一类三维分段光滑系统的穿越极限环

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摘 要：本文研究了一类三维分段光滑系统的穿越极限环. 由于相空间被一个超平面分成两个区域，因而系统呈现两个不同的向量场. 此外，系统还具有 two-fold 点，且在该点处两个向量场都与该超平面相切. 本文证明系统穿越极限环的最大个数是 2，给出了存在一个和两个穿越极限环的充要条件，并确定其周期及在切换流形上的穿越位置.

关键词：穿越极限环；分段光滑系统；切换流形；two-fold
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Crossing limit cycles of a 3D piecewise-smooth system

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Abstract: In this paper we investigate the crossing limit cycles of a 3D discontinuous piecewise-smooth system. In this system, the phase space is divided into two regions by a hypersurface and thus the system presents two different vector fields. Meanwhile, the system presents two-fold in which both vector fields are tangent to the hypersurface. We prove that the maximum number of crossing limit cycles is 2 and give necessary and sufficient conditions for one and two crossing limit cycles respectively. Furthermore, the crossing locations of the crossing limit cycles are determined as well as their periods.

Keywords: Crossing limit cycle; Piecewise-smooth system; Switching manifold; Two-fold
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1 Introduction

In recent years, discontinuous piecewise-smooth (DPWS) systems have enjoyed widespread application such as mechanical systems with friction, switched electronic systems, and control systems[1, 2]. For DPWS systems, bifurcations of limit cycle are also an important problem[3, 4]. In this paper we consider a 3D discontinuous piecewise-smooth system defined by ordinary differential equations. Here the vector field is discontinuous along a hypersurface in its 3D phase space and, sometimes, this hypersurface is called a *switching manifold*. For 3D discontinuous piecewise-smooth systems, an important and interesting type of singularity is two- fold singularity, which is a point on the switching manifold and at which the vector fields on both sides are tangent with the switching manifold. Actually, as indicated in Refs. [5-7] each sub-vector field on different sides has a set of tangential singularities on the switching manifold and this set is a curve,

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which is also called *tangency curve*. So, a two-fold is actually an intersection point of two tangency curves of sub-vector fields. Additionally, under some non-degenerate conditions, this two-fold is said to be *regular*, otherwise, we say this two-fold is *degenerate*.

The dynamical behavior was analyzed near a regular two-fold for a 3D discontinuous piecewise-smooth system in Refs. [8-10] by using Filippov's convention[11]. Considering the degeneracy of two-fold, Q2-*singularity* had been presented in Ref. [12]. Further, by the tangency degree of those two tangency curves, the classification of degenerate two-fold was given in Ref. [5] as 1-degenerate two-fold (Q2-*singularity*) and 2-degenerate two-fold, the normal form for systems having degenerate two-fold was also provided, which shows that under perturbations a 1-degenerate two-fold may disappear or split into two regular two-folds and of course some interesting sliding bifurcations happen. A truncation of generic unfolding of a normal form having a 1-degenerate two-fold was reduced as

$$\dot{X}=\begin{cases} F^{+}(X)=\begin{pmatrix} c_1 \\ 1 \\ -y+x^2-\mu \end{pmatrix}, & \text{if} \quad z>0, \\ F^{-}(X)=\begin{pmatrix} c_2 \\ -1 \\ -y-x^2+\mu \end{pmatrix}, & \text{if} \quad z<0 \end{cases} \tag{1}$$

in Ref. [13], where $X=(x, y, z)^T$, c_1, c_2, $\mu\in\mathbf{R}$. It was proved in Ref. [13] that system (1) has a family of non-isolated crossing periodic orbits when $c_1+c_2=0$. Fixing $c_1=-c_2=-1/4$ and adding terms ax, az in $F^{+}(X)$ in (1), it was obtained in Ref. [7] that

$$\dot{X}=\begin{cases} F^{+}(X)=\begin{pmatrix} -1/4-ax \\ 1 \\ -y+x^2-az-\mu \end{pmatrix}, \text{if } z>0, \\ F^{-}(X)=\begin{pmatrix} 1/4 \\ -1 \\ -y-x^2+\mu \end{pmatrix}, \text{if } z<0 \end{cases} \tag{2}$$

where $a,\mu\in\mathbf{R}$ and $a\neq 0$. The existence and number of crossing limit cycles (CLCs for abbreviation) for system (2) were investigated in Ref. [7]. Although some terms are added in (2), c_1 and c_2 are restricted as $-1/4$ and $1/4$ respectively. We wonder whether there are still CLCs without this restriction, i. e, general c_1, c_2 satisfying $c_1+c_2=0$.

In this paper, we study the existence and number of CLCs for system

$$\dot{X}=\begin{cases} F^{+}(X)=\begin{pmatrix} c-ax \\ 1 \\ -y+x^2-az-\mu \end{pmatrix}, \text{if } z>0, \\ F^{-}(X)=\begin{pmatrix} -c \\ -1 \\ -y-x^2+\mu \end{pmatrix}, \text{if } z<0 \end{cases} \tag{3}$$

where $c, a, \mu\in\in\mathbf{R}$ and $a\neq 0$. The switching manifold of system (3) is $\Sigma:=\{X\in\mathbf{R}^3: h(X)=0\}$, where $h(X):=z$, and the two tangency curves are

$$T^{+}:=\{X\in\Sigma: y-x^2+\mu=0\},$$
$$T^{-}:=\{X\in\Sigma: y+x^2-\mu=0\},$$

respectively. It is not hard to check that system (3) has no two-folds when $\mu<0$, one 1-degenerate invisible-invisible two-fold $O:(0,0,0)$ when $\mu=0$, and two regular invisible-invisible two-folds $P^{\pm}:(\pm\sqrt{\mu},0,0)$, when $0<\mu\leqslant 1$. Moreover, the crossing regions and sliding regions are

$$\Sigma_c^{+}:=\{X\in\Sigma: y<\min\{x^2-\mu, -x^2+\mu\}\},$$
$$\Sigma_c^{-}:=\{X\in\Sigma: y>\max\{x^2-\mu, -x^2+\mu\}\},$$
$$\Sigma_s^{a}:=\{X\in\Sigma: x^2-\mu<y<-x^2+\mu\},$$
$$\Sigma_s^{r}:=\{X\in\Sigma: -x^2+\mu<y<x^2-\mu\}.$$

We show the crossing and sliding regions when $\mu<0$, $\mu=0$, and $\mu>0$ respectively in Fig. 1.

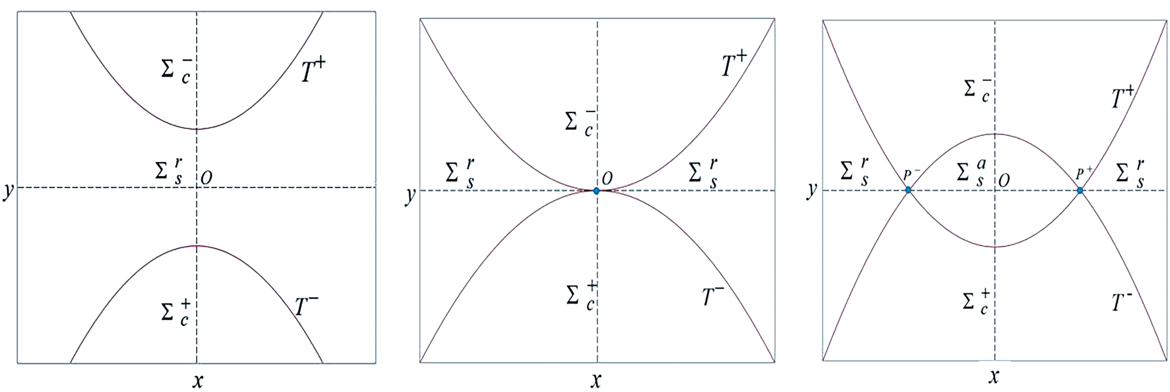
We get the results about CLCs of system (3) in the following theorem.

Theorem 1.1 For system (3), the maximum number of CLCs is 2 and it is reachable. Moreover, there exists CLCs if and only if there exists $\tau\in\Omega_{(c,a)}:=\{\tau>0: g_1(\tau)>0, g_2(\tau)<0\}$ such that $\varphi(\tau)$ has zeros. Each zero τ corresponds to one CLC with period 2τ and crossing Σ at two points $\hat{P}_0:(\hat{x}_0, \hat{y}_0, 0)$, $\hat{P}_1:(\hat{x}_1, \hat{y}_1, 0)$, considering $\tau\in(0,+\infty)$,

$$\begin{cases} g_1(\tau):=\dfrac{\tau(a+2c^2)}{2a}+\dfrac{\tau^2c^2(2e^{a\tau}+1)}{3(1-e^{a\tau})}, \\ g_2(\tau):=-\dfrac{1}{a}-\dfrac{\tau(a-2c^2)}{a(1-e^{a\tau})}+\dfrac{\tau^2c^2(e^{a\tau}+1)}{(1-e^{a\tau})^2}, \\ \varphi(\tau):=\dfrac{a(3+c^2a\tau^2)+6c^2}{6a^2}+ \\ \quad \dfrac{\tau(a+2c^2)(1+e^{a\tau})}{4a(1-e^{a\tau})}-\mu \end{cases} \tag{4}$$

and

$$\begin{cases} \hat{x}_0:=\dfrac{c((a\tau-1)e^{a\tau}+1)}{a(1-e^{a\tau})}, \\ \hat{y}_0:=\dfrac{(-2c^2a\tau^2+(6c^2-9a)\tau+6)e^{2a\tau}-4(2c^2a\tau^2-3a\tau+3)e^{a\tau}-2c^2a\tau^2-(6c^2+3a)\tau+6}{12a(1-e^{a\tau})^2}, \\ \hat{x}_1:=\hat{x}_0+c\tau, \\ \hat{y}_1:=\hat{y}_0+\tau \end{cases} \tag{5}$$



(a) $\mu<0$　　(b) $\mu=0$　　(c) $\mu>0$

Fig. 1　The crossing regions and sliding regions on the switching manifold $z=0$

In Theorem 1.1, results about the existence and number of CLCs as well as their birth given in Ref. [7] are generalized from the case $c_1=-c_2=-1/4$ to the case $c_1+c_2=0$.

This paper is organized as following. In Section 2, we introduce some basic definitions of DPWS systems. In Section 3, we provide a proof for Theorem 1.1. In Section 4, we present some remarks and give examples to show the existence of CLCs.

2　Preliminaries

In this section, we introduce some basic definitions about DPWS systems. For a 3D DPWS system

$$\dot{X}=F(X)=\begin{cases} F^+(X), \text{ if } X\in\Sigma_+, \\ F^-(X), \text{ if } X\in\Sigma_- \end{cases} \tag{6}$$

the switching manifold is $\Sigma:=\{X\in\mathbf{R}^3: h(X)=0\}$, which separates the phase space into two regions $\Sigma_+:=\{X\in\mathbf{R}^3: h(X)>0\}$ and $\Sigma_-:=\{X\in\mathbf{R}^3: h(X)<0\}$. Here $F^\pm:=(f_1^\pm, f_2^\pm, f_3^\pm)$. As in Ref. [7], the Lie derivatives $L_{F^\pm}h:=\langle\nabla h, F^\pm\rangle$ denote the contact type of vector field $F^\pm$ with Σ, where $\nabla h\neq 0$ is the gradient of $h(X)$ and $\langle .\,, .\rangle$ denotes the canonical inner product. Similarly, higher order Lie derivatives are given as $L_{F^\pm}^n h:=\langle\nabla L_{F^\pm}^{n-1}h, F^\pm\rangle$ for $n\geqslant 2$. Using the Lie derivatives, we state the sliding regions and the crossing regions on switching manifold as follows

$$\Sigma_c^+:=\{X\in\Sigma: L_{F^+}h>0, L_{F^-}h>0\},$$
$$\Sigma_c^-:=\{X\in\Sigma: L_{F^+}h<0, L_{F^-}h<0\},$$
$$\Sigma_s^a:=\{X\in\Sigma: L_{F^+}h<0, L_{F^-}h>0\},$$
$$\Sigma_s^r:=\{X\in\Sigma: L_{F^+}h>0, L_{F^-}h<0\}.$$

For $X\in\Sigma_s^a\cup\Sigma_s^r$, by Filippov's convention[11] we obtain the sliding vector field $\hat{F}^s$ as

$$\hat{F}^s(X):=\frac{L_{F^-}hF^+-L_{F^+}hF^-}{L_{F^-}h-L_{F^+}h}(X),$$

which is tangent to the Σ. Moreover

$$F^s(X):=(L_{F^-}hF^+-L_{F^+}hF^-)(X),$$

is called the desingularized planar sliding vector field of $\hat{F}^s$. Clearly, $(L_{F^-}h-L_{F^+}h)(X)>0$ (resp. <0) when $X\in\Sigma_s^a$ (resp. $X\in\Sigma_s^r$). Thus, we usually consider $F^s(X)$ for $X\in\Sigma_s^a\cup\Sigma_s^r$. As in Ref. [5], the tangency curves of vector fields $F^\pm$ are $T^\pm:=\{X\in\Sigma:L_{F^\pm}h(X)=0\}$, respectively.

Definition 2.1 A point $p\in T^\pm$ is called a fold of $F^\pm$ if $L_{F^\pm}^2h(p)\neq 0$. Moreover, a fold $p\in T^+$ (resp. T^-) is visible if $L_{F^+}^2h(p)>0$ (resp. $L_{F^-}^2h(p)<0$) and invisible if $L_{F^+}^2h(p)<0$ (resp. $L_{F^-}^2h(p)>0$).

When a point p is a fold for both F^+ and F^-, the point is called two-fold. Generically, T^+ and T^- are transverse at this point, and more precisely classification is shown as follows.

Definition 2.2[5] A point $p\in T^+\cap T^-$ is called a regular two-fold (resp. 1-degenerate two-fold) if the contact between T^+ and T^- at p is transverse and p is a hyperbolic critical point (resp. quadratic and $L_{F^+}^2h(p)\neq L_{F^-}^2h(p)$) and the eigenvectors of the Jacobian matrix of F^s at p are transverse to T^+ and T^-.

Let $\varphi(t,p_0)$ be the solution of system (6) satisfying the initial condition $\varphi(0,p_0)=p_0$. As indicated in Ref. [5], by the implicit function theorem, for each p_0 in a neighborhood $U(p)$ of invisible-invisible two-fold p, there exists a unique positive time $t(p_0)$ such that $\varphi(t,p_0)$ return to Σ at point $\varphi(t(p_0),p_0)$. Note that $\varphi(t(p_0),p_0)=\varphi^+(t(p_0),p_0)$ (resp. $=\varphi^-(t(p_0),p_0)$) if $p_0\in\Sigma_c^+$ (resp. $\in\Sigma_c^-$), where $\varphi^\pm$ are the solutions for $\dot{X}=F^\pm(X)$, respectively.

Definition 2.3 The half-return map for F^+ is defined as $\Phi^+:p_0\to\varphi^+(t_0,p_0)\in\Sigma_c^-$ for $p_0\in\Sigma_c^+$, The half-return map for F^- is defined as Φ^-: $p_1\to\varphi^+(t_1,p_1)\in\Sigma_c^+$ for $p_1\in\Sigma_c^-$. The first return map $\Phi:\Sigma_c^+\to\Sigma_c^+$ is defined by the composition $\Phi^-\circ\Phi^+$.

3 Proof of the main result

In this section, we give a detailed proof of Theorem 1.1.

Lemma 3.1 The return map Φ of system (3) has a fixed point $\hat{p}_0:(\hat{x}_0,\hat{y}_0,0)$ in Σ_c^+ and $t_0=t_1=\tau>0$, where $\hat{x}_0$, $\hat{y}_0$ are given by (5) and τ is the zero of φ. Here φ is defined in (4).

Proof We solve the differential equation $\dot{X}=F^+(X)$ directly and get the solution when $z>0$ with initial condition $(x_0,y_0,0)\in\Sigma_c^+$, that is

$$x^+(t)=\frac{(ax_0-c)+ce^{at}}{ae^{at}},$$

$$y^+(t)=y_0+t,$$

$$z^+(t)=\frac{(c^2+a(1-at))e^{2at}-a(2c^2t+1)e^{at}-c^2}{a^3e^{2at}}+\frac{2cx_0(1+(at-1)e^{at})}{a^2e^{2at}}+\frac{(1-e^{at})((y_0+\mu)e^{at}-x_0^2)}{ae^{2at}}.$$

The return time $t_0>0$ satisfies $z^+(t_0)/(e^{-at_0}-1)=0$, then $(x^+(t_0),y^+(t_0),z^+(t_0))=(x_1,y_1,0)\in\Sigma_c^-$. The same to the vector field F^-, the corresponding solution with initial condition $(x_1,y_1,0)\in\Sigma_c^-$ is

$$x^-(t)=x_1-ct,$$

$$y^-(t)=y_1-t,$$

$$z^-(t)=-\frac{1}{3}c^2t^3+\frac{1}{2}(1+2x_1c)t^2-(y_1+x_1^2-\mu)t.$$

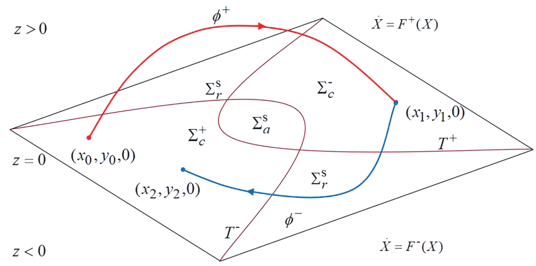


Fig. 2 The trajectories and return map for system (3)

The return time $t_1>0$ satisfies $z^-(t_1)/t_1=0$, then $(x^-(t_1),y^-(t_1),z^-(t_1))=(x_2,y_2,0)\in\Sigma_c^+$. We obtain the return map $\Phi:(x_0,y_0,0)\to(x_2,y_2,0)$, whose fixed point may represent a closed orbit. We solve these equations $(x_0,y_0)=(x_2,y_2)$,

$z^+(t_0)/(e^{-at_0}-1)=0$, $z^-(t_1)/t_1=0$. From $y_0=y_2$, we get $t_0=t_1$, and define $\tau=t_0=t_1$ as an auxiliary parameter. By substituting it into the remaining equations, we obtain the expressi ons (5) and τ satisfies $\varphi(\tau)=0$, and the period of closed orbit is $t_0+t_1=2\tau$.

A fixed point represents a CLC which is in crossing region, therefore Lie derivatives need to satisfy these conditions: $L_{F^\pm}h(\hat{x}_0,\hat{y}_0,0)(\tau)>0$ and $L_{F^\pm}h(\hat{x}_1,\hat{y}_1,0)(\tau)<0$, namely

$$L_{F^-}h(\hat{x}_0,\hat{y}_0,0)(\tau)=g_1(\tau)>0,$$
$$L_{F^+}h(\hat{x}_1,\hat{y}_1,0)(\tau)=g_2(\tau)<0,$$
$$L_{F^+}h(\hat{x}_0,\hat{y}_0,0)(\tau)=-\frac{1}{a}+\frac{(a-2c^2)\tau e^{a\tau}}{a(e^{a\tau}-1)}+\frac{\tau^2c^2e^{a\tau}(e^{a\tau}+1)}{(1-e^{a\tau})^2}>0,$$
$$L_{F^-}h(\hat{x}_1,\hat{y}_1,0)(\tau)=-\frac{\tau(a+2c^2)}{2a}-\frac{\tau^2c^2(2+e^{a\tau})}{3(1-e^{a\tau})}<0 \qquad (7)$$

here $g_1(\tau),g_2(\tau)$ are defined in (4). Because

$$L_{F^-}h(\hat{x}_1,\hat{y}_1,0)<-L_{F^-}h(\hat{x}_0,\hat{y}_0,0),$$

fixed point only needs to satisfy first three conditions. For writing conveniently, we write the third formula in (7) as $g_3(\tau)$. Considering function $\varphi(\tau)$, we obtain the derivative

$$\frac{\partial\varphi(\tau)}{\partial\tau}=\frac{c^2\tau}{3}+\frac{(a+2c^2)(e^{a\tau}+1)}{4a(1-e^{a\tau})}+\frac{\tau(a+2c^2)e^{a\tau}}{2(1-e^{a\tau})^2},\quad \lim_{\tau\to0^+}\frac{\partial\varphi(\tau)}{\partial\tau}=0.$$

Lemma 3.2 For $a>2c^2$, there exists a constant τ_{sn} such that $\varphi(\tau)$ decreases for $0<\tau<\tau_{sn}$ and increases for $\tau>\tau_{sn}$. For $a\leqslant2c^2$, the function $\varphi(\tau)$ increases.

Proof By the definition of φ, we get

$$\lim_{\tau\to0^+}\varphi(\tau)=-\mu$$

and

$$\frac{\partial\varphi(\tau)}{\partial\tau}=\frac{e^{a\tau}(2c^2+3a+2c^2(e^{-a\tau}+e^{a\tau}))}{6(e^{a\tau}-1)^2}\cdot(f_1(\tau)-f_2(\tau)),$$

where

$$f_1(\tau):=\tau,\ f_2(\tau):=\frac{3(e^{a\tau}-e^{-a\tau})(a+2c^2)}{2a(2c^2+3a+2c^2(e^{-a\tau}+e^{a\tau}))}.$$

Consider the case $a>2c^2$ firstly, Since $2c^2+3a+2c^2(e^{-a\tau}+e^{a\tau})>0$ for $\tau\in(0,+\infty)$, the zeros of $\partial\varphi(\tau)/\partial\tau$ are given by the intersection of functions f_1, f_2. Clearly,

$$\lim_{\tau\to+\infty}f_2(\tau)=\frac{3(a+2c^2)}{4ac^2}.$$

The derivatives of $f_2(\tau)$ about τ are

$$f_2'(\tau)=\frac{3((2c^2+3a)(e^{-a\tau}+e^{a\tau})+8c^2)(a+2c^2)}{2(2c^2+3a+2c^2(e^{-a\tau}+e^{a\tau}))^2},$$

$$f_2''(\tau)=\frac{3a(a+2c^2)(e^{a\tau}-e^{-a\tau})(-2c^2(2c^2+3a)(e^{-a\tau}+e^{a\tau})+12ac^2+9a^2-28c^4)}{2(2c^2+3a+2c^2(e^{-a\tau}+e^{a\tau}))^3},$$

which implies that $f_2'(0)=1, f_2''(0)=0$. Moreover, $f_2'(\tau)>0$ for the case $a>2c^2$, and then, $f_2(\tau)$ is increasing. We find that $f_2''(\tau)$ has a unique zero, denoted by τ_1^*. It is not hard to check that $f_2''(\tau)>0$ (resp. $f_2''(\tau)<0$) for $0<\tau<\tau_1^*$ (resp. $\tau>\tau_1^*$). Thus $f_1(\tau)$ and $f_2(\tau)$ have a unique intersection, denoted by τ_{sn}. Then $\partial\varphi(\tau)/\partial\tau$ has a unique zero τ_{sn} and $\partial\varphi(\tau)/\partial\tau<0$ (resp. >0) for $0<\tau<\tau_{sn}$ (resp. $\tau>\tau_{sn}$). That is, $\varphi(\tau)$ has a minimum at τ_{sn}.

Consider the case $a\leqslant2c^2$ secondly. We also find that when $a>-2c^2$, $2c^2+3a+2c^2(e^{-a\tau}+e^{a\tau})>0$ for $\tau\in(0,+\infty)$, the zeros of $\partial\varphi(\tau)/\partial\tau$ are given by the intersection of functions f_1, f_2. When $a\in[-2c^2/3, 2c^2]$, we get that $f_2'(\tau)>0$ and $f_2''(\tau)<0$, which implies that $f_1(\tau)$ and $f_2(\tau)$ have no intersections. Therefore function $\varphi(\tau)$ is increasing. When $a\in(-2c^2, -2c^2/3)$, we obtain that $f_2'(\tau)$ has a unique zero, denoted by τ_2^* and $f_2'(\tau)>0$ (resp. $f_2'(\tau)<0$) for $0<\tau<\tau_2^*$ (resp. $\tau>\tau_2^*$). On the other hand, $f_2''(\tau)$ has a unique zero, denoted by τ_3^*, and $f_2''(\tau)<0$ (resp. $f_2''(\tau)>0$) for $0<\tau<\tau_3^*$ (resp. $\tau>\tau_3^*$). Thus $f_1(\tau)-f_2(\tau)>0$ for $\tau>0$, which implies that $\partial\varphi(\tau)/\partial\tau>0$. That is, function $\varphi(\tau)$ is increasing. When $a=-2c^2$, we compute that $\partial\varphi(\tau)/\partial\tau=c^2\tau/3>0$.

Thus function $\varphi(\tau)$ is increasing for $a\in[-2c^2, -2c^2/3)$. When $a\in(-\infty,-2c^2)$, we get $f_2'(\tau)>0$ and find that the denominator of f_2' has a unique zero, denoted by τ_*, and it is also the unique zero of the denominator of about f_2''. Further, $f_2''>0$ for $0<\tau<\tau_*$ and $f_2''<0$ for $\tau>\tau_*$, and $f_2(\tau)>0$ (resp. $f_2(\tau)<0$) for $0<\tau<\tau_*$ (resp. $\tau>\tau_*$). Then $f_2>f_1$ for $\tau<\tau_*$, $f_2<f_1$ for $\tau>\tau_*$, f_1 and f_2 do not have intersections. On the other hand, τ_* is the unique zero of $2c^2+3a+2c^2(e^{-a\tau}+e^{a\tau})$, and $2c^2+3a+2c^2(e^{-a\tau}+e^{a\tau})<0$ (>0) for $0<\tau<\tau_*$ (resp. $\tau>\tau_*$) when $a<-2c^2$. Thus $\partial\varphi(\tau)/\partial\tau>0$, then $\varphi(\tau)$ is increasing for $a\in(-\infty,-2c^2)$. Finally we get that function $\varphi(\tau)$ is increasing for $a\leqslant 2c^2$.

On the other hand, since $\lim_{\tau\to+\infty}\varphi(\tau)=+\infty$, function $\varphi(\tau)+\mu$ has a unique zero, called τ_0, and $\tau_{sn}<\tau_0$. After we prove the monotonicity of function $\varphi(\tau)$, we analyze the number of CLCs, which is equivalent to the number of zeros of $\varphi(\tau)$. In the following, we give a proof of Theorem 1.1.

Proof of Theorem 1.1 By the definition of $g_3(\tau)$, we rewrite $g_3(\tau)$ as

$$g_3(\tau)=-\frac{1}{a}+\frac{\tau}{(1-e^{-a\tau})}+\frac{c^2\tau}{(1-e^{-a\tau})^2}h_1(\tau),$$

where

$$h_1(\tau):=\tau(1+e^{-a\tau})+\frac{2(e^{-a\tau}-1)}{a}.$$

Then, for $\tau\in(0,+\infty)$ we have

$$\lim_{\tau\to0^+}h_1=0,\ \lim_{\tau\to0^+}\left(-\frac{1}{a}+\frac{\tau}{(1-e^{-a\tau})}\right)=0,$$

$$\frac{\partial h_1(\tau)}{\partial\tau}>0,\ \frac{\partial\left(-\frac{1}{a}+\frac{\tau}{(1-e^{-a\tau})}\right)}{\partial\tau}>0.$$

We obtain that function $g_3(\tau)>0$. Therefore, in order to make sure that the fixed points lie in the crossing region, we only require that $\tau\in\Omega_{(c,a)}$, where $\Omega_{(c,a)}$ is defined in the statement of this theorem. Further, by Lemma 3.1 the necessary and sufficient condition given in this theorem is obtained for the existence of CLCs.

In the following, we prove that the maximum number of CLCs of system (3) is 2 and it is reachable. For $a\leqslant 2c^2$, $\varphi(\tau)$ is an increasing function. Thus $\varphi(\tau)$ has at most one zero, *i.e.*, system (3) has at most one CLC in this case.

For $a>2c^2$, $\varphi(\tau)$ has at most two zeros, which implies that there exists at most two CLCs. By the proof of Lemma 3.2, we get $\tau_{sn}=f_2(\tau_{sn})$. Associating with expressions of g_1, g_2 and f_2, we obtain

$$g_1(\tau_{sn})=\frac{\tau_{sn}(a+2c^2)(3a-c^2+c^2e^{-a\tau_{sn}})}{2a(2c^2+3a+2c^2(e^{-a\tau_{sn}}+e^{a\tau_{sn}}))}>0,$$

$$g_2(\tau_{sn})=\frac{1-e^{a\tau_{sn}}}{4a^2\ (2c^2e^{2a\tau_{sn}}+2c^2e^{a\tau_{sn}}+3ae^{a\tau_{sn}}+2c^2)^2}\cdot H(\tau_{sn})<0,$$

where

$$H(\tau_{sn})=16ac^4e^{3a\tau_{sn}}+3c^2(4c^4+4ac^2+9a^2)e^{2a\tau_{sn}}+6a(4c^4+4ac^2+3a^2)e^{a\tau_{sn}}-12c^6+20ac^4+21a^2c^2.$$

Therefore, $\tau_{sn}\in\Omega_{(c,a)}$.

On the other hand, by the expression of $\varphi(\tau)$ given in(4)we get that $\varphi(\tau_{sn})=-\varepsilon$ when

$$\mu=\frac{a(3+c^2a\tau_{sn}^2)+6c^2}{6a^2}+\frac{\tau_{sn}(a+2c^2)(1+e^{a\tau_{sn}})}{4a(1-e^{a\tau_{sn}})}+\varepsilon.$$

Since $\varphi(\tau_{sn})$ is the minimum of φ, we obtain that $\varphi(\tau)=0$ has two solutions $\hat{\tau},\tilde{\tau}$ in a small neighborhood of τ_{sn} when $0<\varepsilon\ll1$. Because of the continuity, we get $g_1(\hat{\tau})>0, g_1(\tilde{\tau})>0$ and $g_2(\hat{\tau})<0, g_2(\tilde{\tau})<0$. Thus, there exists two CLCs. That is, the maximum number 2 of CLCs is reachable.

4 Remarks and examples

In this section, we provide some remarks to understand more details about the birth and disappearance of CLCs. By the expressions of $g_1(\tau)$, $g_2(\tau)$ and $\varphi(\tau)$, it is not hard to check that

$$\lim_{a\to0}g_1(\tau)=\frac{\tau}{6}(3-c^2\tau),$$

$$\lim_{a\to0}g_2(\tau)=\frac{\tau}{6}(c^2\tau-3),$$

$$\lim_{a\to-\infty}g_1(\tau)=\frac{\tau}{6}(3+2c^2\tau),$$

$$\lim_{a\to-\infty}g_2(\tau)=\tau(c^2\tau-1),$$

$$\lim_{a\to+\infty}g_1(\tau)=\frac{\tau}{6}(3-4c^2\tau),$$

$$\lim_{a\to+\infty} g_2(\tau)=0,$$

$$\lim_{a\to+\infty} \varphi(\tau)+\mu=\frac{\tau}{12}(2c^2\tau-3),$$

$$\lim_{a\to+\infty} \frac{\partial\varphi(\tau)}{\partial\tau}=\frac{1}{3}c^2\tau-\frac{1}{4},$$

which implies that

(i) zeros of $g_1(\tau),\varphi(\tau)+\mu,\ \partial\varphi(\tau)/\partial\tau$ tend to $3/(4c^2),3/(2c^2),3/(4c^2)$ respectively as $a\to+\infty$;

(ii) zeros of $g_2(\tau)$ tend to $1/c^2$ as $a\to-\infty$;

(iii) zeros of $g_1(\tau),g_2(\tau)$ tend to $3/c^2$ as $a\to 0$.

This is consistent with the properties of these functions in our example in the end of this paper.

Another thing, by the expressions of $\hat{x}_0,\hat{y}_0,\hat{x}_1,\hat{y}_1$ given in (5), we have

$$\lim_{\tau\to0^+}(\hat{x}_0,\ \hat{y}_0,\ \hat{x}_1,\ \hat{y}_1,\ \varphi(\tau))=(0,\ 0,\ 0,\ 0,\ -\mu).$$

When $\mu=0$, we get $\varphi(0^+)=0$. On the other hand, by the definition of $\varphi(\tau)$, we have

$$\lim_{\tau\to0^+}\frac{\partial\varphi(\tau)}{\partial\tau}=0,\ \lim_{\tau\to0^+}\frac{\partial^2\varphi(\tau)}{\partial\tau^2}=\frac{1}{12}(2c^2-a).$$

Thus, for the case that $a<2c^2$ (resp. $a>2c^2$), function $\varphi(\tau)$ has a unique positive zero in a small neighborhood of 0 when $0<\mu\leqslant1$ (resp. $-1\leqslant\mu<0$) and this zero tends to 0 as $\mu\to0$. Moreover, according to expressions of $g_1(\tau),g_2(\tau)$ given in (4), we have

$$\lim_{\tau\to0^+} g_1(\tau)=0,\ \lim_{\tau\to0^+}\frac{\partial g_1(\tau)}{\partial\tau}=\frac{1}{2},$$

$$\lim_{\tau\to0^+} g_2(\tau)=0,\ \lim_{\tau\to0^+}\frac{\partial g_2(\tau)}{\partial\tau}=-\frac{1}{2}.$$

This implies that this zero is in $\Omega_{(c,a)}$ for $|\mu|\leqslant1$. Therefore, by Theorem 1.1 there exists a unique CLC corresponding to this zero and this CLC births from a 1-degenerate two-fold when μ changes from 0 to a positive (resp. negative) number.

To end this paper, we take some values for c and μ in system (3) as examples. When $c=1/3$ and $\mu=-0.1$, we get Fig. 3, where

$\Gamma_1 := \{(a,\ \tau):\ g_1(\tau)\ =0\}$,

$\Gamma_2 := \{(a,\ \tau): g_2(\tau)=0\}$,

$\Gamma_3 := \{(a,\tau):\varphi(\tau)=0\}$.

Here, the curve Γ_1 (resp. Γ_2) has a horizontal asymptotic line $a=27/4$ (resp. $a=9$) as $a\to+\infty$ (resp. $a\to-\infty$). The curves Γ_1 and Γ_2 intersect at point $(a,\tau)=(0,27)$. This is consistent with the analysis in the first paragraph of this section.

For a fixed a, set $\Omega_{(c,a)}$ is the segment below curves Γ_1 and Γ_2 and above line $\tau=0$. From Fig. 3, we obtain that there exists no CLCs, one CLC, two CLCs when $a\in(-\infty,0.31)$, $a\in\{0.31\}\cup[0.34,+\infty)$, $a\in(0.31,0.34)$ respectively.

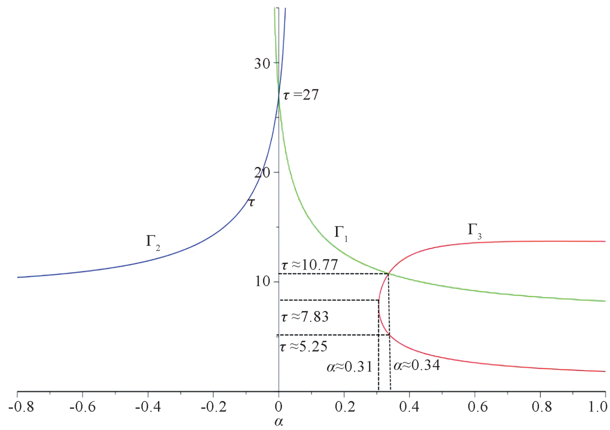


Fig. 3　Domain $\Omega_{(c,a)}$ and zeros of $\varphi(\tau)$ in (a,τ) plane of system (3) when $c=1/3$, $\mu=-0.1$

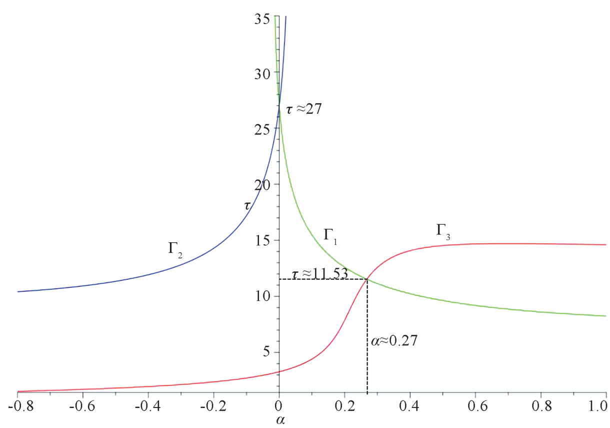


Fig. 4　Domain $\Omega_{(c,a)}$ and zeros of $\varphi(\tau)$ in (a,τ) plane of system (3) when $c=1/3$, $\mu=0.1$

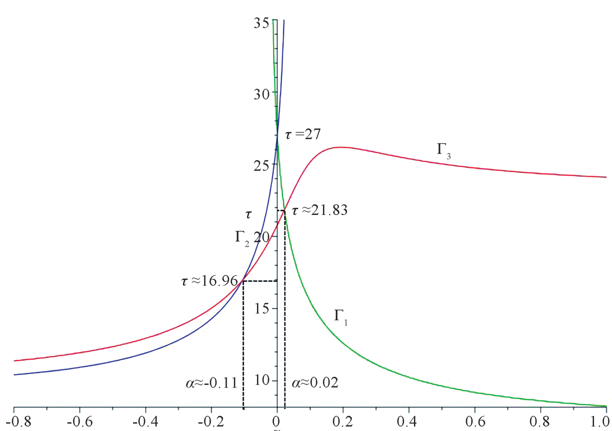


Fig. 5　Domain $\Omega_{(c,a)}$ and zeros of $\varphi(\tau)$ in (a,τ) plane of system (3) when $c=1/3$, $\mu=4$

To observe more cases, we take $c = 1/3$ and $\mu = 0.1$, $\mu = 4$, $\mu = 7.5$ in system (3) respectively and get Figs. 4, 5 and 6. One can obtain that there exists no CLCs, one CLC when $a \in [0.27, +\infty)$, $a \in (-\infty, 0.27)$ in the case that $(c, \mu) = (1/3, 0.1)$. There exists no CLCs, one CLC when $a \in (-\infty, -0.11] \cup [0.02, +\infty)$, $a \in (-0.11, 0.02)$ in the case that $(c, \mu) = (1/3, 4)$. There exists no CLCs in the case that $(c, \mu) = (1/3, 7.5)$.

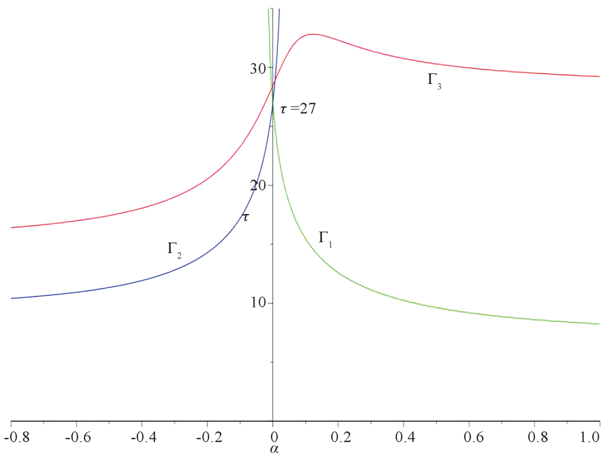


Fig. 6 Domain $\Omega_{(c,a)}$ and zeros of $\varphi(\tau)$ in (a, τ) plane of system (3) when $c = 1/3, \mu = 7.5$

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