Rosenau-KdV 方程初边值问题的一个高精度线性守恒差分格式

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摘　要：本文针对 Rosenau-KdV 方程的初边值问题提出了一个具有 $O(\tau^2+h^4)$精度的三层线性差分格式,该格式能够较好地保持两个守恒不变量. 此外,本文还得到了差分解的存在唯一性和先验误差估计,并通过能量方法证明了数值格式的收敛性和稳定性. 数值算例验证了理论结果.

关键词：Rosenau-KdV 方程；差分格式；守恒律；收敛性；稳定性
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A high-accuracy linear conservative difference scheme for the initial-boundary value problem of Rosenau-KdV equation

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Abstract: In this paper, a three-level linear finite difference scheme with theoretical accuracy of $O(\tau^2+h^4)$ is proposed for the initial-boundary value problem of Rosenau-KdV equation. This scheme simulates two conservative properties very well. The existence, uniqueness of the difference solution and prior estimates are obtained. Then the convergence and stability of the scheme are analyzed by using the energy method. Numerical examples verify the theoretical results.

Keywords: Rosenau-KdV equation; Difference scheme; Conservative law; Convergence; Stability
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1 Introduction

In the study of the dynamics of compact discrete systems, wave-wave and wave-wall interactions cannot be described by the well known KdV equation. To overcome this shortcoming of KdV equation, Rosenau proposed the following Rosenau equation[1,2]

$$u_t+u_{xxxxt}+u_x+uu_x=0 \tag{1}$$

The existence and uniqueness of solution of (1) were proved by Park[3]. As the further consideration of nonlinear wave, Zuo[4] added a viscous term u_{xxx} to (1) and discussed the Rosenau-KdV equa-

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tion

$$u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0,\ x \in \mathbf{R},\ t > 0 \tag{2}$$

Accordingly, the solitary wave solution and periodic solution of Rosenau-KdV of (2) are also investigated. As a more general case, Esfahani[5], Razborova and coworkers[6] discussed the solitary solution of the generalized Rosenau-KdV equation with usual power nonlinearity[7]. Moreover, the two invariants quantities of the Rosenau-KDV equation are also presented. In this paper, we consider the follwing initial-boundary value problem of the Rosenau-KdV equation

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0, \\ \quad x \in (x_L, x_R),\ t \in (0, T], \\ u(x,0) = u_0(x),\ x \in [x_L, x_R], \\ u(x_L, t) = u(x_R, t) = 0, \\ u_x(x_L, t) = u_x(x_R, t) = 0, \\ u_{xx}(x_L, t) = u_{xx}(x_R, t) = 0,\ t \in [0, T] \end{cases} \tag{3}$$

As the solitary wave solution of (2) is[5]

$$u(x,t) = \left(-\frac{35}{24} + \frac{35}{312}\sqrt{313}\right)\operatorname{sech}^4 \left[\frac{1}{24}\sqrt{-26 + 2\sqrt{313}}\left(x - \left(\frac{1}{2} + \frac{1}{26}\sqrt{313}\right)t\right)\right],$$

(3) is as the same as Cauchy problem of (2) when $-x_L \gg 0, x_R \gg 0$. It is easy to verified that (3) satisfies the following conservative laws[5, 6, 8]

$$Q(t) = \int_{x_L}^{x_R} u(x,t)\mathrm{d}x = \int_{x_L}^{x_R} u_0(x)\mathrm{d}x = Q(0) \tag{4}$$

$$E(t) = \| u \|_{L_2}^2 + \| u_{xx} \|_{L_2}^2 = E(0) \tag{5}$$

where $Q(0)$ and $E(0)$ are constants depend only on initial data.

It is difficult to obtain the analytic solution of a Rosenau-KDV equation, thus many studies consider the numerical methods. Since the Rosenau-KDV equation is a conservative physical system, numerical schemes with conservation properties are particularly necessary. As Li and Vu-Quoc[9] pointed, in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation. Similarly, Zhang and coworkers[10] show thata conservative difference scheme can simulate the conservative law of initial problem well and avoid the nonlinear blow-up. Hu and coworkers[8] proposed a three-level linear conservative difference scheme for (3) with theoretical accuracy is $O(\tau^2 + h^2)$. Wongsaijai and Poochinapan[11] proposed a three-level average finite difference scheme by coupling the Rosenau-KdV and the Rosenau-RLW equations. A three-level average implicit finite difference scheme is proposed by Mohebbi and Faraz[12] and stability and convergence of $O(\tau^2 + h^2)$ are proved. Using cubic B-spline functions, Ucar and coworkers[13] discussed a Galerkin finite element method. Based on subdomain method, Karakoc and Ak[14] use sextic B-spline functions to simulate the motion of single solitary wave and derive the numerical solution of the Rosenau-KdV equation. Meanwhile, the invariants of motion verify the conservation properties. Kutluay and coworkers[15] studied the operator time-splitting techniques combined with quantic B-spline collocation method for the generalized Rosenau-KdV equation in which conservative properties of the discrete mass and energy are considered.

On the other hand, most second order accuracy schemes are not satisfactory in practical computations, in particular due to the large time scale. Hence, in this paper, by using the Richardson extrapolation technique[16], we propose a three-level linear difference scheme with theoretic accuracy of $O(\tau^2 + h^4)$ and without refined mesh. Furthermore, the proposed scheme can simulate the two conservative laws (4) and (5) well. Mean while, some numerical analysis such as the prior estimate, the existence and uniqueness of the difference solution, the convergence and stability of the scheme are studied.

The rest of this paper is organized as follows. The conservative difference scheme is proposed in Section 2. The existence and uniqueness of numerical solutions are proved in Section 3. Section 4 is devoted to the prior estimate, convergence and stability of the difference scheme. In Section 5, we verify our theoretical analysis by numerical examples.

2 The difference scheme

Let h and τ be the uniform step size in the spatial and temporal directions, respectively. Denote $h=\frac{x_R-x_L}{J}$, $x_j=x_L+jh$, $j=-2,-1,0,\ldots,J,J+1,J+2$; $t_n=n\tau$, $n=0,1,2,\ldots,N$, $N=\left[\frac{T}{\tau}\right]$. In this paper, C is a positive constant which has different value in difference place. Let $u_j^n\equiv u(x_j,t_n)$ be the value of $u(x,t)$ at (x_j,t_n); and $U_j^n\approx u(x_j,t_n)$ be the approximation of $u(x,t)$ at (x_j,t_n), $e_j^n=u_j^n-U_j^n$.

Define

$$Z_h^0=\{U=(U_j)\mid U_{-2}=U_{-1}=U_0=U_J=U_{J+1}=U_{J+2}=0,\ j=-2,-1,0,1,\ldots,J,J+1,J+2\}$$

and

$$(U_j^n)_x=\frac{U_{j+1}^n-U_j^n}{h},\quad (U_j^n)_{\bar{x}}=\frac{U_j^n-U_{j-1}^n}{h},$$

$$(U_j^n)_{\hat{x}}=\frac{U_{j+1}^n-U_{j-1}^n}{2h},\quad (U_j^n)_{\ddot{x}}=\frac{U_{j+2}^n-U_{j-2}^n}{4h},$$

$$(U_j^n)_{\hat{t}}=\frac{U_j^{n+1}-U_j^{n-1}}{2\tau},\quad (\bar{U}_j^n)=\frac{U_j^{n+1}+U_j^{n-1}}{2},$$

$$\langle U^n,V^n\rangle=h\sum_{J-1}^{j=1}U_j^nV_j^n,\quad \|U^n\|^2=\langle U^n,U^n\rangle,$$

$$\|U^n\|_\infty=\max_{1\leqslant j\leqslant J-1}|U_j^n|.$$

Consider the following difference scheme for (3):

$$(U_j^n)_{\hat{t}}+\frac{5}{3}(U_j^n)_{xx\bar{x}\bar{x}\hat{t}}-\frac{2}{3}(U_j^n)_{x\bar{x}\hat{x}\hat{x}\hat{t}}+\frac{4}{3}(\bar{U}_j^n)_{\hat{x}}-\frac{1}{3}(\bar{U}_j^n)_{\ddot{x}}+\frac{3}{2}(\bar{U}_j^n)_{x\bar{x}\hat{x}}-\frac{1}{2}(\bar{U}_j^n)_{x\bar{x}\ddot{x}}+\frac{4}{9}[U_j^n(\bar{U}_j^n)_{\hat{x}}+(U_j^n\bar{U}_j^n)_{\hat{x}}]-\frac{1}{9}[U_j^n(\bar{U}_j^n)_{\ddot{x}}+(U_j^n\bar{U}_j^n)_{\ddot{x}}]=0,$$

$$j=1,2,\ldots,J-1;n=1,2,\ldots,N-1 \tag{6}$$

$$U_j^0=u_0(x_j),\ j=0,1,2,\ldots,J \tag{7}$$

$$U_j^1+\frac{5}{3}(U_j^1)_{xx\bar{x}\bar{x}}-\frac{2}{3}(U_j^1)_{x\bar{x}\hat{x}\hat{x}}=u_0(x_j)+\frac{\partial^4u_0}{\partial x^4}(x_j)-\tau\frac{\partial u_0}{\partial x}(x_j)-\tau\frac{\partial^3u_0}{\partial x^3}(x_j)-\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j),j=1,2,\ldots,J-1 \tag{8}$$

$$U^n\in Z_h^0,(U_0^n)_{\hat{x}}=(U_J^n)_{\hat{x}}=0,(U_0^n)_{x\bar{x}}=(U_J^n)_{x\bar{x}}=0,n=0,1,2,\ldots,N \tag{9}$$

The discrete boundary condition (9) is reasonable from the homogeneous boundary condition in (3). Define the following two bilinear functions:

$$\psi(U_j^n,\bar{U}_j^n)=\frac{4}{9}[U_j^n(\bar{U}_j^n)_{\hat{x}}+(U_j^n\bar{U}_j^n)_{\hat{x}}],$$

$$\xi(U_j^n,\bar{U}_j^n)=\frac{1}{9}[U_j^n(\bar{U}_j^n)_{\ddot{x}}+(U_j^n\bar{U}_j^n)_{\ddot{x}}].$$

Lemma 2.1[17] For any discrete functions $U,V\in Z_h^0$, we have

$$\langle U_x,V\rangle=-\langle U,V_{\bar{x}}\rangle,\ \langle U_{x\bar{x}},V\rangle=-\langle U_x,V_x\rangle.$$

Therefore,

$$\|U_x\|^2=\|U_{\bar{x}}\|^2,\ \langle U_{x\bar{x}},U\rangle=-\langle U_x,U_x\rangle=-\|U_x\|^2 \tag{10}$$

Moreover, if $(U_0)_{x\bar{x}}=(U_J)_{x\bar{x}}=0$ then

$$\langle U_{xx\bar{x}\bar{x}},U\rangle=\|U_{xx}\|^2.$$

Lemma 2.2[18] If $U\in Z_h^0$, then from Cauchy-Schwarz inequality and summation by parts[16], we have $\|U_{\ddot{x}}\|^2\leqslant\|U_{\hat{x}}\|^2\leqslant\|U_x\|^2$.

The following theorem shows how the difference scheme (6)~(9) simulates the conservative law numerically.

Theorem 2.3 The difference scheme (6)~(9) is conservative for discrete energy, that is

$$Q^n=\frac{h}{2}\sum_{j=1}^{J-1}(U_j^{n+1}+U_j^n)+\frac{2h}{9}\tau\sum_{j=1}^{J-1}(U_j^nU_j^{n+1})_{\hat{x}}-\frac{h}{18}\tau\sum_{j=1}^{J-1}(U_j^nU_j^{n+1})_{\ddot{x}}=Q^{n-1}=\cdots=Q^0 \tag{11}$$

$$E^n=\frac{1}{2}(\|U^{n+1}\|^2+\frac{5}{3}\|U_{xx}^{n+1}\|^2-\frac{2}{3}\|U_{x\hat{x}}^{n+1}\|^2+\|U^n\|^2+\frac{5}{3}\|U_{xx}^n\|^2-\frac{2}{3}\|U_{x\hat{x}}^n\|^2)=E^{n-1}=\cdots=E^0 \tag{12}$$

Proof Multiplying h on both sides of (6) and taking summation of j, we obtain from (9) and Lemma 2.1 that

$$h\sum_{j=1}^{J-1}\frac{U_j^{n+1}-U_j^{n-1}}{2\tau}+\frac{4}{9}h\sum_{j=1}^{J-1}U_j^n(\bar{U}_j^n)_{\hat{x}}-\frac{1}{9}h\sum_{j=1}^{J-1}U_j^n(\bar{U}_j^n)_{\ddot{x}}=0 \tag{13}$$

On the other hand,

$$h\sum_{j=1}^{J-1}U_j^n(\bar{U}_j^n)_{\hat{x}}=\frac{h}{2}\sum_{j=1}^{J-1}U_j^n(U_j^{n+1})_{\hat{x}}+\frac{h}{2}\sum_{j=1}^{J-1}U_j^n(U_j^{n-1})_{\hat{x}}=\frac{h}{2}\sum_{j=1}^{J-1}U_j^n(U_j^{n+1})_{\hat{x}}-$$

$$\frac{h}{2}\sum_{j=1}^{J-1}U_j^{n-1}(U_j^n)_{\hat{x}},$$

$$h\sum_{j=1}^{J-1}U_j^n(\overline{U_j}^n)_{\ddot{x}}=\frac{h}{2}\sum_{j=1}^{J-1}U_j^n(U_j^{n+1})_{\ddot{x}}+$$

$$\frac{h}{2}\sum_{j=1}^{J-1}U_j^n(U_j^{n-1})_{\ddot{x}}=\frac{h}{2}\sum_{j=1}^{J-1}U_j^n(U_j^{n+1})_{\ddot{x}}-$$

$$\frac{h}{2}\sum_{j=1}^{J-1}U_j^{n-1}(U_j^n)_{\ddot{x}}.$$

Substituting them into (13), we obtain (11) by deducing.

Then, by taking the inner product of (6) with $2\bar{U}^n$, it follows from (9) and Lemma 2.1 that

$$\|U^n\|_{\hat{t}}^2+\frac{5}{3}\|U_{xx}^n\|_{\hat{t}}^2-\frac{2}{3}\|U_{x\hat{x}}^n\|_{\hat{t}}^2+$$

$$\frac{8}{3}\langle\bar{U}_{\hat{x}}^n,\bar{U}^n\rangle-\frac{2}{3}\langle\bar{U}_{\ddot{x}}^n,\bar{U}^n\rangle+3\langle\bar{U}_{x\bar{x}\hat{x}}^n,\bar{U}^n\rangle-$$

$$\langle\bar{U}_{x\bar{x}\ddot{x}}^n,\bar{U}^n\rangle+2\langle\psi(U^n,\bar{U}^n),\bar{U}_n\rangle-$$

$$2\langle\xi(U^n,\bar{U}^n),\bar{U}_n\rangle=0 \tag{14}$$

Note that

$$\langle\bar{U}_{\hat{x}}^n,\bar{U}^n\rangle=0,\ \langle\bar{U}_{\ddot{x}}^n,\bar{U}^n\rangle=0,$$

$$\langle\bar{U}_{x\bar{x}\hat{x}}^n,\bar{U}^n\rangle=0,\ \langle\bar{U}_{x\bar{x}\ddot{x}}^n,\bar{U}^n\rangle=0,$$

$$\langle\psi(U^n,\bar{U}^n),\bar{U}^n\rangle=$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}[U_j^n(\bar{U}_j^n)_{\hat{x}}+(U_j^n\bar{U}_j^n)_{\hat{x}}]\bar{U}_j^n=$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}U_j^n\bar{U}_j^n(\bar{U}_j^n)_{\hat{x}}+$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}(U_j^n\bar{U}_j^n)_{\hat{x}}\bar{U}_j^n=$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}(U_j^n\bar{U}_j^n)(\bar{U}_j^n)_{\hat{x}}-$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}U_j^n\bar{U}_j^n(\bar{U}_j^n)_{\hat{x}}=0,$$

and

$$\langle\xi(U^n,\bar{U}^n),\bar{U}^n\rangle=$$

$$\frac{1}{9}h\sum_{j=1}^{J-1}[U_j^n(\bar{U}_j^n)_{\ddot{x}}+(U_j^n\bar{U}_j^n)_{\ddot{x}}]\bar{U}_j^n=$$

$$\frac{1}{9}h\sum_{j=1}^{J-1}U_j^n\bar{U}_j^n(\bar{U}_j^n)_{\ddot{x}}+\frac{1}{9}h\sum_{j=1}^{J-1}(U_j^n\bar{U}_j^n)_{\ddot{x}}\bar{U}_j^n=$$

$$\frac{1}{9}h\sum_{j=1}^{J-1}(U_j^n\bar{U}_j^n)(\bar{U}_j^n)_{\ddot{x}}-$$

$$\frac{1}{9}h\sum_{j=1}^{J-1}U_j^n\bar{U}_j^n(\bar{U}_j^n)_{\ddot{x}}=0.$$

So one can obtain by taking them into (14) that

$$\frac{1}{2\tau}(\|U^{n+1}\|^2-\|U^{n-1}\|^2)+$$

$$\frac{5}{6\tau}(\|U_{xx}^{n+1}\|^2-\|U_{xx}^{n-1}\|^2)-$$

$$\frac{1}{3\tau}(\|U_{x\hat{x}}^{n+1}\|^2-\|U_{x\hat{x}}^{n-1}\|^2)=0 \tag{15}$$

By the definition of E^n, we obtain (12) by deducing (15) about n.

3 Solvability of the difference scheme

Theorem 3.1 The difference scheme (6)~(9) is uniquely solvable.

Proof We will use the mathematical induction to prove the result. It is easy to see that U^0 and U^1 are determined uniquely by (7) and (8), respectively. Let $U^0, U^1, \cdots, U^{n-1}, U^n\ (n\leqslant N-1)$ be the unique solution of difference scheme (6)~(9). Now we consider U^{n+1} in (6). We have

$$\frac{1}{2\tau}U_j^{n+1}+\frac{5}{6\tau}(U_j^{n+1})_{xx\bar{x}\bar{x}}-\frac{1}{3\tau}(U_j^{n+1})_{x\bar{x}\hat{x}\hat{x}}+$$

$$\frac{2}{3}(U_j^{n+1})_{\hat{x}}-\frac{1}{6}(U_j^{n+1})_{\ddot{x}}+\frac{3}{4}(u_j^{n+1})_{x\bar{x}x}-$$

$$\frac{1}{4}(u_j^{n+1})_{x\bar{x}x}+\frac{1}{2}\psi(U_j^n,U_j^{n+1})-$$

$$\frac{1}{2}\xi(U_j^n,U_j^{n+1})=0 \tag{16}$$

Taking the inner product of (16) with U^{n+1}, it follows from the boundary condition (9) and Lemma 2.1 that

$$\frac{1}{2\tau}\|U^{n+1}\|^2+\frac{5}{6\tau}\|U_{xx}^{n+1}\|^2-\frac{1}{3\tau}\|U_{x\hat{x}}^{n+1}\|^2+$$

$$\frac{2}{3}\langle U_{\hat{x}}^{n+1},U^{n+1}\rangle-\frac{1}{6}\langle U_{\ddot{x}}^{n+1},U^{n+1}\rangle+$$

$$\frac{3}{4}\langle U_{x\bar{x}\hat{x}}^{n+1},U^{n+1}\rangle-\frac{1}{4}\langle U_{x\bar{x}\ddot{x}}^{n+1},U^{n+1}\rangle+$$

$$\frac{1}{2}\langle\psi(U^n,U^{n+1}),U^{n+1}\rangle-$$

$$\frac{1}{2}\langle\xi(U^n,U^{n+1}),U^{n+1}\rangle=0 \tag{17}$$

Noticing that

$$\langle U_{\hat{x}}^{n+1},U^{n+1}\rangle=0,\langle U_{\ddot{x}}^{n+1},U^{n+1}\rangle=0,$$

$$\langle U_{x\bar{x}\hat{x}}^{n+1},U^{n+1}\rangle=0,\langle U_{x\bar{x}\ddot{x}}^{n+1},U^{n+1}\rangle=0,$$

$$\langle\psi(U^n,U^{n+1}),U^{n+1}\rangle=$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}U_j^nU_j^{n+1}(U_j^{n+1})_{\hat{x}}+$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}(U_j^nU_j^{n+1})_{\hat{x}}U_j^{n+1}=$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}U_j^nU_j^{n+1}(U_j^{n+1})_{\hat{x}}-$$

$$\frac{4}{9}h\sum_{j=1}^{J-1}U_j^nU_j^{n+1}(U_j^{n+1})_{\hat{x}}=0,$$

and

$$\langle\xi(U^n,U^{n+1}),U^{n+1}\rangle=$$
$$\frac{1}{9}h\sum_{j=1}^{J-1}U_j^nU_j^{n+1}(U_j^{n+1})_{\ddot{x}}+$$
$$\frac{1}{9}h\sum_{j=1}^{J-1}(U_j^nU_j^{n+1})_{\ddot{x}}U_j^{n+1}=$$
$$\frac{1}{9}h\sum_{j=1}^{J-1}(U_j^nU_j^{n+1})(U_j^{n+1})_{\ddot{x}}-$$
$$\frac{1}{9}h\sum_{j=1}^{J-1}U_j^nU_j^{n+1}(U_j^{n+1})_{\ddot{x}}=0,$$

by substituting above results into (17), we get

$$\|U^{n+1}\|^2+\frac{5}{3}\|U_{xx}^{n+1}\|^2-\frac{2}{3}\|U_{x\hat{x}}^{n+1}\|^2=0.$$

Then, from Lemma 2.2, we have

$$\|U_{x\hat{x}}^{n+1}\|^2\leqslant\|U_{xx}^{n+1}\|^2,$$

that is to say,

$$\|U^{n+1}\|^2+\|U_{xx}^{n+1}\|^2\leqslant 0.$$

Thus (16) only admits zero solution and there exits unique U_j^{n+1} satisfies (6)～(9).

4　Convergence and stability of the difference scheme

In this seetion we study the convergence and stability of the difference scheme (6)～(9) by using the discrete functional analysis method. The truncation error of the difference scheme (6)～(9) is defined as follows.

$$r_j^n=(u_j^n)_{\hat{t}}+\frac{5}{3}(u_j^n)_{x\bar{x}\bar{x}\hat{t}}-\frac{2}{3}(u_j^n)_{x\bar{x}\hat{x}\hat{x}\hat{t}}+$$
$$\frac{4}{3}(\bar{U}_j^n)_{\hat{x}}-\frac{1}{3}(\bar{U}_j^n)_{\ddot{x}}+\frac{3}{2}(\bar{U}_j^n)_{x\bar{x}\hat{x}}-$$
$$\frac{1}{2}(\bar{U}_j^n)_{x\bar{x}\ddot{x}}+\psi(u_j^n,\bar{U}_j^n)-\xi(u_j^n,\bar{U}_j^n),$$
$$j=1,2,\ldots,J-1;n=1,2,\ldots,N-1 \tag{18}$$

$$u_j^0=u_0(x_j),j=0,1,2,\ldots,J \tag{19}$$

$$u_j^1+\frac{5}{3}(u_j^1)_{x\bar{x}x\bar{x}}-\frac{2}{3}(u_j^1)_{x\bar{x}\hat{x}\hat{x}}=u_0(x_j)+$$
$$\frac{\partial^4u_0}{\partial x^4}(x_j)-\tau\frac{\partial u_0}{\partial x}(x_j)-\tau\frac{\partial^3u_0}{\partial x^3}(x_j)-$$
$$\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j)+r_j^0,j=1,2,\ldots,J-1 \tag{20}$$

$$u^n\in Z_h^0,(u_0^n)_{\hat{x}}=(u_J^n)_{\hat{x}}=0,$$
$$(u_0^n)_{x\bar{x}}=(u_J^n)_{x\bar{x}}=0,n=0,1,2,\ldots,N \tag{21}$$

Suppose that the solution of (3) is smooth sufficiently. By using the Taylor expansion, we know that

$$|r_j^n|=O(\tau^2+h^4) \tag{22}$$

Lemma 4.1[8]　Suppose that $u_0\in H_0^2[x_L,x_R]$. Then the solution of (3) satisfies

$$\|u\|_{L_2}\leqslant C,\ \|u_x\|_{L_2}\leqslant C,\ \|u_{xx}\|_{L_2}\leqslant C,$$
$$\|u\|_{L_\infty}\leqslant C,\ \|u_x\|_{L_\infty}\leqslant C.$$

Theorem 4.2　Suppose that $u_0\in H_0^2[x_L,x_R]$. Then the solution to difference scheme (6)～(9) satisfies

$$\|U^n\|\leqslant C,\ \|U_x^n\|\leqslant C,\ \|U_{xx}^n\|\leqslant C,$$
$$\|U^n\|_\infty\leqslant C,\ \|u_x^n\|_\infty\leqslant C,n=1,2,\ldots,N.$$

Proof　From Lemma 2.2 we have

$$\|U_{x\hat{x}}^n\|^2\leqslant\|U_{xx}^n\|^2,\ \|U_{x\hat{x}}^{n+1}\|^2\leqslant\|U_{xx}^{n+1}\|^2.$$

It follows from Theorem 2.3 that

$$\frac{1}{2}(\|U^{n+1}\|^2+\|U_{xx}^{n+1}\|^2+\|U^n\|^2+$$
$$\|U_{xx}^n\|^2)\leqslant E^n=E^0=C,$$

that is, $\|U^n\|\leqslant C$, $\|U_{xx}^n\|\leqslant C$. Then, from (10) and Cauchy-Schwarz inequality, one obtains

$$\|U_x^n\|^2\leqslant\|U^n\|\cdot\|U_{xx}^n\|\leqslant$$
$$\frac{1}{2}(\|U^n\|^2+\|U_{xx}^n\|^2),$$

which yields $\|U_x^n\|\leqslant C$. Therefore, from discrete Sobolev inequality[17], we get $\|U^n\|_\infty\leqslant C$, $\|U_x^n\|_\infty\leqslant C$.

Theorem 4.3　Suppose that the solution of (3) is smooth sufficiently and $u_0\in H_0^2[x_L,x_R]$. Then the solution $\{U^n\}$ converges to the solution of (3) with convergent rate $O(\tau^2+h^4)$ in the sense of norm $\|\cdot\|_\infty$.

Proof　Subtracting (6)～(9) from (18)～(21), we get

$$r_j^n=(e_j^n)_{\hat{t}}+\frac{5}{3}(e_j^n)_{x\bar{x}\bar{x}\hat{t}}-\frac{2}{3}(e_j^n)_{x\bar{x}\hat{x}\hat{x}\hat{t}}+$$
$$\frac{4}{3}(\bar{e}_j^n)_{\hat{x}}-\frac{1}{3}(\bar{e}_j^n)_{\ddot{x}}+\frac{3}{2}(\bar{e}_j^n)_{x\bar{x}\hat{x}}-$$
$$\frac{1}{2}(\bar{e}_j^n)_{x\bar{x}\ddot{x}}+\psi(u_j^n,\bar{U}_j^n)-\psi(U_j^n,\bar{U}_j^n)-$$
$$\xi(u_j^n,\bar{U}_j^n)+\xi(U_j^n,\bar{U}_j^n)$$
$$j=1,2,\ldots,J-1,n=1,2,\ldots,N-1 \tag{23}$$

$$e_j^0=0,j=0,1,2,\ldots,J-1 \tag{24}$$

$$e_j^1+\frac{5}{3}(e_j^1)_{xx\bar{x}\bar{x}}-\frac{2}{3}(e_j^1)_{x\bar{x}\hat{x}\hat{x}}=r_j^0,$$

$$j=1,2,\ldots,J-1 \tag{25}$$

$$e^n\in Z_h^0,(e_0^n)_{\hat{x}}=(e_J^n)_{\hat{x}}=0,(e_0^n)_{x\bar{x}}=(e_J^n)_{x\bar{x}}=0,\ n=0,1,2,\ldots,N \tag{26}$$

Taking the inner product on both sides of (24) with e^1, we obtain from boundary condition (23) and Lemma 2.1 that

$$\|e^1\|^2+\frac{5}{3}\|e_{xx}^1\|^2-\frac{2}{3}\|e_{x\hat{x}}^1\|^2=\langle r^0,e^1\rangle.$$

From Lemma 2.2, we have

$$\|e_{x\hat{x}}^n\|^2\leqslant\|e_{xx}^n\|^2 \tag{27}$$

Again, from (22) the Cauchy-Schwarz inequality and (27), one concludes that

$$\|e^1\|^2+\|e_{xx}^1\|^2\leqslant O(\tau^2+h^4)^2 \tag{28}$$

Taking the inner product on both sides of (23) with $2\bar{e}^n$, we get from boundary condition (26) and Lemma 2.1 that

$$\begin{aligned}\langle r^n,2\bar{e}^n\rangle=&\|e^n\|_{\hat{t}}^2+\frac{5}{3}\|e_{xx}^n\|_{\hat{t}}^2-\frac{2}{3}\|e_{x\hat{x}}^n\|_{\hat{t}}^2+\\&\frac{8}{3}\langle\bar{e}_{\hat{x}}^n,\bar{e}^n\rangle-\frac{2}{3}\langle\bar{e}_{\ddot{x}}^n,\bar{e}^n\rangle+3\langle\bar{e}_{x\bar{x}\hat{x}}^n,\bar{e}^n\rangle-\\&\langle\bar{e}_{x\bar{x}\ddot{x}}^n,\bar{e}^n\rangle+2\langle\psi(u^n,\bar{U}^n)-\psi(U^n,\bar{U}^n),\\&\bar{e}^n\rangle-2\langle\xi(u^n,\bar{U}^n)-\xi(U^n,\bar{U}^n),\bar{e}^n\rangle\end{aligned} \tag{29}$$

Similar to (14), we get

$$\langle\bar{e}_{\hat{x}}^n,\bar{e}^n\rangle=0,\langle\bar{e}_{\ddot{x}}^n,\bar{e}^n\rangle=0,\ \langle\bar{e}_{x\bar{x}\hat{x}}^n,\bar{e}^n\rangle=0,\langle\bar{e}_{x\bar{x}\ddot{x}}^n,\bar{e}^n\rangle=0 \tag{30}$$

From Lemma 4.1, Theorem 4.2, Lemma 2.2 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\langle\psi(u^n,\bar{U}^n)-\psi(U^n,\bar{U}^n),\bar{e}^n\rangle=\\&\frac{4}{9}h\sum_{j=1}^{J-1}[u_j^n(\bar{U}_j^n)_{\hat{x}}-U_j^n(\bar{U}_j^n)_{\hat{x}}]\bar{e}_j^n+\\&\frac{4}{9}h\sum_{j=1}^{J-1}[(u_j^n\bar{U}_j^n)_{\hat{x}}-(U_j^n\bar{U}_j^n)_{\hat{x}}]\bar{e}_j^n=\\&\frac{4}{9}h\sum_{j=1}^{J-1}[e_j^n(\bar{U}_j^n)_{\hat{x}}+U_j^n(\bar{e}_j^n)_{\hat{x}}]\bar{e}_j^n-\\&\frac{4}{9}h\sum_{j=1}^{J-1}(e_j^n\bar{U}_j^n+U_j^n\bar{e}_j^n)(\bar{e}_j^n)_{\hat{x}}\leqslant\\&C(\|e^n\|^2+\|\bar{e}^n\|^2+\|\bar{e}_{\hat{x}}^n\|^2)\leqslant\\&C(\|e^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2+\\&\|e_x^{n+1}\|^2+\|e_x^{n-1}\|^2)\end{aligned} \tag{31}$$

$$\begin{aligned}&\langle\xi(u^n,\bar{U}^n)-\xi(U^n,\bar{U}^n),\bar{e}^n\rangle=\\&\frac{1}{9}h\sum_{j=1}^{J-1}[u_j^n(\bar{U}_j^n)_{\ddot{x}}-U_j^n(\bar{U}_j^n)_{\ddot{x}}]\bar{e}_j^n+\\&\frac{1}{9}h\sum_{j=1}^{J-1}[(u_j^n\bar{U}_j^n)_{\ddot{x}}-(U_j^n\bar{U}_j^n)_{\ddot{x}}]\bar{e}_j^n=\\&\frac{1}{9}h\sum_{j=1}^{J-1}[u_j^n(\bar{U}_j^n)_{\ddot{x}}-U_j^n(\bar{U}_j^n)_{\ddot{x}}]\bar{e}_j^n-\\&\frac{1}{9}h\sum_{j=1}^{J-1}(u_j^n\bar{U}_j^n-U_j^n\bar{U}_j^n)(\bar{e}_j^n)_{\ddot{x}}=\\&\frac{1}{9}h\sum_{j=1}^{J-1}[e_j^n(\bar{U}_j^n)_{\ddot{x}}+U_j^n(\bar{e}_j^n)_{\ddot{x}}]\bar{e}_j^n-\\&\frac{1}{9}h\sum_{j=1}^{J-1}(e_j^n\bar{U}_j^n+U_j^n\bar{e}_j^n)(\bar{e}_j^n)_{\ddot{x}}\leqslant\\&C(\|e^n\|^2+\|\bar{e}^n\|^2+\|e_x^n\|^2)\leqslant\\&C(\|e^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2+\\&\|e_x^{n+1}\|^2+\|e_x^{n-1}\|^2)\end{aligned} \tag{32}$$

and

$$\langle r^n,2\bar{e}^n\rangle=\langle r^n,e^{n+1}+e^{n-1}\rangle\leqslant\|r^n\|^2+\|e^{n+1}\|^2+\|e^{n-1}\|^2 \tag{33}$$

Substituting (30)~(33) into (29), one gets

$$\begin{aligned}&\|e^n\|_{\hat{t}}^2+\frac{5}{3}\|e_{xx}^n\|_{\hat{t}}^2-\frac{2}{3}\|e_{x\hat{x}}^n\|_{\hat{t}}^2\leqslant\|r^n\|^2+\\&C(\|e^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2+\\&\|e_x^{n+1}\|^2+\|e_x^{n-1}\|^2)\end{aligned} \tag{34}$$

Similarly, we have

$$\|e_x^{n-1}\|^2\leqslant\frac{1}{2}(\|e^{n-1}\|^2+\|e_{xx}^{n-1}\|^2),$$

$$\|e_x^n\|^2\leqslant\frac{1}{2}(\|e^n\|^2+\|e_{xx}^n\|^2),$$

$$\|e_x^{n+1}\|^2\leqslant\frac{1}{2}(\|e^{n+1}\|^2+\|e_{xx}^{n+1}\|^2) \tag{35}$$

As a result, (34) can be rewritten into

$$\begin{aligned}&\|e^n\|_{\hat{t}}^2+\frac{5}{3}\|e_{xx}^n\|_{\hat{t}}^2-\frac{2}{3}\|e_{x\hat{x}}^n\|_{\hat{t}}^2\leqslant\|r^n\|^2+\\&C(\|e^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2+\\&\|e_{xx}^{n+1}\|^2+\|e_{xx}^{n-1}\|^2)\end{aligned} \tag{36}$$

Let

$$\begin{aligned}B^n=&\|e^{n+1}\|^2+\|e^n\|^2+\frac{5}{3}\|e_{xx}^{n+1}\|^2+\\&\frac{5}{3}\|e_{xx}^n\|^2-\frac{2}{3}\|e_{x\hat{x}}^{n+1}\|^2-\frac{2}{3}\|e_{x\hat{x}}^n\|^2.\end{aligned}$$

Multiplying 2τ on both sides of (36) and taking summation from 1 to n, we get

$$\begin{aligned}B^n\leqslant&B^0+2\tau\sum_{l=1}^{n}\|r^l\|^2+\\&C\tau\sum_{l=0}^{n+1}(\|e^l\|^2+\|e_{xx}^l\|^2)\end{aligned} \tag{37}$$

From (22), we have

$$\tau\sum_{l=1}^{n}\|r^l\|^2\leqslant n\tau\max_{1\leqslant l\leqslant n}\|r^l\|^2\leqslant T\cdot(\tau^2+h^4)^2.$$

On the other hand, it follows from (24) and (28) that $B^0=O(\tau^2+h^4)^2$. Similar to the proof of Theorem 4.2, we have

$$\|e_{\hat{x}\hat{x}}^{n+1}\|\leqslant\|e_{xx}^{n+1}\|,\ \|e_{\hat{x}\hat{x}}^{n}\|\leqslant\|e_{xx}^{n}\|.$$

It follows from inequality (37) that

$$\|e^{n+1}\|^2+\|e^n\|^2+\|e_{xx}^{n+1}\|^2+\|e_{xx}^n\|^2\leqslant B^n\leqslant O(\tau^2+h^4)^2+C\tau\sum_{l=0}^{n+1}(\|e^l\|^2+\|e_{xx}^l\|^2).$$

By the discrete Gronwall inequality[16],

$$\|e^n\|\leqslant O(\tau^2+h^4),\ \|e_{xx}^n\|\leqslant O(\tau^2+h^4).$$

From (35), we have $\|e_x^n\|\leqslant O(\tau^2+h^4)$. From the discrete Sobolev inequality[17], we get $\|e^n\|_\infty\leqslant O(\tau^2+h^4)$ and end the proof.

We can prove the following theorem in a similar way of Theorem 4.3.

Theorem 4.4 Under the hypotheses of Theorem 4.3, $\{U^n\}$ is stable in the sense of norm $\|\cdot\|_\infty$.

5 Numerical examples

As the difference scheme (6)~(9) is a linear system about u_j^{n+1}, it does not need any iteration. Obviously, the advantage of this scheme is that it can greatly save calculation time. Let $x_L=-70$, $x_R=100$, $T=40$, and

$$u_0(x)=\left(-\frac{35}{24}+\frac{35}{312}\sqrt{313}\right)\cdot\operatorname{sech}^4\left(\frac{1}{24}\sqrt{-26+2\sqrt{313}}\,x\right).$$

For some different value of τ and h, we list errors at several time in Tab. 1 and verify the accuracy of the difference scheme in Tab. 2. The numerical simulation of two conservative quantities (4) and (5) is listed in Tab. 3. The stability and convergence of the scheme are verified by these numerical examples. It shows that our proposed scheme is effective and reliable.

Tab. 1　The error estimates of the numerical solution at different time

	$\tau=0.4,\ h=0.2$		$\tau=h=0.1$		$\tau=0.025,\ h=0.05$	
	$\|e^n\|$	$\|e^n\|_\infty$	$\|e^n\|$	$\|e^n\|_\infty$	$\|e^n\|$	$\|e^n\|_\infty$
$t=10$	1.12908e−2	4.52718e−3	6.98904e−4	2.79441e−4	4.7300e−5	1.75040e−5
$t=20$	1.87579e−2	7.08299e−3	1.15263e−3	4.32932e−4	7.19301e−5	2.70500e−5
$t=30$	2.40018e−2	8.81077e−3	1.47057e−3	5.36978e−4	9.16277e−5	3.34997e−5
$t=40$	2.79798e−2	1.01168e−2	1.71214e−3	6.15605e−4	1.06520e−4	3.83553e−5

Tab. 2　The numerical verification of the theoretical accuracy $O(\tau^2+h^4)$

	$\|e^n(h,\tau)\|/\|e^{4n}\left(\frac{h}{2},\frac{\tau}{4}\right)\|$			$\|e^n(h,\tau)\|_\infty/\|e^{4n}\left(\frac{h}{2},\frac{\tau}{4}\right)\|_\infty$		
	$\tau=0.4$ $h=0.2$	$\tau=0.1$ $h=0.1$	$\tau=0.025$ $h=0.05$	$\tau=0.4$ $h=0.2$	$\tau=0.1$ $h=0.1$	$\tau=0.025$ $h=0.05$
$t=10$	—	16.1551	15.9822	—	16.2008	15.9644
$t=20$	—	16.2740	16.0243	—	16.3605	16.0048
$t=30$	—	16.3214	16.0494	—	16.4080	16.0293
$t=40$	—	16.3420	16.0734	—	16.4340	16.0500

Tab. 3　The numerical conservative quantities Q^n and E^n

	$\tau=0.1,\ h=0.1$		$\tau=0.025,\ h=0.05$	
	Q^n	E^n	Q^n	E^n
$t=10$	5.498 286 66	1.989 783 48	5.498 179 66	1.989 782 39
$t=20$	5.498 286 71	1.989 783 33	5.498 180 24	1.989 782 58
$t=30$	5.498 286 71	1.989 783 69	5.498 179 74	1.989 782 22
$t=40$	5.498 286 56	1.989 783 97	5.498 179 23	1.989 781 86

6 Conclusions

The purpose of this paper is to study the conservative difference scheme for the initial-boundary value problem of Rosenau-KDV equation. By using the Richardson extrapolation, this scheme can improve the spatial accuracy to the fourth order. At the same time, due to the three-layer framework, the scheme does not require any nonlinear iteration, and thus greatly improves the computational efficiency. Theoretically, the energy stability, existence and uniqueness of the numerical solution are given, and the convergence and stability of the numerical scheme are also proved. Finally, numerical examples also verify the theoretical analysis of the proposed scheme.

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