-q)+(s-t)\in\mathbf{N}^n\cup(-\mathbf{N})^n$ and

$$r^{|-d+(p-q)+(s-t)|}\xi^{-d+(p-q)+(s-t)}\in \mathrm{Ran}[T_{\xi^p\bar\xi^q\varphi_1}, T_{\xi^s\bar\xi^t\varphi_2}].$$

Thus the commutator $[T_{\xi^p\bar\xi^q\varphi_1}, T_{\xi^s\bar\xi^t\varphi_2}]$ cannot have an odd rank, which corresponds to Ref. [8, Theorem 4]. We will call $\frac{-(p-q)-(s-t)}{2}$ the symmetry multi-index of the commutator $[T_{\xi^p\bar\xi^q\varphi_1}, T_{\xi^s\bar\xi^t\varphi_2}]$ on b_α^2.

Proposition 2.4 Let $p,q,s,t\in\mathbf{N}^n$. Suppose $\xi^p\bar\xi^q\varphi_1(r_1,\cdots,r_m)$, $\xi^s\bar\xi^t\varphi_2(r_1,\cdots,r_m)$, and $\xi^{p+s}\bar\xi^{q+t}\psi(r_1,\cdots,r_m)$ are bounded k-quasi-homogeneous functions on B_n. Then for any $l\in\mathbf{N}^n\cup(-\mathbf{N})^n$ the following properties hold:

(a) If $-l-(p-q)-(s-t)\notin\mathbf{N}^n\cup(-\mathbf{N})^n$, then

$$(T_{\xi^p\bar\xi^q\varphi_1}T_{\xi^s\bar\xi^t\varphi_2}-T_{\xi^{p+s}\bar\xi^{q+t}\psi})(r^{|l|}\xi^l)=(T_{\xi^s\bar\xi^t\varphi_2}T_{\xi^p\bar\xi^q\varphi_1}-T_{\xi^{p+s}\bar\xi^{q+t}\psi})(r^{|l|}\xi^l)=0;$$

(b) If $-l-(p-q)-(s-t)\in\mathbf{N}^n\cup(-\mathbf{N})^n$, then

$$(T_{\xi^p\bar\xi^q\varphi_1}T_{\xi^s\bar\xi^t\varphi_2}-T_{\xi^{p+s}\bar\xi^{q+t}\psi})(r^{|l|}\xi^l)=0$$

if and only if

$$(T_{\xi^s\bar\xi^t\varphi_2}T_{\xi^p\bar\xi^q\varphi_1}-T_{\xi^{p+s}\bar\xi^{q+t}\psi})(r^{|-l-(p-q)-(s-t)|}\xi^{-l-(p-q)-(s-t)})=0.$$

Proof First we assume that $-l-(p-q)-(s-t)\notin\mathbf{N}^n\cup(-\mathbf{N})^n$. It is then clear from Lemma 2.2 that

$$T_{\xi^p\bar\xi^q\varphi_1}T_{\xi^s\bar\xi^t\varphi_2}(r^{|l|}\xi^l)=T_{\xi^s\bar\xi^t\varphi_2}T_{\xi^p\bar\xi^q\varphi_1}(r^{|l|}\xi^l)=T_{\xi^{p+s}\bar\xi^{q+t}\psi}(r^{|l|}\xi^l)=0,$$

and the result follows.

Next we assume that $-l-(p-q)-(s-t)\in\mathbf{N}^n\cup(-\mathbf{N})^n$. If $l+s-t\in\mathbf{N}^n\cup(-\mathbf{N})^n$ as well, then by Lemma 2.2,

$$\begin{aligned}&(T_{\xi^s\bar\xi^t\varphi_2}T_{\xi^p\bar\xi^q\varphi_1}-T_{\xi^{p+s}\bar\xi^{q+t}\psi})(r^{|-l-(p-q)-(s-t)|}\xi^{-l-(p-q)-(s-t)})=0\\ \Leftrightarrow\ &C_{s,t,-l-(s-t)}C_{p,q,-l-(p-q)-(s-t)}-C_{p+s,q+t,-l-(p+s)+(q+t)}=0\\ \Leftrightarrow\ &C_{s,t,l}C_{p,q,l+s-t}-C_{p+s,q+t,l}=0\\ \Leftrightarrow\ &(T_{\xi^p\bar\xi^q\varphi_1}T_{\xi^s\bar\xi^t\varphi_2}-T_{\xi^{p+s}\bar\xi^{q+t}\psi})(r^{|l|}\xi^l)=0.\end{aligned}$$

If $l+s-t\notin\mathbf{N}^n\cup(-\mathbf{N})^n$, then Lemma 2.2 implies

$$\begin{aligned}&(T_{\xi^s\bar\xi^t\varphi_2}T_{\xi^p\bar\xi^q\varphi_1}-T_{\xi^{p+s}\bar\xi^{q+t}\psi})(r^{|-l-(p-q)-(s-t)|}\xi^{-l-(p-q)-(s-t)})=0\\ \Leftrightarrow\ &C_{p+s,q+t,-l-(p+s)+(q+t)}=0\Leftrightarrow C_{p+s,q+t,l}=0\\ \Leftrightarrow\ &(T_{\xi^p\bar\xi^q\varphi_1}T_{\xi^s\bar\xi^t\varphi_2}-T_{\xi^{p+s}\bar\xi^{q+t}\psi})(r^{|l|}\xi^l)=0.\end{aligned}$$

The result follows.

3 Monomial-type Toeplitz operators

In this section we will show when the commutators and semi-commutators of two monomial-type Toeplitz operators on b_α^2 have finite rank. Clearly, the symmetries in previous section are very useful for the study of the rank of the commutator and semi-commutator of two k-quasi-homogeneous Toeplitz operators on b_α^2.

Theorem 3.1 Let $l,k\in\mathbf{R}_+$, $p,q,s,t\in\mathbf{N}^n$. Then the following statements are equivalent for Toeplitz operators on b_α^2.

(a) The commutator $[T_{r^l\xi^p\bar\xi^q}, T_{r^k\xi^s\bar\xi^t}]$ has finite rank.

(b) The following two conditions hold.

(b1) For each $i\in\{1,\cdots,n\}$, at least one of the following conditions holds:

(i) $p_i=q_i=0$,

(ii) $s_i=t_i=0$,

(iii) $p_i=s_i=0$,

(iv) $q_i=t_i=0$,

(v) $p_i=q_i$ and $s_i=t_i$,

(vi) $p_i=s_i$ and $q_i=t_i$;

(b2) There exist some real numbers μ,ν and $a\geqslant\mu/2, b\geqslant\nu/2$ such that

$$\frac{\Gamma(\eta+a+\alpha+1)\Gamma(\eta+b+\mu+\alpha+1)\Gamma(\eta+b)\Gamma(\eta+a+\nu)}{\Gamma(\eta+a)\Gamma(\eta+b+\mu)\Gamma(\eta+b+\alpha+1)\Gamma(\eta+a+\nu+\alpha+1)}=\frac{\Gamma(\eta+|s|)\Gamma(\eta+\mu+\alpha+1)\Gamma(\eta+\nu+|p|)}{\Gamma(\eta+|p|)\Gamma(\eta+\nu+\alpha+1)\Gamma(\eta+\mu+|s|)}\quad(5)$$

for any $\eta\in\mathbf{C}$ on some right half-plane.

Proof Observe that $r^l\xi^p\bar\xi^q$ is a k-quasi-homogeneous function with $k=(n)$. Thus for any $\lambda\in\mathbf{N}^n$, (1) becomes

$$\widehat{r^l\circ\kappa}(\lambda)=\int_0^1(1-|r|^2)^{\alpha}r^{|\lambda|+l+2n-1}\mathrm{d}r=$$

$$\frac{\Gamma(\alpha+1)\Gamma(n+\frac{|\lambda|+l}{2})}{2\Gamma(n+\frac{|\lambda|+l}{2}+\alpha+1)}\tag{6}$$

To prove that (a) implies (b), we simply write

$\mu=|p|-|q|,\nu=|s|-|t|$,

$$a=\frac{l}{2}+\frac{|p|}{2}-\frac{|q|}{2},$$

$$b=\frac{k}{2}+\frac{|s|}{2}-\frac{|t|}{2},$$

and define

$$H_{p,q,a}(\zeta)=\frac{\Gamma(\sum_{i=1}^n\zeta_i+\mu+n+\alpha+1)\Gamma(\sum_{i=1}^n\zeta_i+a+n)\prod_{i=1}^n\Gamma(\zeta_i+p_i+1)}{\Gamma(\sum_{i=1}^n\zeta_i+|p|+n)\Gamma(\sum_{i=1}^n\zeta_i+a+n+\alpha+1)\prod_{i=1}^n\Gamma(\zeta_i+p_i-q_i+1)}\tag{7}$$

Then, for each $\beta\in\mathbf{N}^n$, with $\beta\geqslant\gamma$, where

$\gamma_i=\max\{0,-p_i+q_i,-s_i+t_i,-p_i+q_i-s_i+t_i\}$

for each $i\in\{1,\cdots,n\}$. It follows from (6) and Lemma 2.1 that

$$[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](z^\beta)=[H_{s,t,b}(\beta)H_{p,q,a}(\beta+s-t)-H_{p,q,a}(\beta)H_{s,t,b}(\beta+p-q)]z^{\beta+p-q+s-t}.$$

Assume $[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}]$ has finite rank. Then, by the same argument as in the proof of Ref. [13, Theorem 5] we see that

$$H_{s,t,b}(\zeta)H_{p,q,a}(\zeta+s-t)-H_{p,q,a}(\zeta)H_{s,t,b}(\zeta+p-q)=0$$

for all $\zeta\in\mathbf{C}^n$ with Re $\zeta_i\geqslant\gamma_i,1\leqslant i\leqslant n$. Combining this with Ref. [19, Proposition 2.1], we obtain (5) holds on $\{\eta\in\mathbf{C}:\mathrm{Re}\eta\geqslant|\gamma+1|\}$ and

$$\frac{\Gamma(\eta_i+p_i)\Gamma(\eta_i+s_i-t_i)\Gamma(\eta_i+p_i-q_i+s_i)}{\Gamma(\eta_i+s_i)\Gamma(\eta_i+p_i-q_i)\Gamma(\eta_i+s_i-t_i+p_i)}=1$$

on $\{\eta_i\in\mathbf{C}:\mathrm{Re}\ \eta_i\geqslant\gamma_i+1\}$ for each $i\in\{1,\cdots,n\}$, which implies that (b1) holds. This completes the proof that (a) implies (b).

Next, we assume that Condition (b) holds. If $\beta\in\mathbf{N}^n$ and $\beta\geqslant\gamma$, it is clear that

$$[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](z^\beta)=0\tag{8}$$

If $\beta\in\mathbf{N}^n$ and $\beta\geqslant-\gamma'$, where $r_i{}'=\min\{0,-p_i+q_i,-s_i+t_i,-p_i+q_i-s_i+t_i\}$ for each $i\in\{1,\cdots,n\}$, then $\gamma'=-\gamma-(p-q)-(s-t)$ and $\beta-(p-q)-(s-t)\geqslant\gamma$. Therefore, from Condition (b) of Proposition 2.3 and (8) we obtain

$$[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](\bar{z}^\beta)=[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](r^{|-\beta|}\xi^{-\beta})=$$

$$[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](r^{|\beta-(p-q)-(s-t)|}\xi^{\beta-(p-q)-(s-t)})=0.$$

If $\beta\in\mathbf{N}^n$, $\beta\ngeqq\gamma$, and $\beta\nleqq\gamma$, then $\beta_i<\gamma_i$ and $\beta_j>\gamma_j$ for some $i,j\in\{1,\cdots,n\}$. From Condition (a) of Ref. [13, Lemma 1] and the definition of γ_j we deduce that

$$\beta_i<-p_i+q_i-s_i+t_i,\beta_j>-p_j+q_j-s_j+t_j.$$

Consequently, $-\beta-(p-q)-(s-t)\notin\mathbf{N}^n\cup(-\mathbf{N})^n$. Thus Condition (a) of Proposition 2.3 implies $[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](z^\beta)=0$. If $\beta\in\mathbf{N}^n$, $\beta\ngeqq\gamma'$, and $\beta\nleqq\gamma'$, then $\beta_i<-\gamma_i{}'$ and $\beta_j>-\gamma_j{}'$ for some $i,j\in\{1,\cdots,n\}$. It follows from Condition (b) of Ref. [13, Lemma 1] and the definition of $\gamma_j{}'$ that

$$\beta_i<p_i-q_i+s_i-t_i,\beta_j>p_j-q_j+s_j-t_j,$$

which gives $[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](\bar{z}^\beta)=0$.

If $\frac{-(p-q)-(s-t)}{2}\in\mathbf{N}^n\cup(-\mathbf{N})^n$, then Condition (c) of Proposition 2.3 shows that

$$[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](r^{|\frac{-|p-q|-|s-t|}{2}|}\xi^{\frac{-|p-q|-|s-t|}{2}})=0.$$

Thus we arrive at the conclusion that

$$\mathrm{Ran}[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}]=$$

$$\mathrm{Span}\{[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}](r^{|l|}\xi^l):$$

$$l\in\mathbf{N}^n\cup(-\mathbf{N})^n\backslash\frac{-p+q-s+t}{2}\},\gamma'\nleqq l\nleqq\gamma\}.$$

This together with Lemma 2.2 yields that the commutator $[T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}]$ has finite rank, and hence condition (a) holds.

Theorem 3.2 Let $l,k\in\mathbf{R}_+$, $p,q,s,t\in\mathbf{N}^n$. Then the following statements are equivalent for Toeplitz operators on b_α^2.

(a) The semi-commutator$(T_{r^l\xi^p\bar{\xi}^q},T_{r^k\xi^s\bar{\xi}^t}]$has finite rank k.

(b) The semi-commutator$(T_{r^k\xi^s\bar{\xi}^t},T_{r^l\xi^p\bar{\xi}^q}]$ has finite rank k.

(c) The following two conditions hold:

(c1) $p_i=q_i=0,s_i=t_i=0,p_i=s_i=0$, or $q_i=t_i$

$=0$ for all $1\leqslant i\leqslant n$;

(c2) There exist some real numbers μ, ν and $a\geqslant\frac{\mu}{2}, b\geqslant\frac{\nu}{2}$ such that both (5) and

$$\frac{\Gamma(\eta+b)\Gamma(\eta+\nu+\alpha+1)\Gamma(\eta+a+\nu)}{\Gamma(\eta+|s|)\Gamma(\eta+b+\alpha+1)\Gamma(\eta+\nu+|p|)\Gamma(\eta+a+\nu+\alpha+1)}=\frac{\Gamma(\eta+a+b)}{\Gamma(\eta+|p|+|s|)\Gamma(\eta+a+b+\alpha+1)} \tag{9}$$

hold for any $\eta\in\mathbf{C}$ on some right half-plane.

Proof It is obvious from Proposition 2.4 that (a) is equivalent to (b).

To prove that (a) implies (c), we consider each $\beta\in\mathbf{N}^n$ with $\beta\geqslant\delta$, where

$$\delta_i=\max\{0,-s_i+t_i,-p_i+q_i-s_i+t_i\}.$$

Then we deduce from Lemma 2.1 and the notation of (7) that

$$[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}](z^\beta)=[H_{s,t,b}(\beta)H_{p,q,a}(\beta+s-t)-H_{p+s,q+t,a+b}(\beta)](z^{\beta+p-q+s-t}).$$

By the same argument as in the proof of Theorem 3.1, we get (9) holds on $\{\eta\in\boldsymbol{C}:\boldsymbol{Re}\eta\geqslant|\delta+1|\}$ and

$$\frac{\Gamma(\eta_i+s_i-t_i)\Gamma(\eta_i+p_i+s_i)}{\Gamma(\eta_i+s_i)\Gamma(\eta_i+s_i-t_i+p_i)}=1$$

on $\{\eta_i\in\mathbf{C}:\mathrm{Re}\,\eta_i\geqslant\delta_i+1\}$ for each $i\in\{1,\cdots,n\}$, which implies either $p_i=0$ or $t_i=0$

$$\frac{\Gamma(\eta+b)\Gamma(\eta+\nu+\alpha+1)\Gamma(\eta+a+\nu)}{\Gamma(\eta+|s|)\Gamma(\eta+b+\alpha+1)\Gamma(\eta+\nu+|p|)\Gamma(\eta+a+\nu+\alpha+1)}=\frac{\Gamma(\eta+a+b)}{\Gamma(\eta+|p|+|s|)\Gamma(\eta+a+b+\alpha+1)}.$$

Moreover, it follows from Ref. [13, Corollary 14] that the commutator $[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}]$ also has finite rank. Then, by Theorem 3.1, we have that Condition (c) holds. Thus Condition (a) implies (c).

Conversely, if Condition (c) holds, then for each $\beta\in\mathbf{N}^n$ with $\beta\geqslant\delta$, it is clear that

$$[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}](z^\beta)=0 \tag{10}$$

Next, we consider $[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}](\bar{z}^\beta)$ for each $\beta\in\mathbf{N}^n$ with $\beta\geqslant-\delta'$, where $\delta_i'=\min\{0,-s_i+t_i,-p_i+q_i-s_i+t_i\}$ for each $i\in\{1,\cdots,n\}$. Obviously, $\beta-(p-q)-(s-t)\geqslant\delta''$ with $\delta_i''=\max\{0,-p_i+q_i,-p_i+q_i-s_i+t_i\}$. Since Condition (c1) holds, it is easy to check that

$$\gamma=\delta=\delta''=\max\{0,-p_i+q_i-s_i+t_i\} \tag{11}$$

Combining this with Condition (c), we obtain

$$T_{r^l\xi^p\xi^q}T_{r^k\xi^s\xi^t}(z^{\beta-(p-q)-(s-t)})=T_{r^k\xi^s\xi^t}T_{r^l\xi^p\xi^q}(z^{\beta-(p-q)-(s-t)}).$$

This together with (10) implies that $[T_{r^k\xi^s\xi^t},T_{r^l\xi^p\xi^q}](z^{\beta-(p-q)-(s-t)})=0$ for each $\beta\in\mathbf{N}^n$ with $\beta\geqslant-\delta'$. Then from Condition (b) of Proposition 2.4 we deduce that

$$\begin{aligned}[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}](\bar{z}^\beta)&=[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}](r^{|-\beta|}\xi^{-\beta})\\&=[T_{r^k\xi^s\xi^t},T_{r^l\xi^p\xi^q}](r^{|\beta-(p-q)-(s-t)|}\xi^{\beta-(p-q)-(s-t)})=0.\end{aligned}$$

Now consider $\beta\in\mathbf{N}^n$ with $\beta\ngeqq\delta$ and $\beta\nleqq\delta$. Then $\beta_i<\delta_i$ and $\beta_j>\delta_j$ for some $i,j\in\{1,\cdots,n\}$. From (11) and the definition of δ_j we deduce that

$$\beta_i<-p_i+q_i-s_i+t_i,\beta_j>-p_j+q_j-s_j+t_j.$$

By Condition (a) of Proposition 2.4, we have $[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}](z^\beta)=0$. Finally, if $\beta\in\mathbf{N}^n$ with $\beta\ngeq\delta'$ and $\beta\nleq\delta'$, a similar argument shows that $[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}](\bar{z}^\beta)=0$. Therefore, we conclude that

$$\begin{aligned}\mathrm{Ran}[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}]=&\\ \mathrm{Span}\{&[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}](r^{|l|}\xi^l):\\ &l\in\mathbf{N}^n\cup(-\mathbf{N})^n\},\delta'\nleqq l\nleqq\delta\}\end{aligned}$$

which shows that $[T_{r^l\xi^p\xi^q},T_{r^k\xi^s\xi^t}]$ has finite rank and Condition (a) holds.

Remark Jiang, Zhou and the second author completely characterize the finite rank commutator and semi-commutator of two monomial-type Toeplitz operators on the un-weighted Bergman space of certain weakly pseudoconvex domains in Ref. [19]. The case $\alpha=0$ in (5) is just the same as Ref. [19, Equation (1.1)], which have many nontrivial solutions. Thus that there exist too many cases for the tuple $(|p|,|s|,\mu,\nu,a,b)$ satisfying (5) or (9). In this case, $|q|=|p|-\mu$, $|t|=|s|-\nu$, $l=2a-\mu$ and $k=2b-\nu$.

As a direct consequence of Theorems 3.1 and 3.2, one can easily get the corresponding results for monomial Toeplitz operators in Ref. [13]. It is also worth to mention that the technique and discussion here are different, which simplify the proof even though the symbols seems complicated than the monomial.

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