具有非线性传导率的麦克斯韦方程的一个保能量混合有限元

刘莎莎，吴永科
(电子科技大学数学科学学院，成都 611731)

摘　要：本文针对具有非线性传导率的麦克斯韦方程构造了一个保能量的混合有限元. 其中对麦克斯韦方程的一阶形式，本文直接使用有限元外微分去离散空间变量，得到保能量的半离散格式，进而通过一个二阶连续时间 Galerkin 方法 (CTG)去离散半离散格式的时间变量，得到保能量的全离散格式. 本文中的半离散和全离散格式能够精确地保持磁场的严格无散条件，具有最优收敛阶. 数值算例验证了理论结果.

关键词：麦克斯韦方程；非线性传导率；混合有限元；能量守恒；最优误差估计
中图分类号：O241.82　　**文献标识码**：A　　DOI：10.19907/j.0490-6756.2022.051003

An energy preserving mixed finite element for Maxwell's equations with nonlinear conductivity

LIU Sha-Sha, *WU Yong-Ke*
(School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China)

Abstract: An energy preserving mixed finite element is constructed to solve the Maxwell's equations with nonlinear conductivity. This finite element is obtained by discretizing the first-order formulation of Maxwell's equation in space based on the finite element exterior calculus as well as a continuous-time Galerkin method, which can be viewed as a modification of the Crank-Nicolson method, is used to discretize the time. Then we obtain a full discrete scheme preserving the total energy exactly when the source term is vanished. The mixed finite element method can preserve the magnetic Gauss law exactly. Based on a projection-based quasi-interpolation operator, the optimal order convergence of the method is established. Finally, numerical examples are presented to exemplify the theoretical results.

Keywords: Maxwell's equation; Nonlinear conductivity; Mixed finite element; Energy preserving; Optional error estimate
(2010 MSC 65M60)

1 Introduction

In this paper, we consider the energy preserving mixed finite element to solve the Maxwell's equation. That is to say, find

$$E:(0,T]\rightarrow H_0(\mathrm{curl})\cap H(\mathrm{div})$$

and

$$B:(0,T]\rightarrow H_0(\mathrm{div})\cap H(\mathrm{curl})$$

收稿日期：2022-03-08
基金项目：四川省中央引导地方科技发展专项项目(2021ZYD0006)
作者简介：刘莎莎(1997－)，女，四川广安人，硕士研究生，主要研究方向为偏微分方程数值解. E-mail：1258246754@qq.com
通讯作者：吴永科. E-mail：wuyongke1982@sina.com

satisfying

$$\begin{cases} \partial_t E + \sigma(|E|)E - \nabla \times B = f \text{ in } \Omega \times (0,T], \\ \partial_t B + \nabla \times E = 0 \text{ in } \Omega \times (0,T], \\ \nabla \cdot E = \rho \text{ in } \Omega \times (0,T], \\ \nabla \cdot B = 0 \text{ in } \Omega \times (0,T] \end{cases} \quad (1)$$

with homogeneous boundary conditions

$$E \times n = 0,\ B \cdot n = 0,\ \text{on } \partial\Omega \times (0,T],$$

and initial conditions $E(\cdot,0) = E_0(\cdot)$ and $B(\cdot,0) = B_0(\cdot)$ in Ω. Here $\Omega \subset \mathbf{R}^3$ is a bounded domain homomorphism to a ball with Lipschitz piecewise smooth boundary and n denotes the outward unite norm vector of $\partial\Omega$. The unknown E and B are time dependent functions on the domain Ω, which denote the electric field and the magnetic field respectively. The known functions f and ρ denote the conductivity current density and the charge density respectively. T is a finite positive real number denoting the final time. The vector function $B_0(\cdot)$ satisfying the condition div $B_0 = 0$. We assume that Ω is occupied by a nonlinear conducting material with electric conductivity $\sigma(|E|)$ which is supposed to be a monotone function of the power law form $\sigma(E) = |E|^{\alpha-1}$ with $\alpha \in (0,1)$. As proved in Ref. [1], we have

$$(\sigma(|E_1|)E_1 - \sigma(|E_2|)E_2, E_1 - E_2) \geqslant 0, \quad \forall E_1, E_2 \in (L^2(\Omega))^3 \quad (2)$$

Maxwell′s equations with a power law form for the conductivity is used in a variety of physical models such as the conductive law for type-II superconductors[2] and modeling of the nonlinear conductivity of the charge density wave state of $NbSe_3$[3]. The existence and uniqueness of weak solution for the Maxwell′s equations (1) with a power law form for the conductivity $\sigma(|E|)$ was discussed in Ref. [4]. However, since the Maxwell′s equations (1) are coupled nonlinear partial differential equations, it is usually difficult to obtain their analytical solutions, and the only way to solve them is to numerically seek some approximation solutions. Fortunately, there are many works in this research fields [1, 5-8]. In Ref. [5], the authors gave a numerical scheme based on backward Euler discretization in time and curl-conforming finite element methods in space to solve (1). They proved its convergence based on the boundedness of the second derivative in the dual space by the Minty-Browder technique. In Ref. [1], the authors developed a fully discrete A-φ finite element method to solve (1) with power law form of conductivity based on backward Euler discretization in time and nodal finite element methods in space. In Ref. [7], the authors analyzed a second order numerical scheme for (1) by using the Nédelec finite element method. They treated the nonlinear conductivity explicitly and obtained an $O(\Delta t^2 + h^s)$ error estimate in the L^2 norm. Most of the existing works except Ref. [7] deal with (1) by eliminating the magnetic field B, and then transforming the Maxwell′s equation into curl-curl system with unknown electric field E. This seems difficult to preserve the total energy and the magnetic Gauss law exactly.

In this paper, we try to develop a natural-total-energy-preserving and magnetic-Gauss-law-preserving mixed finite element method for (1). We discretize the first order formulation of (1) directly by using the finite element exterior calculus in space and a continuous time Galerkin method which can be viewed as a modification of the Crank-Nicolson method to discrete the time variable. Comparing with the exiting work[7], our finite element can exactly preserve the total energy of the system for both semi- and full-discrete schemes. Also, our mixed finite element preserves the divergence-free condition of the magnetic field B exactly. Such condition is very important for Maxwell′s equations, since it means there is no magnetic monopole in the world.

The rest of this paper is organized as follows. In Section 2, we introduce several Sobolev spaces and give weak formulation of the equation (1) and obtain the energy inequality. In Section 3, we recall some finite element spaces and give the semi-discrete scheme of the equation. We show that the semi-discrete scheme preserves an energy similar as the continuous one and preserves the divergence-free condition of the magnetic field B exactly. We also give the optimal order error estimates for the semi-discrete scheme. In Section 4, the full-discrete scheme is given by employing a continuous times Galerkin method. We show that the full-discrete scheme preserves the energy and the divergence-free condition of the magnetic field exactly, and give the optimal order error estimates. In Section 5, some numerical examples are given to verify the theoretical results. In Section 6 we summarize our findings.

2 Preliminaries

In this section, we introduce several Sobolev spaces and give the weak formulation of the equation. For some nonnegative integer m, we denote by $H^m(\Omega)$ the usual m-th order Sobolev space with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$. In particular, $H^0(\Omega)=L^2(\Omega)$ denotes the space of square integrable functions on Ω, with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. For the vector spaces $(H^m(\Omega))^3$ and $(L^2(\Omega))^3$, we use the same notations of norm, semi-norm and inner product as those for the scalar cases. We also introduce the spaces

$$H(\mathrm{curl})=\{v\in(L^2(\Omega))^3: \mathrm{curl}v\in(L^2(\Omega))^3\},$$
$$H(\mathrm{div})=\{v\in(L^2(\Omega))^3: \mathrm{div}v\in L^2(\Omega)\}$$

and set

$$S:=H_0^1(\Omega)=\{v\in H^1(\Omega): v=0 \text{ on } \partial\Omega\},$$
$$U:=H_0(\mathrm{curl})=\{v\in H(\mathrm{curl}): v\times n=0 \text{ on } \partial\Omega\},$$
$$V:=H_0(\mathrm{div})=\{v\in H(\mathrm{div}): v\cdot n=0 \text{ on } \partial\Omega\},$$
$$W:=L_0^2(\Omega)=\{v\in L^2(\Omega): \int_\Omega v\,\mathrm{d}x=0\}.$$

For any scalar- or vector-valued space X defined on Ω, with norm $\|\cdot\|_X$, we set

$$L^p([0,T],X):=\{v:[0,T]\to X; \|v\|_{L^p(X)}<\infty\},$$

where

$$\|v\|_{L^p(X)}:=\begin{cases}\left(\int_0^T \|v(\cdot,t)\|_X^p\right)^{1/p}, & \text{if } 1\leqslant p<\infty,\\ \operatorname*{ess\,sup}_{0\leqslant t\leqslant T}\|v(\cdot,t)\|_X, & \text{if } p=\infty.\end{cases}$$

For simplicity, we set

$$L^p(X):=L^p([0,T];X).$$

For any integer $r\geqslant 0$, the spaces $H^r(X):=H^r([0,T];X)$ and $C^r(X):=C^r([0,T];X)$ can be defined similarly.

The weak formulation of the equation (1) reads: for given $f\in C^0((0,T),(L^2(\Omega))^3)$, find $(E,B)\in U\times V$ such that

$$\begin{cases}(\partial_t E,\psi)+(\sigma(|E|)E,\psi)-(B,\nabla\times\psi)=\\ \quad (f,\psi)\ \forall\psi\in U,\\ (\partial_t B,\varphi)+(\nabla\times E,\varphi)=0\ \ \forall\varphi\in V\end{cases}\tag{3}$$

with initial condition

$$E(\cdot,0)=E_0(\cdot),\ B(\cdot,0)=B_0(\cdot) \text{ in } \Omega.$$

Define

$$A: (U\cap H(\mathrm{div}))\times(V\cap H(\mathrm{curl}))\to (L^2(\Omega))^3\times(L^2(\Omega))^3$$

as

$$A=\begin{pmatrix}0 & -\nabla\times\\ \nabla\times & 0\end{pmatrix}\tag{4}$$

Denote $U=(B,E)^T\in U\times V$, $V=(\psi,\varphi)^T\in U\times V$, $g=(\sigma(|E|)E,0)^T$ and $F=(f,0)^T$. The weak formulation of (3) can be rewritten as

$$(U_t,V)+(AU,V)+(g,V)=(F,V),\quad \forall V\in U\times V\tag{5}$$

with the initial value

$$U(\cdot,0)=U_0(\cdot)=(E_0(\cdot),B_0(\cdot))^T.$$

Remark 1 Using the fact that $\nabla\times U\subset V$ and the second equation of (3), we have

$$\partial_t B+\nabla\times E=0\in V.$$

Taking divergence on the two sides of the above equation and note the fact $\nabla\cdot(\nabla\times)=0$, we have

$$\partial_t(\nabla\cdot B)=0.$$

Together with the assumption on B_0, we have $\nabla\cdot B(\cdot,t)=0$ for any $t\in[0,T]$.

Now we turn to the energy estimates of the weak formulation. We first follow Ref. [9] to introduce an inequality.

Lemma 2.1[9] Suppose that a real number x satisfies the quadratic inequality $x^2\leqslant\alpha x+\beta^2$ for α, $\beta\geqslant 0$ and $\alpha^2+\beta^2>0$. Then $x\leqslant\alpha+\beta$.

Theorem 2.2 Let $U=(E,B)^T$ be the solution of the weak formulation (3) or (5). Provided that $f\in L^1([0,T],(L^2(\Omega))^3)$, we have the following stability bound:

$$\sup_{0\leqslant s\leqslant T}\|U(\cdot,s)\|+\sqrt{2}\int_0^T(\sigma(|E|)E,E)\,\mathrm{d}s\leqslant \|U_0\|+2\int_0^T\|f(\cdot,s)\|\,\mathrm{d}s\tag{6}$$

Proof　Taking $V=U$ in (5), we have

$$(U_t(\cdot,s),U(\cdot,s))+(AU,U)+(g,U)=(F,U).$$

Using the fact that A is a skew-symmetric operator, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|U(\cdot,s)\|^2+(\sigma(|E|)E,E)=(F,U).$$

Integrating the above equation in the interval $(0,t)$ for any $t\in(0,T]$, we have

$$\|U(\cdot,s)\|^2+2\int_0^t(\sigma(|E|)E,E)\mathrm{d}s=$$

$$\|U_0\|^2+2\int_0^t(F(\cdot,s),U(\cdot,s))\mathrm{d}s\leqslant$$

$$\|U_0\|^2+2\sup_{0\leqslant s\leqslant T}\|U(\cdot,s)\|\int_0^T\|F(\cdot,s)\|\mathrm{d}s.$$

Using Lemma 2.2 with

$$x=\sup_{0\leqslant s\leqslant T}\|U(\cdot,s)\|+\sqrt{2}(\int_0^T(\sigma(|E|E,E)\mathrm{d}s)^{\frac{1}{2}},$$

the desired result follows.

Remark 2　From the proof of Theorem 2.3, we can see that when the source term f vanished, we always have

$$\|U(\cdot,s)\|^2+2\int_0^t(\sigma(|E|)E,E)\mathrm{d}s=\|U_0\|^2$$

for any $s\in[0,T]$, which means the energy of the system are preserved exactly.

3　Semi-discretization

In this section, we recall some finite element spaces and give the semi-discrete scheme of (1). We will show that the semi-discrete scheme preserves an energy similar to the continuous one and preserves the divergence-free condition of the magnetic field B exactly. We also give the optimal order error estimates for the semi-discrete scheme. It should be point out that, in the existing work[7], the authors ignored the semi-discretization of the nonlinear Maxwell's equations.

3.1　Finite element spaces

Let T_h be a quasi-uniform shape regular tetrahedron triangulation of Ω, we have the following finite element spaces with respect to the partition T_h:

$S_h\subset S$ is the Lagrange elements space with continuous piecewise polynomials[10];

$U_h\subset U$ is the edge element space[11, 12];

$V_h\subset V$ is the face element space[12-14];

$W_h\subset W$ is the discontinuous piecewise polynomial space[10];

We assume that the finite element spaces S_h, U_h, V_h and W_h contain all piecewise polynomials of order up to $l\geqslant 0$. Choosing proper types of the above finite element spaces such that the following diagram

$$\begin{array}{ccccccc} S & \xrightarrow{\text{grad}} & U & \xrightarrow{\text{curl}} & V & \xrightarrow{\text{div}} & W \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S_h & \xrightarrow[\text{grad}]{} & U_h & \xrightarrow[\text{curl}]{} & V_h & \xrightarrow[\text{div}]{} & W_h \end{array} \qquad (7)$$

is commute exact sequence in the sense that

ker(curl)=img(grad)

and

ker(div)=img(curl).

There are many finite element spaces satisfying (7), for example we can see Ref. [15]. We define the discrete weak operators $\nabla_h\cdot:U_h\rightarrow S_h$ and $\nabla_h\times:V_h\rightarrow U_h$ as the adjoint operators of $-\nabla$and $\nabla\times$, respectively, *i.e.*, define $\nabla_h u_h\in S_h$, subject to

$$(\nabla_h\cdot u_h,s_h)=-(u_h,\nabla s_h),\forall s_h\in S_h \qquad (8)$$

define $\nabla_h\times v_h\in U_h$, subject to

$$(\nabla_h\times v_h,u_h)=(v_h,\nabla\times u_h),\forall u_h\in U_h \qquad (9)$$

We introduce the projection-based quasi-interpolation operators $I_h^c:U\rightarrow U_h$ and $I_h^d:V\rightarrow V_h$ follows Refs. [16, 17]. These operators will play an important role in the error analysis and have the following properties.

Lemma 3.1[17]　The projection-based quasi-interpolation operators I_h^c and I_h^d have the following properties: For any $u\in U$ and $v\in V$,

$$(I_h^c u,\nabla s_h)=(u,\nabla s_h),\ \forall s_h\in S_h,$$

$$(I_h^d v,\nabla\times\varphi_h)=(v,\nabla\times\varphi_h),\ \forall\varphi_h\in U_h,$$

and

$$(\nabla\times I_h^c u,\nabla\times\psi_h)=(\nabla\times u,\nabla\times\psi_h)\ \forall\psi_h\in U_h,$$

$$(\nabla\cdot I_h^d v,\nabla\cdot\varphi_h)=(\nabla\cdot v,\nabla\cdot\varphi_h),\forall\varphi_h\in V_h.$$

For any $u\in U$ and $v\in V$,

$$\|\nabla\times I_h^c u\|\leqslant\|\nabla\times u\|,\ \|\nabla\cdot I_h^d v\|\leqslant\|\nabla\cdot v\|.$$

For any $u\in U\cap H(\mathrm{div})$ and $v\in V\cap H(\mathrm{curl})$,

$$\nabla_h\cdot I_h^c u=Q_h^g\nabla\cdot u,\nabla_h\times I_h^d v=Q_h^c\nabla\times v,$$

where $Q_h^g:L^2(\Omega)\to S_h$ and $Q_h^c:(L^2(\Omega))^3\to U_h$ are L^2 orthogonal projection operators. Therefore

$$\|\nabla_h\cdot I_h^c u\|\leqslant\|\nabla\cdot u\|,\|\nabla_h\times I_h^d v\|\leqslant\|\nabla\times v\|.$$

For any $u\in U\cap H^{l+1}(\Omega)$ and $v\in V\cap H^{l+1}(\Omega)$,

$$\|u-I_h^c u\|\lesssim h^r\|u\|_r,\ \text{for } 1\leqslant r\leqslant l+1,$$

$$\|v-I_h^d v\|\lesssim h^r\|v\|_r,\text{for } 1\leqslant r\leqslant l+1.$$

Furthermore, if $\nabla\times u\in H^{l+1}(\Omega)$ and $\nabla\cdot v\in H^{l+1}(\Omega)$, we have

$$\|\nabla\times(u-I_h^c u)\|\lesssim h^r\|\nabla\times u\|_r,\ \text{for } 1\leqslant r\leqslant l+1,$$

$$\|\nabla\cdot(v-I_h^d v)\|\lesssim h^r\|\nabla\cdot v\|_r,\ \text{for } 1\leqslant r\leqslant l+1.$$

3.2 Semi-discrete scheme

In this part we will give the discretization of (1) in space based on the finite element exterior calculus[18, 19]. The semi-discrete scheme reads as: find $U_h=(E_h,B_h)^T\in U_h\times V_h$ such that

$$\begin{cases}(\partial_t E_h,\psi_h)+(\sigma(|E_h|)E_h,\psi_h)-\\ \quad(B_h,\nabla\times\psi_h)=(f,\psi_h),\ \forall\psi_h\in U_h,\\ (\partial_t B_h,\varphi_h)+(\nabla\times E_h,\varphi_h)=0,\ \forall\varphi_h\in V_h\end{cases}\tag{10}$$

with initial condition

$$E_h(\cdot,0)=I_h^c E_0(\cdot),\ B_h(\cdot,0)=I_h^d B_0(\cdot)\text{ in }\Omega.$$

Define A_h: $U_h\times V_h\to U_h\times V_h$ as

$$A_h=\begin{pmatrix}0&-\nabla_h\times\\ \nabla\times&0\end{pmatrix}.$$

Denote

$$V_h=(\psi_h,\varphi_h)^T\in U_h\times V_h$$

and $g_h=(\sigma(|E_h|)E_h,0)^T$. The semi-discrete scheme (10) can be rewritten as: find $U_h=(E_h,B_h)^T\in U_h\times V_h$ such that

$$(\partial_t U_h,V_h)+(A_h U_h,V_h)+(g_h,V_h)=(F,V_h),\forall V_h\in U_h\times V_h\tag{11}$$

with the initial value

$$U_h(\cdot,0)=U_{h0}(\cdot)=(I_h^c E_0(\cdot),I_h^d B_0(\cdot))^T.$$

Same line as the proof of Theorem 2.3, we have the following energy estimate for the semi-discrete scheme (10).

Theorem 3.2 Let $U_h=(E_h,B_h)^T$ be the solution of the weak formulation (10) or (11). Provided that $f\in L^1([0,T],(L^2(\Omega))^3)$, we have the stability bound

$$\sup_{0\leqslant s\leqslant T}\|U_h(\cdot,s)\|+\sqrt{2}\int_0^T(\sigma(|E_h|)E_h,E_h)\mathrm{d}s\leqslant\|U_{h0}\|+2\int_0^T\|f(\cdot,s)\|\mathrm{d}s\tag{12}$$

Furthermore, if the source term $f=0$, we have the estimate

$$\|U_h(\cdot,s)\|^2+2\int_0^s(\sigma(|E_h|)E_h,E_h)\mathrm{d}s=\|U_{h0}\|^2,\ \forall s\in[0,T].$$

We also have the following divergence-free condition of the magnetic field B_h.

Theorem 3.3 Let $U_h=(E_h,B_h)^T$ be the solution of the weak formulation (10) or (11). For any $t\in[0,T]$, we have $\nabla\cdot B_h(\cdot,t)=0$.

Proof Using the fact that $\nabla\times E_h\in V_h$ and taking $\varphi_h=\partial_t B_h+\nabla\times E_h\in V_h$ in the second equation of (10), we get $\partial_t B_h+\nabla\times E_h=0$. Taking divergence on the two sides of the above equation and using the fact that $\nabla\cdot I_h^d B_0=Q_h\nabla\cdot B_0=0$, the desired result follows.

3.3 Error estimates

In this subsection, we will give the optimal order error estimates for the semi-discrete scheme. Firstly, we denote

$$I_h=\begin{pmatrix}I_h^c&0\\0&I_h^d\end{pmatrix},\ \xi=I_hU-U_h,\eta=I_hU-U.$$

Then, by the weak formulation (3), we have

$$(\partial_t I_hU,V_h)+(A_hI_hU,V_h)+(g,V_h)=(\partial_t\eta,V_h)+(A_hI_hU-AU,V_h)+(F,V_h)\tag{13}$$

for all $V_h=(\psi_h,\varphi_h)\in U_h\times V_h$. Using Lemma 3.1, we have

$$(A_hI_hU-AU,V_h)=(\nabla\times(I_h^cE-E),\varphi_h)-(I_h^dB-B,\nabla\times\psi_h)=(\nabla\times(I_h^cE-E),\varphi_h):=(G,V_h),$$

where $G=(\nabla\times(I_h^cE-E),\ 0)^T$. Subtracting (11) from (13), we get that $\xi\in U_h\times V_h$ satisfies

$$(\partial_t\xi,V_h)+(A_h\xi,V_h)+(g-g_h,V_h)=(\partial_t\eta+G,V_h),\forall V_h\in U_h\times V_h\tag{14}$$

with $\xi(\cdot,0)=U_0-I_hU_0$. We have the following estimates of the quality ξ.

Lemma 3.4 Let $U=(E,B)^T$ and $U_h=$

$(E_h, B_h)^T$ be the solutions of (3) (or (5)) and (10) (or (11)), respectively. Assume that $U_0 \in H^{l+1}(\Omega), U \in W^{1,1}([0,T], H^{l+1}(\Omega))$ and $\nabla \times E \in H^{l+1}(\Omega)$. For any $t \in (0,T]$, we have the following estimate

$$\sup_{0\leqslant s\leqslant t}\|\xi(\cdot,s)\| \lesssim h^{l+1}\|U_0\|_{l+1} + h^{l+1}\int_0^t (\|\partial_t U\|_{l+1} + \|\nabla\times E\|_{l+1} + \|\sigma(|E|)\|_\infty \|E\|_{l+1} + |\sigma'(c)|\,\|E\|_\infty \|E\|_{l+1})\mathrm{d}s,$$

where $c>0$ is a constant between $|E|$ and $|I_h^c E|$.

Proof Taking $V_h = \xi$ in (14) and using the fact that A_h is skew-symmetric, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 + (g-g_h, \xi) = (\partial_t \eta + G, \xi) \qquad (15)$$

Note that

$$\begin{aligned}(g-g_h,\xi) = \\ &(\sigma(|E|)E - \sigma(|E_h|)E_h, I_h^c E - E_h) = \\ &(\sigma(|E|)(E - I_h^c E), I_h^c E - E_h) + \\ &((\sigma(|E|) - \sigma(|I_h^c E|))I_h^c E, I_h^c E - E_h) + \\ &(\sigma(|I_h^c E|)I_h^c E - \sigma(|E_h|)E_h, I_h^c E_h - E_h).\end{aligned}$$

For any $t\in(0,T]$, integrating (15) in the interval $(0,t)$ and using (2), we obtain

$$\begin{aligned}\|\xi(\cdot,t)\|^2 \leqslant \|\xi_0\|^2 + \int_0^t |\partial_t \eta + G, \xi|\,\mathrm{d}s + \\ \int_0^t |(\sigma(|E|)(E - I_h^c E), I_h^c E - E_h)|\,\mathrm{d}s + \\ \int_0^t |((\sigma(|E|) - \sigma(|I_h^c E|))I_h^c E, I_h^c E - E_h)|\,\mathrm{d}s := \\ \|\xi_0\|^2 + \sum\nolimits_{i=1}^3 T_i.\end{aligned}$$

Now, we focus on the estimates of $T_i (i=1,2,3)$. Using the fourth item of Lemma 3.1, we have

$$T_1 = \int_0^t |\partial_t \eta, \xi|\,\mathrm{d}s + \int_0^t |G,\xi|\,\mathrm{d}s \leqslant Ch^{l+1}\int_0^t (\|\partial_t U\|_{l+1} + \|\nabla\times E\|_{l+1})\|\xi\|\,\mathrm{d}s$$

and

$$T_2 \leqslant \int_0^t \|\sigma(|E|)(E - I_h^c E)\|\,\|\xi\|\,\mathrm{d}s \leqslant Ch^{l+1}\int_0^t \|\sigma(|E|)\|_\infty \|E\|_{l+1}\|\xi\|\,\mathrm{d}s.$$

Using the mean value theorem, there exists a constant c which is between $|E|$ and $|I_h^c E|$ such that

$$\sigma(|E|) - \sigma(|I_h^c E|) = \sigma'(c)(|E| - |I_h^c E|).$$

Then we have

$$T_3 \leqslant Ch^{l+1}\int_0^t |\sigma'(c)|\,\|E\|_\infty \|E\|_{l+1}\|\xi\|\,\mathrm{d}s.$$

Adding the above equations together, we have

$$\|\xi(\cdot,t)\|^2 \lesssim h^{2(l+1)}\|U_0\|_{l+1}^2 + h^{l+1}\sup_{0\leqslant s\leqslant t}\|\xi(\cdot,s)\| \times \int_0^t (\|\partial_t U\|_{l+1} + \|\nabla\times E\|_{l+1} + \|\sigma(|E|)\|_\infty \|E\|_{l+1} + |\sigma'(c)|\,\|E\|_\infty \|E\|_{l+1})\mathrm{d}s.$$

Then the desired result follows by Lemma 2.2 with $x = \sup_{0\leqslant s\leqslant t}\|\xi(\cdot,s)\|$.

Using the triangular inequality and the fourth item of Lemma 3.1, we have

Theorem 3.5 Let $U = (E,B)^T$ and $U_h = (E_h, B_h)^T$ be the solutions of (3) (or (5)) and (10) (or (11)), respectively. Assume that $U_0 \in H^{l+1}(\Omega), U \in W^{1,1}([0,T], H^{l+1}(\Omega))$ and $\nabla\times E \in H^{l+1}(\Omega)$. For any $t\in(0,T]$, we have the following estimate

$$\sup_{0\leqslant s\leqslant t}\|U(\cdot,s) - U_h(\cdot,s)\| \lesssim h^{l+1}\|U_0\|_{l+1} + h^{l+1}\|U\|_{L^\infty((0,T),H^{l+1})} + h^{l+1}\int_0^t (\|\partial_t U\|_{l+1} + \|\nabla\times E\|_{l+1} + \|\sigma(|E|)\|_\infty \|E\|_{l+1} + |\sigma'(c)|\,\|E\|_\infty \|E\|_{l+1})\mathrm{d}s,$$

where $c > 0$ is a constant between $|E|$ and $|I_h^c E|$.

4 Full discrete scheme

In this section, the full-discrete scheme is given for the nonlinear Maxwell′s equation by employing a continuous time Galerkin method. We show that the full-discrete scheme preserves the energy and the divergence-free condition of the magnetic field exactly. Finally, we give the optimal order error estimates for the full-discrete scheme.

4.1 Full-discrete scheme

Let $J_{\Delta t}$ denote the equispaced partition of the time interval $(0,T)$ with $\Delta t = \frac{T}{N}$ and N the number of elements in $J_{\Delta t}$. For $1\leqslant n\leqslant N$, we denote $t_n = n\Delta t$ and $I_n = (t_{n-1}, t_n)$ with $t_0 = 0$. For any quantity $v(t)$, we denote $v^n = v(t_n)$. For any Sobolev space S associates with the spatial variables, define $P_1(J_{\Delta t}, S)$ (abbr. $P_1(S)$) as the set of continuous piecewise linear polynomials with respect to the time variable t on $J_{\Delta t}$ and in the Sobolev space S about the spatial variables. Define $P_0(J_{\Delta t}, S)$ ($P_0(S)$) as the set of piecewise constant with respect to the time variable t on $J_{\Delta t}$ and in the Sobolev space S about the spatial variables.

The full-discrete scheme of the nonlinear Maxwell′s equations reads as: find $\widetilde{U}_h = (\widetilde{E}_h, \widetilde{B}_h)^T \in P_1(U_h) \times P_1(V_h)$ such that

$$\int_0^T (\partial_t \widetilde{U}_h, V_h)\mathrm{d}t + \int_0^t (A_h \widetilde{U}_h, V_h)\mathrm{d}t + \int_0^T (\widetilde{g}_h, V_h)\mathrm{d}t = \int_0^T (F, V_h)\mathrm{d}t, \forall\ V_h \in P_0(U_h) \times P_0(V_h) \tag{16}$$

with initial value $\widetilde{U}_h^0 = (I_h^c E_0, I_h^d B_0)^T$, here $\widetilde{g}_h = (\sigma(|\widetilde{E}_h|)\widetilde{E}_h$.

Let $Q_{\Delta t}: L^2(0,T) \to P_0$ be the L^2 orthogonal projection operator with respect to the time variable. We have the following energy estimates for the full-discrete scheme.

Theorem 4.1 Let $\widetilde{U}_h = (\widetilde{E}_h, \widetilde{B}_h)^T$ be the solution of the full-discrete scheme(16). Provided $f \in L^1((0,T), (L^2(\Omega))^3)$. For any integer $0 \leqslant m \leqslant N$, we have the following stability bound

$$\max_{0\leqslant n\leqslant m} \|\widetilde{U}_h^n\| + \sqrt{2}\left(\int_0^{t_m} (\sigma(|\widetilde{E}_h|)\widetilde{E}_h, Q_{\Delta t}\widetilde{E}_h)\mathrm{d}t\right)^{\frac{1}{2}} \leqslant \|\widetilde{U}_h^0\| + 2\int_0^{t_m} \|f\|\mathrm{d}t \tag{17}$$

Proof Taking $V_h|_{I_n} = \widetilde{U}_h^n + \widetilde{U}_h^{n-1}$ and $V_h|_{(0,T)\backslash I_n} = 0$ in the full-discrete scheme (16), we get

$$\|\widetilde{U}_h^n\|^2 - \|\widetilde{U}_h^{n-1}\|^2 + \int_{I_n} (A_h \widetilde{U}_h^n, \widetilde{U}_h^n + \widetilde{U}_h^{n-1})\mathrm{d}t + \int_{I_n} (\widetilde{g}_h, \widetilde{U}_h^n + \widetilde{U}_h^{n-1})\mathrm{d}t = \int_{I_n} (F, \widetilde{U}_h^n + \widetilde{U}_h^{n-1})\mathrm{d}t \tag{18}$$

Using the skew-symmetric property of A_h and trapezoid formula, we get

$$\|\widetilde{U}_h^n\|^2 - \|\widetilde{U}_h^{n-1}\|^2 + \int_{I_n} (\widetilde{g}_h, \widetilde{U}_h^n + \widetilde{U}_h^{n-1})\mathrm{d}t = \int_{I_n} (F, \widetilde{U}_h^n + \widetilde{U}_h^{n-1})\mathrm{d}t \tag{19}$$

The definition of $Q_{\Delta t}$ implies $Q_{\Delta t}\widetilde{U}_h|_{I_n} = \frac{1}{2}(\widetilde{U}_h^n + \widetilde{U}_h^{n-1})$, thus we have

$$\|\widetilde{U}_h^m\|^2 + 2\int_0^{t_m} (\widetilde{g}_h, Q_{\Delta t}\widetilde{U}_h)\mathrm{d}t = \|\widetilde{U}_h^0\|^2 + 2\int_0^{t_m} (F, Q_{\Delta t}\widetilde{U}_h)\mathrm{d}t \leqslant \|\widetilde{U}_h^0\|^2 + 2\max_{0\leqslant n\leqslant m} \|\widetilde{U}_h^n\| \int_0^{t_m} \|F\|\mathrm{d}t.$$

Then, Lemma 2.2 implies the desired result.

Remark 3 From the proof of Theorem 4.1, we can see if the source term f vanished, we always have

$$\max_{0\leqslant n\leqslant m} \|\widetilde{U}_h^n\| + \sqrt{2}\left(\int_0^{t_m} (\sigma(|\widetilde{E}_h|)\widetilde{E}_h, Q_{\Delta t}\widetilde{E}_h)\mathrm{d}t\right)^{\frac{1}{2}} = \|\widetilde{U}_h^0\|,$$

which means the energy is preserved exactly.

Theorem 4.2 Let $\widetilde{U}_h = (\widetilde{E}_h, \widetilde{B}_h)^T$ be the solution of the full-discrete scheme(16). Then, for any $0 \leqslant n \leqslant N$, we have $\nabla \cdot \widetilde{B}_h^n = 0$.

Proof On the interval I_n, the second equation of (16) can be rewritten as

$$(\widetilde{B}_h^n - \widetilde{B}_h^{n-1}, \varphi_h) + \frac{\Delta t}{2}(\nabla \times (\widetilde{E}_h^n + \widetilde{E}_h^{n-1}), \varphi_h) = 0.\ \forall\, \varphi_h \in V_h.$$

Since $\nabla \times (\widetilde{E}_h^n + \widetilde{E}_h^{n-1}) \in V_h$, taking

$$\varphi_h = \widetilde{B}_h^n - \widetilde{B}_h^{n-1} + \frac{\Delta t}{2}\nabla \times (\widetilde{E}_h^n + \widetilde{E}_h^{n-1})$$

in the above equation, we get

$$\widetilde{B}_h^n - \widetilde{B}_h^{n-1} + \frac{\Delta t}{2}\nabla \times (\widetilde{E}_h^n + \widetilde{E}_h^{n-1}) = 0.$$

Taking divergence on the two sides of the above equation, and using the divergence-free assumption on the initial data of the magnetic field B, the desired result then follows.

4.2 Error analysis of the full discretization

In this subsection, we will focus on the error analysis of the full-discrete scheme (16). We use $I_{\Delta t}^t: H^1(0,T) \to P_1$ to denote the stand interpolation operator for the time variable t. From Ref. [10], $I_{\Delta t}^t$ has the following property

$$\|u-I_{\Delta}^{t}u\|\lesssim\Delta t^{2}\|u\|_{H^{2}(0,T)},\ \forall u\in H^{2}(0,T).$$

Denote $\zeta=I_hU-\widetilde{U}_h$. Simple calculation shows that for any $V_h\in P_0(U_h)\times P_0(V_h)$, ζ satisfies the following equation:

$$\int_{I_n}(\partial_t\zeta,V_h)+\int_{I_n}(A_h\zeta,V_h)\mathrm{d}t+\int_{I_n}(g-\tilde{g}_h,V_h)\mathrm{d}t=\int_{I_n}(\partial_t\eta+G,V_h)\mathrm{d}t \tag{20}$$

with initial value $\zeta=U_0-I_hU_0$. We have the following estimate about ζ.

Lemma 4.3 Let $U=(E,B)^T$ and $\widetilde{U}_h=(\widetilde{E}_h,\widetilde{B}_h)^T$ be the solution of equation (3) and (16), respectively. Assume that $U_0\in H^{l+1}(\Omega)$, $U\in W^{1,1}([0,T],H^{l+1}(\Omega))$ and $\nabla\times E\in H^{l+1}(\Omega)$. For any $0\leqslant m\leqslant N$, we have the estimate

$$\max_{0\leqslant n\leqslant m}\|\zeta^n\|\lesssim\Delta t^2\int_0^{t_m}\|AU\|\mathrm{d}t+h^{l+1}\|U_0\|_{l+1}+h^{l+1}\int_0^{t_m}(\|\partial_tU\|_{l+1}+\|\nabla\times E\|_{l+1}+\|\sigma(|E|)\|_\infty\|E\|_{l+1}+|\sigma'(c)|\|E\|_\infty\|E\|_{l+1})\mathrm{d}s.$$

Proof Taking V_h in (20) as

$$V_h|_{I_n}=\zeta^n+\zeta^{n-1},V_h|_{(0,T)/I_n}=0,$$

we obtain

$$\int_{I_n}(\partial_t\zeta,\zeta^n+\zeta^{n-1})\mathrm{d}t+\int_{I_n}(A_h\zeta,\zeta^n+\zeta^{n-1})\mathrm{d}t+\int_{I_n}(g-\tilde{g}_h,\zeta^n+\zeta^{n-1})\mathrm{d}t=\int_{I_n}(\partial_t\eta+G,\zeta^n+\zeta^{n-1})\mathrm{d}t.$$

Summing over n from 1to $m\leqslant N$, we obtain

$$\|\zeta^m\|^2+2\int_0^{t_m}(A_h\zeta,Q_\Delta\zeta)\mathrm{d}t+2\int_0^{t_m}(g-\tilde{g}_h,Q_\Delta\zeta)\mathrm{d}t=\|\zeta^0\|^2+2\int_0^{t_m}(\partial_t\eta+G,Q_\Delta\zeta)\mathrm{d}t.$$

Noting that

$$\int_0^{t_m}(A_h\zeta,Q_\Delta\zeta)\mathrm{d}t=\int_0^{t_m}(A_h(I-I_\Delta^t)\zeta,Q_\Delta\zeta)\mathrm{d}t\lesssim\Delta t^2\max_{0\leqslant n\leqslant m}\|\zeta^n\|\int_0^{t_m}\|AU_{tt}\|\mathrm{d}t,$$

by using the similar discuss as in the proof of Lemma 3.4, the desired result then follows.

Triangular inequality and Lemma 4.4 imply

Theorem 4.4 Let $U=(E,B)^T$ and $\widetilde{U}_h=(\widetilde{E}_h,\widetilde{B}_h)^T$ be the solution of equation (3) and (16), respectively. Assume that $U_0\in H^{l+1}(\Omega)$, $U\in W^{1,1}([0,T],H^{l+1}(\Omega))$ and $\nabla\times E\in H^{l+1}(\Omega)$. For any $0\leqslant m\leqslant N$, we have the estimate

$$\max_{0\leqslant n\leqslant m}\|U^n-\widetilde{U}_h^n\|\lesssim\Delta t^2\int_0^{t_m}\|AU\|\mathrm{d}t+h^{l+1}(\|U_0\|_{l+1}+\|U\|_{L^\infty((0,T),H^{l+1})})+h^{l+1}\int_0^{t_m}(\|\partial_tU\|_{l+1}+\|\nabla\times E\|_{l+1}+\|\sigma(|E|)\|_\infty\|E\|_{l+1}+|\sigma'(c)|\|E\|_\infty\|E\|_{l+1})\mathrm{d}t.$$

5 Numerical examples

In this section, we will give a numerical example to illustrate the theoretical results. We display the numerical example by using the iFEM-package [20]. The domain Ω is choosing as the unite cubic $(0,1)^3$. We using the NE_0 finite element space to discretize the electric variable E and RT_0 element to discretize the magnetic field B. In this case $l=0$, and the optimal convergence order is 1. The exact solutions we choose are

$$E(x,y,z,t)=\sin t\begin{pmatrix}\sin\pi y\sin\pi z\\ \sin\pi x\sin\pi z\\ \sin\pi x\sin\pi y\end{pmatrix},$$

$$B(x,y,z,t)=(\cos t-1)\pi\begin{pmatrix}\sin\pi x(\cos\pi y-\cos\pi z)\\ \sin\pi y(\cos\pi z-\cos\pi x)\\ \sin\pi z(\cos\pi x-\cos\pi y)\end{pmatrix}.$$

The ending time $T=1$. The numerical results are listed in Tab. 1.

Tab. 1 Errors and convergence orders in various norms with $\Delta t=\sqrt{h}$

h	$\frac{\|E-\widetilde{E}_h\|}{\|E\|}$	$\frac{\|\nabla\times(E-\widetilde{E}_h)\|}{\|\nabla\times E\|}$	$\frac{\|B-\widetilde{B}_h\|}{\|B\|}$	$\|\nabla\cdot\widetilde{B}_h\|_\infty$
1/4	0.3591	0.2897	0.2882	0.0071e−11
1/8	0.1905	0.1506	0.1429	0.0338e−11
1/16	0.0945	0.0777	0.0716	0.1684e−11
1/32	0.0467	0.0383	0.0354	0.7745e−11
order	0.9811	0.9725	1.0089	—

From this example, we have the following observations:

The mixed finite element method is of first order convergence. All of the variables have optimal convergence order.

The divergence-free condition of the magnetic field B is preserved exactly.

6 Conclusions

In this paper, we develop a class of energy-preserving mixed finite element methods for the nonlinear Maxwell's equations. Our methods preserve the energy and the divergence-free condition of magnetic field, and have optimal convergence order.

References:

[1] Kang T, Chen T, Zhang H, *et al*. Fully discrete A-φ finite element method for Maxwell's equations with nonlinear conductivity [J]. Numer Meth Part Diff Equat, 2014, 30: 2083.

[2] Rhyner J. Magnetic properties and AC losses of superconductors with power law current-voltage characteristics [J]. Physica C, 1993, 212: 292.

[3] Grüner G, Zawadowski A, Chaikin P M. Nonlinear conductivity and noise due to charge density wave depinning in $NbSe_3$ [J]. Phys Rev Lett, 1981, 46: 511.

[4] Yin H M. On a singular limit problem for nonlinear Maxwell's equations [J]. J Differ Equat, 1999, 156: 355.

[5] Durand S, Slŏdicka M. Fully discrete finite element method for Maxwell's equations with nonlinear conductivity [J]. IMA J Numer Anal, 2011, 31: 1713.

[6] Durand S, Slodička M. Convergence of the mixed finite element method for Maxwell's equations with nonlinear conductivity [J]. Math Meth Appl Sci, 2012, 35: 1489.

[7] Yao C, Jia S. A second order numerical scheme for nonlinear Maxwell's equations using conforming finite element [J]. Appl Math Comput, 2020, 371: 1.

[8] Zhou Y J, Chen Y M. A hybrid stress finite element method with mass lumping for Maxwell elastodynamic problems of wave propagation [J]. J Sichuan Univ: Nat Sci Ed, 2021, 58: 061001(in Chinese).

[9] Kirby R C, Kieu T T. Symplectic-mixed finite element approximation of linear acoustic wave equations [J]. Numer Math, 2015, 130: 257.

[10] Ciarlet P G. The finite element method for elliptic problems [M]. Amsterdam: North-Holland, 1978.

[11] Nédélec J C. Mixed finite elements in $\mathbf{R}^3$ [J]. Numer Math, 1980, 35: 315.

[12] Nédélec J C. A new family of mixed finite elements in $\mathbf{R}^3$ [J]. Numer Math, 1986, 50: 57.

[13] Galligani I, Magenes E. Mathematical aspects of finite element methods [M]. New York: Springer, 1977.

[14] Douglas B F, Marini L D. Two families of mixed finite elements for second order elliptic problems [J]. Numer Math, 1985, 47: 217.

[15] Hiptmair R. Finite elements in computational electromagnetism [J]. Acta Numer, 2002, 11: 237.

[16] Demkowicz L, Buffa A. H^1, H(curl) and H(div)-conforming projection-based interpolation in three dimensions Quasi-optimal p-interpolation estimates [J]. Comput Methods Appl Mech Engng, 2005, 194: 267.

[17] Wu Y K, Bai Y H. Error analysis of energy-preserving mixed finite element methods for the hodge wave equation [J]. SIAM J Numer Anal, 2021, 59: 1433.

[18] Arnold D N, Falk R S, Winther R W. Finite element exterior calculus, homological techniques, and applications [J]. Acta Numer, 2006, 15: 1.

[19] Arnold D N, Falk R S, Winther R. Finite element exterior calculus: from hodge theory to numerical stability [J]. B AM Math Soc, 2010, 47: 281.

[20] Chen L. iFEM: an integrated finite element methods package in Matlab [R]. Irvine: University of California, 2009.

引用本文格式：

中　文：刘莎莎，吴永科. 具有非线性传导率的麦克斯韦方程的一个保能量混合有限元[J]. 四川大学学报：自然科学版，2022，59：051003.

英　文：Liu S S, Wu Y K. An energy preserving mixed finite element for Maxwell's equations with nonlinear conductivity [J]. J Sichuan Univ: Nat Sci Ed, 2022, 59: 051003.