含快变时滞的格 FitzHugh-Nagumo 系统的拉回吸引子

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摘　要: 本文研究具有快时滞影响的格 FitzHugh-Nagumo 方程的动力学行为,证明了拉回吸引子的存在和唯一性. 一般来说,研究时滞方程吸引子要求时滞项的导数小于 1(慢时滞),本文则使用差分不等式技术消除了这个约束. 因而本文的方法可被用于处理具有快变延迟的方程.

关键词: 全局吸引子; 格; FitzHugh-Nagumo 系统; 快变时滞
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Pullback attractors for lattice FitzHugh-Nagumo systems with fast-varying delays

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Abstract: We investigate the dynamical behavior of lattice FitzHugh-Nagumo equations with fast-varying delays and prove the existence and uniqueness of pullback attractor for the equations. Generally, studying the attractors of time-varying delay equations require that the derivative of the delay term is less than 1 (slow-varying delay). In this paper, by using some differential inequality techniques, we remove this constraint. Thus our method can be used to deal with equations with fast-varying delays.

Keywords: Global attractor; Lattice; FitzHugh-Nagumo system; Fast-varying delay
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1 Introduction

Lattice differential equations have many applications where the spatial structure has a discrete character. Wang *et al.*[1] used the idea of 'tail ends' estimates on solutions and obtained a result concerning the existence of a global attractor for a class of reaction-diffusion lattice systems. Later on, their results were extended to various problems, see for instance, Refs. [2-11]. The FitzHugh-Nagumo system arises as a model describing the signal transmission across axons in neurobiology[12]. The asymptotic behavior of a FitzHugh-Nagumo system was investigated in Refs. [13-15]. The results were extended to stochastic, see for instance Refs. [16-17]. Since time-delays are frequently encountered in many practical systems, which may induce instability, oscillation and poor performance of systems, delay lattice systems then arise naturally while these

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delays are taken into account. Recently, attractors of delay lattice systems have been considered in Refs. [18-24]. The existing results of studying attractors for time-varying delay equations require that the derivative of the delay term be less than 1 (slow-varying delay). By using differential inequality technique, our results remove the constraints on the delay derivative. So we can deal with the lattice FitzHugh-Nagumo systems with fast-varying delays (without any constraints on the delay derivative).

Motivated by the discussions above, we study the dynamical behavior of the following lattice FitzHugh-Nagumo system with fast-varying delays: for $\tau\in\mathbf{R}$ and $i\in\mathbf{Z}$,

$$\frac{\mathrm{d}u_i}{\mathrm{d}t}+\nu(2u_i-u_{i+1}-u_{i-1})+\lambda u_i=$$

$$h_i(u_i(t-\rho_0(t)))-\alpha v_i+f_i(t),\ t>\tau \tag{1}$$

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}=-\delta v_i+\beta u_i+g_i(t),t>\tau \tag{2}$$

with the initial condition

$$u_i(\tau+s)=\varphi_i(s),v_i(\tau)=\varphi_i,s\in[-\rho,\ 0] \tag{3}$$

where u_i, v_i is the unknow value function, ν, λ, α, δ, β, ρ are positive real constants, $\rho_0\in C(\mathbf{R},\ [0,\ \rho])$is an adequate given delay function $f(t)=(f_i(t))_{i\in\mathbf{Z}}\in L^2_{loc}(\mathbf{R},\ l^2)$ and $g(t)=(g_i(t))_{i\in\mathbf{Z}}\in L^2_{loc}(\mathbf{R},\ l^2)$ (l^2 is defined later) are given time dependent sequences, h_i is a nonlinear function satisfying certain conditions, $\varphi_i\in C(\mathbf{R},\ [0,\ \rho])$ and$\varphi_i\in\mathbf{R}$.

This paper is organized as follows. In Section 2, we prove that the lattice system (1)-(3) generates a non-autonomous dynamical system. In Section 3, we derive a priori estimates on the solutions to (1)-(3). In Section 4, we proof the existence and uniqueness of pull-back attractor for the lattice systems.

2 Priori estimates

In this section, we establish the existence of a continuous non-autonomous dynamical system generated by System (1)-(3) and derive some priori estimates which will be needed for proofing the existence of a global attractor. We formulate System (1)-(3) as an abstract ordinary differential equation. To this end, we denote by l^2 the Hilbert space defined by

$$l^2=\{u=(u_i)_{i\in z}:\sum_{i\in\mathbf{Z}}u_i^2<+\infty\}$$

with the norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ given by $\|u\|=(\sum_{i\in\mathbf{Z}}u_i^2)^{\frac{1}{2}}$, $(u,v)=\sum_{i\in\mathbf{Z}}u_iv_i$ for each $u=(u_i)_{i\in z}\in l^2$, $v=(v_i)_{i\in z}\in l^2$. Define the linear operators $A,B,B^*:l^2\to l^2$ as

$$(Bu)_i=u_{i+1}-u_i,$$

$$(B^*u)_i=u_{i-1}-u_i,$$

$$(Au)_i=-u_{i-1}+2u_i-u_{i+1},$$

for each $i\in\mathbf{Z}$. Then

$$A=BB^*=B^*B,$$

$$(B^*u,v)=(u,Bv),u,v\in l^2.$$

Denote

$$\varphi(s)=\{\varphi_i(s)\}_{i\in\mathbf{Z}},s\in[-\rho,0]$$

and $\varphi=\{\varphi_i\}_{i\in\mathbf{Z}}$. Denote by u_t the function defined on $[-\rho,0]$ according to the relation

$$u_t(s)=(u_{it}(s))_{i\in\mathbf{Z}}=(u_i(t+s))_{i\in\mathbf{Z}}=$$

$$u(t+s),s\in[-\rho,0],$$

and let $C_\rho=C([-\rho,\ 0],\ l^2)$ with the maximum norm

$$\|\psi\|_\rho=\sup_{-\rho\leqslant s\leqslant 0}\|\psi(s)\|,\psi\in C_\rho.$$

Then System(1)-(3) can be rewritten as

$$\frac{\mathrm{d}u}{\mathrm{d}t}+\nu Au+\lambda u=h(u(t-\rho_0(t)))-\alpha v+f(t),$$

$$t>\tau \tag{4}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-\delta v=\beta u+g(t),\ t>\tau \tag{5}$$

with the initial condition

$$u(\tau+s)=\varphi(s),\ v(\tau)=\varphi,\ s\in[-\rho,0] \tag{6}$$

where $u=(u_i)_{i\in\mathbf{z}}$,

$$h(u(t-\rho_0(t)))=h_i\ (u_i(t-\rho_0(t)))_{i\in\mathbf{z}},$$

$$f(t)=(f_i(t))_{i\in\mathbf{z}},g(t)=(g_i(t))_{i\in\mathbf{z}},$$

$\varphi=(\varphi_i)_{i\in\mathbf{z}}$ and $\varphi=(\varphi_i)_{i\in\mathbf{z}}$. We make the following assumptions on h_i, $i\in\mathbf{Z}$. For each $i\in\mathbf{Z}$, h_i is a nonlinear function satisfying the following assumption:

(H) $h_i(0)=0$ and h_i is Lipschitz continuous uniformly with respect to i, that is, there is a positive constant L, independent of i, such that for all $s_1,s_2\in\mathbf{R}$,

$$|h_i(s_1)-h_i(s_2)|\leqslant L|s_1-s_2|.$$

In fact, by (H) we find that

$$\| h(u)-h(v) \| \leqslant L \| u-v \|, \ u,v\in l^2.$$

Then it follows from the standard theory of ordinary differential equations that there exists a unique local solution (u,v) for System (4)-(6). The following estimates imply that the local solution is actually defined globally. In the sequence, we assume that

$$\eta=\frac{2L^2}{\sigma\lambda}<1 \tag{7}$$

Lemma 2.1 Assume that (H) and (7) hold. Then for every $\tau\in\mathbf{R}, T>0, \varphi\in C_\rho$ and $\varphi\in l^2$, there exists a positive constant $c=c(\tau,T,\varphi,\varphi)$ such that the solution (u,v) of Problem (4)-(6) satisfies

$$\beta \| u_t \|_\rho^2+\alpha \| v(t) \|^2\leqslant c, t\in[\tau,\tau+T) \tag{8}$$

Proof Taking the inner product of (4) with βu in l^2, we find that

$$\frac{1}{2}\beta\frac{\mathrm{d}}{\mathrm{d}t}\| u \|^2+\beta\nu \| Bu \|^2+\beta\lambda \| u \|^2= \beta(h(u(t-\rho_0(t))),u)-\beta\alpha(u,v)+\beta(u,f(t)) \tag{9}$$

Taking the inner product of (5) with $\alpha\nu$ in l^2, we get that

$$\frac{1}{2}\alpha\frac{\mathrm{d}}{\mathrm{d}t}\| v \|^2=-\alpha\delta \| v \|^2+\beta\alpha(u,v)+\alpha(v,g(t)) \tag{10}$$

Summing up (9) and (10), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\beta \| u \|^2+\alpha \| v \|^2)+\beta\nu \| Bu \|^2+\beta\lambda \| u \|^2+\alpha\delta \| v \|^2=\beta(h(u(t-\rho_0(t))),u)+\beta(u,f(t))+\alpha(v,g(t)) \tag{11}$$

We now estimate the right-hand side of (11). The first term is bounded by

$$|\beta(h(u(t-\rho_0(t))),u)|\leqslant \beta\| h(u(t-\rho_0(t))) \| \| u \|\leqslant \frac{1}{4}\beta\lambda \| u \|^2+\frac{\beta}{\lambda}\| h(u(t-\rho_0(t))) \|^2\leqslant \frac{1}{4}\beta\lambda \| u \|^2+\frac{\beta L^2}{\lambda}\| u(t-\rho_0(t) \|^2 \tag{12}$$

For the left two term on the right-hand side of (11), we have

$$\beta(u,f(t))+\alpha(v,g(t))\leqslant\frac{1}{4}\beta\lambda \| u \|^2+\frac{\beta}{\lambda}\| f(t) \|^2+\frac{1}{2}\alpha\delta \| v \|^2+\frac{\alpha}{2\delta}\| g(t) \|^2 \tag{13}$$

By (11)-(13) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\beta \| u \|^2+\alpha \| v \|^2)\leqslant -(\beta\lambda \| u \|^2+\alpha\delta \| v \|^2)+\frac{2\beta L^2}{\lambda}\| u(t-\rho_0(t) \|^2+\frac{2\beta}{\lambda}\| f(t) \|^2+\frac{\alpha}{\delta}\| g(t) \|^2 \tag{14}$$

Let $\sigma=\min\{\lambda, \delta\}$. Then it follows from (14) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\beta \| u \|^2+\alpha \| v \|^2)\leqslant -\sigma(\beta \| u \|^2+\alpha \| v \|^2)+\frac{2L^2}{\lambda}\beta \| u(t-\rho_0(t) \|^2+\frac{2\beta}{\lambda}\| f(t) \|^2+\frac{\alpha}{\delta}\| g(t) \|^2 \tag{15}$$

By Gronwall inequality, that for $t\geqslant\tau$, we have

$$\beta \| u(t) \|^2+\alpha \| v(t) \|^2\leqslant \mathrm{e}^{-\sigma(t-\tau)}(\beta \| \varphi(0) \|^2+\alpha \| \varphi \|^2)+\frac{2L^2}{\lambda}\int_\tau^t \mathrm{e}^{-\sigma(t-s)}\beta \| u(s-\rho_0(s) \|^2\mathrm{d}s+\frac{2\beta}{\lambda}\int_\tau^t \mathrm{e}^{-\sigma(t-s)} \| f(s) \|^2\mathrm{d}s+\frac{\alpha}{\delta}\int_\tau^t e^{-\sigma(t-s)} \| g(s) \|^2\mathrm{d}s \tag{16}$$

From the condition (7), by using continuity, we obtain that there exist positive constants $\mu<\sigma$ and N such that $\| \varphi \|_\rho+\| \varphi \|\leqslant N$ and

$$\frac{\| \varphi \|_\rho^2+\| \varphi \|^2}{N}+\mathrm{e}^{\mu\rho}\frac{L^2}{(\sigma-\mu)\lambda}<1 \tag{17}$$

hold. Then we prove that for $t\geqslant\tau$

$$\beta \| u(t) \|^2+\alpha \| v(t) \|^2\leqslant dN\mathrm{e}^{-\mu(t-\tau)}+(1-\eta)^{-1}I(t) \tag{18}$$

where

$$I(t)=\max_{\tau\leqslant\xi\leqslant t}\left(\frac{2\beta}{\lambda}\int_\tau^\xi \mathrm{e}^{-\sigma(\xi-s)} \| f(s) \|^2\mathrm{d}s+\frac{\alpha}{\delta}\int_\tau^\xi \mathrm{e}^{-\sigma(\xi-s)} \| g(s) \|^2\mathrm{d}s\right).$$

To this end, we first prove for any $d>1$,

$$\beta \| u(t) \|^2+\alpha \| v(t) \|^2< dN\mathrm{e}^{-\mu(t-\tau)}+(1-\eta)^{-1}I(t),\ t\geqslant\tau \tag{19}$$

If (19) is not true, then, from $\| \varphi \|_\rho+\| \varphi \|\leqslant N$ and $\| u(t) \|$ and $\| v(t) \|$ are continuous, there must be a $t^*>\tau$ such that

$$\beta \| u(t^*) \|^2+\alpha \| v(t^*) \|^2\geqslant dN\mathrm{e}^{-\mu(t^*-\tau)}+(1-\eta)^{-1}I(t^*) \tag{20}$$

and

$$\beta\|u(t)\|< dN\mathrm{e}^{-\mu(t-\tau)}+(1-\eta)^{-1}I(t),\tau-\rho\leqslant t<t^* \tag{21}$$

Hence, it follows from (16) (17) (20) and (21) that

$$\begin{aligned}
&\beta\|u(t^*)\|^2+\alpha\|v(t^*)\|^2\leqslant \\
&\mathrm{e}^{-\sigma(t^*-\tau)}(\beta\|\varphi(0)\|^2+\alpha\|\varphi\|^2)+ \\
&\frac{2L^2}{\lambda}\int_\tau^{t^*}\mathrm{e}^{-\sigma(t^*-s)}\beta\|u(s-\rho_0(s)\|^2\mathrm{d}s+ \\
&\frac{2\beta}{\lambda}\int_\tau^{t^*}\mathrm{e}^{-\sigma(t^*-s)}\|f(s)\|^2\mathrm{d}s+ \\
&\frac{\alpha}{\delta}\int_\tau^{t^*}\mathrm{e}^{-\sigma(t^*-s)}\|g(s)\|^2\mathrm{d}s< \\
&\mathrm{e}^{-\mu(t^*-\tau)}(\beta\|\varphi(0)\|^2+\alpha\|\varphi\|^2)+ \\
&\frac{2L^2}{\lambda}\int_\tau^{t^*}\mathrm{e}^{-\sigma(t^*-s)}(dN\mathrm{e}^{\mu\rho}\mathrm{e}^{-\mu(s-\tau)}+ \\
&(1-\eta)^{-1}I(t^*))\mathrm{d}s+ \\
&\frac{2\beta}{\lambda}\int_\tau^{t^*}\mathrm{e}^{-\sigma(t^*-s)}\|f(s)\|^2\mathrm{d}s+ \\
&\frac{\alpha}{\delta}\int_\tau^{t^*}\mathrm{e}^{-\sigma(t^*-s)}\|g(s)\|^2\mathrm{d}s\leqslant \\
&\mathrm{e}^{-\mu(t^*-\tau)}(\beta\|\varphi(0)\|^2+\alpha\|\varphi\|^2)+ \\
&\frac{2L^2}{\lambda}\int_\tau^{t^*}\mathrm{e}^{-\sigma(t^*-s)}dN\mathrm{e}^{\mu\rho}\mathrm{e}^{-\mu(s-\tau)}\mathrm{d}s+ \\
&\frac{2L^2}{\lambda}(1-\eta)^{-1}I(t^*)\int_\tau^{t^*}\mathrm{e}^{-\sigma(t^*-s)}\mathrm{d}s+I(t^*)\leqslant \\
&\Big[\frac{\beta\|\varphi(0)\|^2+\alpha\|\varphi\|^2}{N}+ \\
&\frac{2L^2}{\lambda}\mathrm{e}^{\mu\rho}\int_\tau^{t^*}\mathrm{e}^{-\sigma(\sigma-\mu)(t^*-s)}\mathrm{d}s\Big]dN\mathrm{e}^{-\mu(t^*-\tau)}+ \\
&\eta(1-\eta)^{-1}I(t^*)+I(t^*)\leqslant \\
&\Big[\frac{\beta\|\varphi(0)\|^2+\alpha\|\varphi\|^2}{N}+ \\
&\mathrm{e}^{\mu\rho}\frac{2L^2}{\lambda(\sigma-\mu)}\Big]dN\mathrm{e}^{-\mu(t^*-\tau)}+(1-\eta)^{-1}I(t^*)\leqslant \\
&dN\mathrm{e}^{-\mu(t^*-\tau)}+(1-\eta)^{-1}I(t^*)
\end{aligned}\tag{22}$$

which contradicts inequality (20). So inequality (19) holds for all $t\geqslant\tau$. Letting $d\to 1$ in inequality (19), we have inequality (18). The proof is complete.

Lemma 2.1 implies that the solution u is defined in any interval of $[\tau,T+\tau)$ for any $T>0$. It means that this local solution is, in fact, a global one.

Given $t\in\mathbf{R}$, define a translation θ_t on $\mathbf{R}$ by

$$\theta_t(\tau)=\tau+t,\ \tau\in\mathbf{R} \tag{23}$$

Then $\{\theta_t\}_{t\in\mathbf{R}}$ is a group acting on $\mathbf{R}$.

We now define amapping $\Phi:\mathbf{R}^+\times\mathbf{R}\times X_\rho\to X_\rho$, for Problem (4)-(6), where $X_\rho=C_\rho\times l^2$. Given $t\in\mathbf{R}^+$, $\tau\in\mathbf{R}$ and $\Psi_\tau=(u_\tau,v_\tau)\in X_\rho$, let

$$\Phi(t,\tau,\Psi_\tau)=(u_{t+\tau}(\cdot,\tau,u_\tau), v(t+\tau,\tau,v_\tau)) \tag{24}$$

where $u_{t+\tau}(s,\tau,u_\tau)=u(t+\tau+s,\tau,u_\tau)$, $s\in[-\rho,0]$. By the uniqueness of solutions, we find that for every $t,s\in\mathbf{R}^+$ and $\tau\in\mathbf{R}$ and $\Psi_\tau\in X_\rho$,

$$\Phi(t+s,\tau,\Psi_\tau)=\Phi(t,s+\tau,(\Phi(s,\tau,\Psi_\tau))).$$

Then we see that Φ is a continuous non-autonomous dynamical system on X_ρ.

In the following two sections, we will investigate the existence of a pullback attractor for Φ. To this end, we need to define an appropriate collection of families of subsets of X_ρ. Let $B_\rho=\{B_\rho(\tau):\tau\in\mathbf{R}\}$ be a family of nonempty subsets of X_ρ. Then B_ρ is called tempered (or subexponentially growing) if for every $c>0$, the following holds:

$$\lim_{t\to-\infty}\mathrm{e}^{ct}\|B_\rho(\tau+t)\|_{X_\rho}=0,$$

where $x=(\varphi,\varphi)$. In the sequel, we denote by D_ρ the collection of all families of tempered nonempty subsets of X_ρ, i.e.,

$$D_\rho=\{B_\rho=\{B_\rho(\tau):\tau\in\mathbf{R}\}:B_\rho \text{ is tempered}\}.$$

From the condition (7), by using continuity, we obtain that there exists a positive constant $\mu<\sigma$ such that

$$\mu-\sigma+\frac{2L^2}{\lambda}\mathrm{e}^{\mu\rho}<0 \tag{25}$$

holds. The following condition will be needed when deriving uniform estimates of solutions:

$$\int_{-\infty}^\tau\mathrm{e}^{\mu s}(\|f(s)\|^2+\|g(s)\|^2)\mathrm{d}s<\infty,\ \forall\tau\in\mathbf{R} \tag{26}$$

3 Uniform estimates of the solutions

In this section, we derive uniform estimates of solutions of Problem(4)~(6) which are needed for proving the existence and uniqueness of pullback attractor for Problem (4)~(6).

The estimates of solutions of Problem (4)~(6) in X_ρ are provided below. The symbol c is a positive constant which may change its value from line to line.

Lemma 3.1　Assume that (H), (7) and (26) hold. Then for every $\tau\in\mathbf{R}$ and $D_\rho=\{D_\rho(\tau):\tau\in\mathbf{R}\}\in D_\rho$, there exists $T=T(\tau, D_\rho)>\rho$ such that for all $t\geqslant T$ and $(\varphi,\varphi)\in D_\rho(\tau-t)$, the solution (u, v) of (4)-(6) satisfies

$$\|u_\tau(\cdot,\tau-t,\varphi), v(\tau,\tau-t,\varphi)\|^2_{X_\rho}\leqslant 2\frac{2\beta}{\chi\lambda}e^{\lambda\rho}\int_{-\infty}^{0}e^{\lambda s}\|f(s+\tau)\|ds+2\frac{\alpha}{\chi\delta}e^{\lambda\rho}\int_{-\infty}^{0}e^{\lambda s}\|g(s+\tau)\|ds\tag{27}$$

where $\chi=\min\{\alpha,\beta\}$.

Proof　Replacing t and τ in (15) by $\tilde\omega$ and $\tau-t$, respectively, we have for $\tilde\omega>\tau-t$,

$$\frac{d}{dt}(\beta\|u(\tilde\omega,\tau-t,\varphi)\|^2+\alpha\|v(\tilde\omega,\tau-t,\varphi)\|^2)\leqslant -\sigma(\beta\|u(\tilde\omega,\tau-t,\varphi)\|^2+\alpha\|v(\tilde\omega,\tau-t,\varphi)\|^2)+\frac{2L^2}{\lambda}\beta\|u(\tilde\omega-\rho(\tilde\omega),\tau-t,\varphi\|^2+\frac{2\beta}{\lambda}\|f(\tilde\omega)\|^2+\frac{\alpha}{\delta}\|g(\tilde\omega)\|^2\tag{28}$$

For simplicity, we denote $u(\tilde\omega)=u(\tilde\omega,\tau-t,\varphi)$ and $v(\tilde\omega)=v(\tilde\omega,\tau-t,\varphi)$. Then, let us define functions

$$V(\tilde\omega)=e^{\mu\tilde\omega}(\beta\|u(\tilde\omega)\|^2+\alpha\|v(\tilde\omega)\|^2),\quad \tilde\omega\geqslant\tau-t-\rho,$$

where $v(\tilde\omega)=0$, $\tilde\omega\in[\tau-t-\rho,\tau-t)$, and

$$U(\tilde\omega)\triangleq\begin{cases}e^{\mu(\tau-t)}(\beta\|\varphi\|_\rho+\alpha\|\varphi\|),\\ \quad\tilde\omega\in[\tau-t-\rho,\tau-t)\\ e^{\mu(\tau-t)}(\beta\|\varphi\|_\rho+\alpha\|\varphi\|)+\\ \quad\dfrac{2\beta}{\lambda}\displaystyle\int_{\tau-t}^{\tilde\omega}e^{\mu s}\|f(s)\|^2ds+\\ \quad\dfrac{\alpha}{\delta}\displaystyle\int_{\tau-t}^{\tilde\omega}e^{\mu s}\|g(s)\|^2ds,\ \tilde\omega\geqslant\tau-t.\end{cases}$$

Now, we claim that

$$V(\tilde\omega)\leqslant U(\tilde\omega),\ \tilde\omega\geqslant\tau-t\tag{29}$$

If inequality (29) is not true, from the fact that $V(\tilde\omega)$ and $U(\tilde\omega)$ are continuous, then there must be a $\tilde\omega^*>\tau-t$ such that

$$V(\tilde\omega)<U(\tilde\omega),\ \tilde\omega\in[\tau-t-\rho,\tilde\omega^*)\tag{30}$$

$$V(\tilde\omega^*)=U(\tilde\omega^*)\tag{31}$$

where

$$\tilde\omega^*\triangleq\inf\{\tilde\omega>\tau-t\,|\,V(\tilde\omega)>U(\tilde\omega)\},$$

and there is a sufficiently small positive constant $\Delta\tilde\omega$ such that

$$V(\tilde\omega)>U(\tilde\omega),\ \tilde\omega\in(\tilde\omega^*,\tilde\omega^*+\Delta\tilde\omega)\tag{32}$$

Calculating the upper right-hand Dini derivative of $V(\tilde\omega)$ at $\tilde\omega$ and considering (31) and (32), we obtain

$$D^+V(\tilde\omega^*)=\limsup_{h\to0^+}\frac{V(\tilde\omega^*+h)-V(\tilde\omega^*)}{h}\geqslant\limsup_{h\to0^+}\frac{U(\tilde\omega^*+h)-U(\tilde\omega^*)}{h}=\frac{2\beta}{\lambda}e^{\mu\tilde\omega^*}\|f(\tilde\omega^*)\|^2+\frac{\alpha}{\delta}e^{\mu\tilde\omega^*}\|g(\tilde\omega^*)\|^2\tag{33}$$

On the other hand, it follows from (28), we have

$$D^+V(\tilde\omega^*)=\mu e^{\mu\tilde\omega^*}(\beta\|u(\tilde\omega^*)\|^2+\alpha\|v(\tilde\omega^*)\|^2)+e^{\mu\tilde\omega^*}D^+(\beta\|u(\tilde\omega^*)\|^2+\alpha\|v(\tilde\omega^*)\|^2)\leqslant(\mu-\sigma)e^{\mu\tilde\omega^*}(\beta\|u(\tilde\omega^*)\|^2+\alpha\|v(\tilde\omega^*)\|^2)+\frac{2L^2}{\lambda}e^{\mu\tilde\omega^*}\beta\|u(\tilde\omega^*-\rho_0(\tilde\omega^*))\|^2+\frac{2\beta}{\lambda}\|f(\tilde\omega^*)\|^2+\frac{\alpha}{\delta}\|g(\tilde\omega^*)\|^2\tag{34}$$

Noticing that $U(\tilde\omega)$ is monotone nondecreasing on $[\tau-t-\rho,+\infty)$, this, together with (30) and (31), yields

$$V(\tilde\omega^*-\rho_0(\tilde\omega^*))<U(\tilde\omega^*-\rho_0(\tilde\omega^*))<U(\tilde\omega^*)=V(\tilde\omega^*)\tag{35}$$

which implies

$$\beta\|u(\tilde\omega^*-\rho_0(\tilde\omega^*))\|^2\leqslant e^{\mu\rho}(\beta\|u(\tilde\omega^*)\|^2+\alpha\|v(\tilde\omega^*)\|^2)\tag{36}$$

It follows from (25) (34) and (36) that

$$D^+V(\tilde\omega^*)<\left(\mu-\sigma+\frac{2L^2}{\lambda}e^{\mu\rho}\right)V(\tilde\omega^*)+\frac{2\beta}{\lambda}\|f(\tilde\omega^*)\|^2+\frac{\alpha}{\delta}e^{\mu\tilde\omega^*}\|g(\tilde\omega^*)\|^2<\frac{2\beta}{\lambda}e^{\mu\tilde\omega^*}\|f(\tilde\omega^*)\|^2+\frac{\alpha}{\delta}e^{\mu\tilde\omega^*}\|g(\tilde\omega^*)\|^2,$$

which contradicts (33). Until now, (29) has been proven to be true. Thus we get for $t>\rho$ and $-\rho\leqslant\xi\leqslant0$,

$$\beta\|u(\tau+\xi,\tau-t,\varphi)\|^2+\alpha\|v(\tau,\tau-t,\varphi)\|^2\leqslant(\|\varphi\|_\rho^2+\|\varphi\|^2)e^{-\lambda(t+\xi)}+e^{-\lambda(t+\xi)}\frac{2\beta}{\lambda}\int_{\tau-t}^{\tau+\xi}e^{\lambda s}\|f(s)\|^2ds+e^{-\lambda(t+\xi)}\frac{\alpha}{\delta}\int_{\tau-t}^{\tau+\xi}e^{\lambda s}\|g(s)\|^2ds\leqslant$$

$$(\|\varphi\|_\rho^2+\|\varphi\|^2)e^{\lambda\rho}e^{-\lambda t}+ e^{\lambda\rho}e^{-\lambda\tau}\frac{2\beta}{\lambda}\int_{\tau-t}^{\tau}e^{\lambda s}\|f(s)\|^2 ds+ e^{\lambda\rho}e^{-\lambda\tau}\frac{\alpha}{\delta}\int_{\tau-t}^{\tau}e^{\lambda s}\|g(s)\|^2 ds.$$

Since $(\varphi,\varphi)\in D_\rho(\tau-t)\in D_\rho$, we find that for every $\tau\in\mathbf{R}$ and $D_\rho\in D_\rho$, there exists $T=T(\tau, D_\rho)>\rho$ such that for all $t\geqslant T$ and $-\rho\leqslant\xi\leqslant 0$,

$$\beta\|u(\tau+\xi,\tau-t,\varphi)\|^2+\alpha\|v(\tau,\tau-t,\varphi)\|^2\leqslant 2\frac{2\beta}{\lambda}e^{\lambda\rho}\int_{-\infty}^{0}e^{\lambda s}\|f(s+\tau)\|^2 ds+2\frac{\alpha}{\delta}e^{\lambda\rho}\int_{-\infty}^{0}e^{\lambda s}\|g(s+\tau)\|^2 ds$$

This completes the proof.

Lemma 3.2 Assume that (H), (7) and (26) hold. Then for every $\tau\in\mathbf{R}$, $D_\rho=\{D_\rho(\tau):\tau\in\mathbf{R}\}\in D_\rho$ and $\varepsilon>0$, there exist $T=T(\tau, D_\rho,\varepsilon)>\rho$ and $N=N(\tau, D_\rho,\varepsilon)$ such that for all $t\geqslant T$ and $(\varphi,\varphi)\in D_\rho(\tau-t)$, the solution (u, v) of (4)~(6) satisfies

$$\sup_{-\rho\leqslant s\leqslant 0}\sum_{|i|\geqslant N}(|u_i(\tau+s,\tau-t,\varphi)|^2+|v_i(\tau,\tau-t,\varphi)|^2)\leqslant\varepsilon \tag{37}$$

Proof We use an idea of cut-off function to establish the uniform estimates on the tails of the solution. Let θ be a smooth cut-off function satisfying $0\leqslant\theta(s)\leqslant 1$ for $s\geqslant 0$ and $\theta(s)=0$ for $0\leqslant s\leqslant 1$; $\theta(s)=1$ for $s\geqslant 2$. Let k be a fixed integer which will be specified later, and set $\tilde{u}=(\tilde{u}_i)_{i\in\mathbf{Z}}$ with $\tilde{u}=\theta\Big(\frac{|i|}{k}\Big)u_i$.

Taking the inner product of (4) with $\beta\tilde{u}$ in l^2, we find that

$$\frac{1}{2}\beta\frac{d}{dt}\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|u_i|^2+\beta\nu(Bu, B\tilde{u})+\beta\lambda\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|u_i|^2=\beta\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)h_i(u_i(t-\rho_0(t)))u_i-\beta\alpha\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)u_iv_i+\beta\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)u_if_i(t) \tag{38}$$

Taking the inner product of (4) with $\tilde{\alpha v}$ in l^2, we get that

$$\frac{1}{2}\alpha\frac{d}{dt}\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|v_i|^2=-\alpha\delta\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|v_i|^2+\beta\alpha\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)u_iv_i+\alpha\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)v_ig_i(t) \tag{39}$$

Summing up (38) and (39), we get

$$\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)(\beta|u_i|^2+\alpha|v_i|^2)+\beta\nu(Bu, B\tilde{u})+\beta\lambda\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|u_i|^2+\alpha\delta\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|v_i|^2\leqslant\beta\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)h_i(u_i(t-\rho_0(t)))u_i+\beta\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)u_if_i(t)v+\alpha\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)v_ig_i(t) \tag{40}$$

We now estimate the terms in (40) as follows. First, we have

$$\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|u_i|^2(Bu, B\tilde{u})=\sum_{i\in\mathbf{Z}}(u_{i+1}-u_i)\Big(\theta\Big(\frac{|i+1|}{k}\Big)u_{i+1}-\theta\Big(\frac{|n|}{k}\Big)u_i\Big)=\sum_{i\in\mathbf{Z}}\Big(\theta\Big(\frac{|i+1|}{k}\Big)-\theta\Big(\frac{|n|}{k}\Big)\Big)(u_{i+1}-u_i)u_{i+1}+\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|u_{i+1}-u_i|^2\geqslant\sum_{i\in\mathbf{Z}}\Big(\theta\Big(\frac{|i+1|}{k}\Big)-\theta\Big(\frac{|i|}{k}\Big)\Big)(u_{i+1}-u_i)u_{i+1}.$$

By the property of the function θ, we have

$$\Big|\sum_{i\in\mathbf{Z}}\Big(\theta\Big(\frac{|i+1|}{k}\Big)-\theta\Big(\frac{|i|}{k}\Big)\Big)(u_{i+1}-u_i)u_{i+1}\Big|\leqslant\sum_{i\in\mathbf{Z}}\frac{|\theta'(\xi_i)|}{k}|u_{i+1}-u_i||u_{i+1}|\leqslant\frac{c_0}{k}\sum_{i\in\mathbf{Z}}|u_{i+1}|^2+|u_i||u_{i+1}|\leqslant\frac{c}{k}\|u\|^2,$$

which implies that

$$-\beta v(Bu,\widetilde{Bu})\leqslant\frac{c}{k}\|u\|^2 \tag{41}$$

We now estimate the right-hand side of (40). The first term is bounded by

$$\Big|\beta\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)h_i(u_i(t-\rho_0(t)))u_i\Big|\leqslant\beta\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|h_i(u_i(t-\rho_0(t)))||u_i|\leqslant\frac{1}{4}\beta\lambda\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|u_i|^2+\frac{\beta}{\lambda}\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|h_i(u_i(t-\rho_0(t)))|^2\leqslant\frac{1}{4}\beta\lambda\sum_{i\in\mathbf{Z}}\theta\Big(\frac{|i|}{k}\Big)|u_i|^2+$$

$$\frac{\beta L^2}{\lambda}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|u_i(t-\rho_0(t))|^2 \tag{42}$$

For the left two term on the right-hand side of (40), we have

$$\beta\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)u_if_i(t)v+\alpha\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)v_ig_i(t)\leqslant \frac{1}{4}\beta\lambda\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|u_i|^2+\frac{\beta}{\lambda}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|f_i(t)|^2+\frac{1}{2}\alpha\delta\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|v_i|^2+\frac{\alpha}{2\delta}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|g_i(t)|^2 \tag{43}$$

By (40)～(43) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)(\beta|u_i|^2+\alpha|v_i|^2)\leqslant -\beta\lambda\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|u_i|^2-\alpha\delta\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|v_i|^2+\frac{2\beta L^2}{\lambda}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|u_i(t-\rho_0(t))|^2+\frac{c}{k}\|u\|^2+\frac{2\beta}{\lambda}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|f_i(t)|^2+\frac{\alpha}{\delta}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|g_i(t)|^2 \tag{44}$$

Let $\sigma=\min\{\lambda,\delta\}$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)(\beta|u_i|^2+\alpha|v_i|^2)\leqslant -\sigma\left[\beta\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|u_i|^2-\alpha\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|v_i|^2\right]+\frac{2\beta L^2}{\lambda}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|u_i(t-\rho_0(t))|^2+\frac{c}{k}\|u\|^2+\frac{2\beta}{\lambda}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|f_i(t)|^2+\frac{\alpha}{\delta}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|g_i(t)|^2 \tag{45}$$

Futher,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)(\beta|u_i|^2+\alpha|v_i|^2)\leqslant -\sigma\left[\beta\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|u_i|^2-\alpha\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|v_i|^2\right]+\frac{2L^2}{\lambda}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)[\beta|u_i(t-\rho_0(t))|^2+\alpha|v_i|^2]+\frac{c}{k}\|u\|^2+\frac{2\beta}{\lambda}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|f_i(t)|^2+\frac{\alpha}{\delta}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)|g_i(t)|^2 \tag{46}$$

By the similar argument as in Lemma 3.1, we get from (46) for any $t>\rho$ and $-\rho\leqslant\xi\leqslant 0$,

$$\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)(\beta|u_i(\tau+\xi,\tau-t,\varphi)|^2+\alpha|v_i(\tau,\tau-t,\varphi)|^2)\leqslant (\beta\|\varphi\|_\rho^2+\alpha\|\varphi\|^2)\mathrm{e}^{-\lambda(t+\xi)}+\frac{c}{k}\mathrm{e}^{-\lambda(t+\xi)}\int_{\tau-t}^{\tau+\xi}\mathrm{e}^{\lambda s}\|u(s,\tau-t,\varphi)\|^2\mathrm{d}s+\frac{2\beta}{\lambda}\mathrm{e}^{-\lambda(t+\xi)}\int_{\tau-t}^{\tau+\xi}\mathrm{e}^{-\lambda s}\sum_{|i|\geqslant k}|f_i(s)|^2\mathrm{d}s+\frac{\alpha}{\delta}\mathrm{e}^{-\lambda(t+\xi)}\int_{\tau-t}^{\tau+\xi}\mathrm{e}^{-\lambda s}\sum_{|i|\geqslant k}|g_i(s)|^2\mathrm{d}s \tag{47}$$

It follows from Lemma 3.1 that for any $\tau\in\mathbf{R}$, $(\varphi,\varphi)\in D_\rho,\varepsilon>0$ there exist $T=T(\tau,D_\rho,\varepsilon)>\rho$ and $K_1=K_1(\tau,D_\rho,\varepsilon)$ such that for $k\geqslant K_1$, $t\geqslant T$ and $-\rho\leqslant\xi\leqslant 0$

$$\frac{c}{k}\mathrm{e}^{-\lambda(t+\xi)}\int_{\tau-t}^{\tau+\xi}\mathrm{e}^{\lambda s}\|u(s,\tau-t,\varphi)\|^2\mathrm{d}r\leqslant\frac{\varepsilon}{3} \tag{48}$$

which, together with (47), implies

$$\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)(\beta|u_i(\tau+\xi,\tau-t,\varphi)|^2+\alpha|v_i(\tau,\tau-t,\varphi)|^2)\leqslant (\beta\|\varphi\|_\rho^2+\alpha\|\varphi\|^2)\mathrm{e}^{-\lambda(t+\xi)}+\frac{\varepsilon}{3}+\frac{2\beta}{\lambda}\mathrm{e}^{-\lambda(t+\xi)}\int_{\tau-t}^{\tau+\xi}\mathrm{e}^{-\lambda s}\sum_{|i|\geqslant k}|f_i(s)|^2\mathrm{d}s+\frac{\alpha}{\delta}\mathrm{e}^{-\lambda(t+\xi)}\int_{\tau-t}^{\tau+\xi}\mathrm{e}^{-\lambda s}\sum_{|i|\geqslant k}|g_i(s)|^2\mathrm{d}s \tag{49}$$

We have from $(\varphi,\varphi)\in D_\rho(\tau-t)$ that there exists $T_1=T_1(\tau,D_\rho,\varepsilon)>0$ such that for all $t\geqslant T_1$ and $-\rho\leqslant\xi\leqslant 0$,

$$(\beta\|\varphi\|_\rho^2+\alpha\|\varphi\|^2)\mathrm{e}^{-\lambda(t+\xi)}\leqslant (\beta\|\varphi\|_\rho^2+\alpha\|\varphi\|^2)\mathrm{e}^{\lambda\rho}\mathrm{e}^{-\lambda t}\leqslant\frac{\varepsilon}{3} \tag{50}$$

We have from (26) that there is a $N_1=N_1(\tau,\varepsilon)>0$ such that for all $k\geqslant N_1$,

$$\frac{2\beta}{\lambda}\mathrm{e}^{\lambda\rho}\mathrm{e}^{-\lambda t}\int_{-\infty}^{0}\mathrm{e}^{-\lambda r}\sum_{|i|\geqslant k}|f_i(s+\tau)|^2\mathrm{d}r+\frac{\alpha}{\delta}\mathrm{e}^{\lambda\rho}\mathrm{e}^{-\lambda t}\int_{-\infty}^{0}\mathrm{e}^{-\lambda r}\sum_{|i|\geqslant k}|f_i(s+\tau)|^2\mathrm{d}r\leqslant\frac{\varepsilon}{3} \tag{51}$$

Note that

$$\sup_{-\rho\leqslant\xi\leqslant 0}\sum_{|i|\geqslant 2k}(\beta|u_i(\tau+\xi,\tau-t,\varphi)|^2+\alpha|v_i(\tau,\tau-t,\varphi)|^2)\leqslant \sup_{-\rho\leqslant\xi\leqslant 0}\sum_{i\in\mathbf{Z}}\theta\left(\frac{|i|}{k}\right)(\beta|u_i(\tau+\xi,\tau-t,\varphi)|^2+\alpha|v_i(\tau,\tau-t,\varphi)|^2),$$

which along with (49)~(51) we conclude the proof.

4 Existence of pullback attractors

In this section, we establish the existence of D_ρ-pullback attractor for the non-autonomous dynamical system Φ associated with the problem (4)~(6).

Lemma 4.1 Assume that (H) (7) and (26) hold. Then for every $\tau\in\mathbf{R}$ and $D_\rho=\{D_\rho(\tau):\tau\in\mathbf{R}\}\in D_\rho$, there exists $T=T(\tau, D_\rho)>\rho$ such that u satisfies that $u_\tau(\cdot, \tau-t, \varphi)$ is equicontinuous in l^2.

Proof Denote by $P_k u=(u_1, u_2, \ldots, u_k, 0, 0, \ldots)$, for $u\in l^2$ and $k\in\mathbf{N}$. By Lemma 3.2, for $\varepsilon>0$, there exists $T=T(\tau, \varepsilon)>\rho$ and large enough integer $N=N(\tau, \varepsilon)$ such that for all $t\geqslant T$,

$$\max_{-\rho\leqslant s\leqslant 0}\|(I-P_N)u(\tau+s, \tau-t, \varphi)\|^2<\frac{\varepsilon}{3} \tag{52}$$

Let $u_1=P_N u$. By Lemma 3.1, it follows from (4) and the equivalence of norm in finite dimensional space that there exists $T=T(\tau)>\rho$ such that for all $t\geqslant T$,

$$\int_{\tau-\rho}^{\tau}\left\|\frac{\mathrm{d}}{\mathrm{d}r}u_1(r, \tau-t, \varphi)\right\|^2\mathrm{d}r\leqslant c \tag{53}$$

where $c=c(\tau)$ is a positive number. Without loss of generality, we assume that $s_1, s_2\in[-\rho, 0]$ with $0<s_1-s_2<1$. Then for any fixed $\tau\in\mathbf{R}$,

$$\|u_1(\tau+s_1, \tau-t, \varphi)-u_1(\tau+s_2, \tau-t, \varphi)\|\leqslant \int_{\tau+s_2}^{\tau+s_1}\left\|\frac{\mathrm{d}u_1(r, \tau-t, \varphi)}{\mathrm{d}r}\right\|\mathrm{d}r\leqslant \left(\int_{\tau-\rho}^{\tau}\left\|\frac{\mathrm{d}u_1(r, \tau-t, \varphi)}{\mathrm{d}r}\right\|^2\mathrm{d}r\right)^{\frac{1}{2}} |s_1-s_2|^{\frac{1}{2}}\leqslant \mathrm{c}\,|s_1-s_2|^{\frac{1}{2}} \tag{54}$$

which implies that there exits a constant $\zeta=\zeta(\varepsilon)>0$ such that if $|s_1-s_2|<\zeta$, then

$$\|u(\tau+s_2, \tau-t, \varphi)- u(\tau+s_1, \tau-t, \varphi)\|<\frac{\varepsilon}{3}$$

which along with (52) implies that for all $t\geqslant T$,

$$\|u(\tau+s_2, \tau-t, \varphi)-u(\tau+s_1, \tau-t, \varphi)\|\leqslant \|P_N u(\tau+s_2, \tau-t, \varphi)- P_N u(\tau+s_1, \tau-t, \varphi)\|+ \|(I-P_N)u(\tau+s_2, \tau-t, \varphi)\|+ \|(I-P_N)u(\tau+s_1, \tau-t, \varphi)\|\leqslant\varepsilon.$$

This completes the proof.

As for the compactness in l^2 in Ref. [16] one can easily verify the the following compactness criteria in $C_\rho=C([-\rho, 0], l^2)$ by means of uniform tail estimates.

Lemma 4.2 Let $\{u^n\}_{n=1}^{\infty}=\{(u_i^n)_{i\in\mathbf{Z}}\}_{n=1}^{\infty}\subseteq C_\rho$. Then $\{u^n\}_{n=1}^{\infty}$ is relative compact in C_ρ if and only if the following conditions are satisfied:

(i) $\{u^n\}_{n=1}^{\infty}$ is bounded in C_ρ;

(ii) $\{u^n\}_{n=1}^{\infty}$ is equicontinuous;

(iii) $\limsup\limits_{k\to\infty}\limsup\limits_{n\to\infty}\sup\limits_{-\rho\leqslant\xi\leqslant 0}\sum\limits_{|i|\geqslant k}|u_i^n|^2=0$.

Theorem 4.3 Assume that (H), (7) and (26) hold. Then, the non-autonomous dynamical system Φ has a unique D_ρ-pullback attractor $A_\rho=\{A_\rho(\tau):\tau\in\mathbf{R}\}\in X_\rho$.

Proof For $\tau\in\mathbf{R}$, denote by

$$K(\tau)=\{(u, v)\in X_\rho:(\|u\|_\rho^2+\|v\|^2)\leqslant M(\tau)\},$$

where

$$M(\tau)=2\frac{2\beta}{\chi\lambda}\mathrm{e}^{\lambda\rho}\int_{-\infty}^{0}\mathrm{e}^{\lambda s}\|f(s+\tau)\|^2\mathrm{d}s+ 2\frac{\alpha}{\chi\delta}\mathrm{e}^{\lambda\rho}\int_{-\infty}^{0}\mathrm{e}^{\lambda s}\|g(s+\tau)\|^2\mathrm{d}s.$$

Firstly, we know from Lemma 3.1 that Φ has a D_ρ-pullback absorbing set $K(\tau)$. Secondly, since Lemma 3.1, 3.2 and 4.1 coincide with all the conditions of Lemma 4.2, Φ is D_ρ-pullback asymptotically compact in X_ρ. Hence the existence of a unique D_ρ-pullback attractor for the non-autonomous dynamical system Φ follows from Proposition 2.7 in Ref. [18] immediately.

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