

一类非对称三次 Liénard 系统的全局动力学

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摘要: 本文研究了一类非对称三次 Liénard 系统的全局动力学, 其中的参数不要求充分小. 在分析了所有平衡点的定性性质并讨论了极限环和异宿轨道的存在性后, 本文在庞加莱圆盘上给出了全局相图的完整分类, 并结合已知结果给出其在参数空间中对应的分岔图.

关键词: 全局相图; 异宿轨; 极限环; Liénard 系统

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Global dynamics of a non-symmetric cubic Liénard system

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Abstract: In this paper, we study global dynamics of a cubic non-symmetric Liénard system with general parameters, *i. e.*, parameters are not required to be sufficiently small. After analyzing qualitative properties of all the equilibria and discussing the existence of limit cycles and heteroclinic orbits, we give a complete classification of the global phase portraits in the Poincaré disc. Finally, associated with the previous results we obtain the bifurcation diagram in the parameter space.

Keywords: Global phase portrait; Heteroclinic orbit; Limit cycle; Liénard system
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1 Introduction

The second order scalar differential equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ usually appears in physical models, biochemical models, chemical models, such as the nonlinear mechanical oscillations and is called a Liénard equation, where

$$\dot{x} = dx/dt, \ddot{x} = d^2x/dt^2.$$

By $y := \dot{x}$ or

$$y := \dot{x} + \int_0^x f(s)ds,$$

this Liénard equation is written as a planar differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - f(x)y \end{cases}$$

or

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases}$$

where $F(x) := \int_0^x f(s)ds$. They are called polynomial Liénard systems when $g(x), f(x)$ are both polynomials.

Dynamical analysis of Liénard systems have been attracting interests of research extensively. When

$$g(x) = -(\mu_1 + \mu_2 x \pm x^3)$$

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and

$$f(x) = -\mu_3 - \mu_4 x + x^2,$$

the system takes form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \mu_1 + \mu_2 x + \mu_3 y + \mu_4 xy \pm x^3 - x^2 y \end{cases} \quad (1)$$

where $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbf{R}$. In the case that $\mu_1, \mu_2, \mu_3, \mu_4$ are sufficiently small, the dynamical behavior of system (1) $|_{-}$ (*i. e.*, choosing “ $-$ ” in system (1)) is analyzed completely as a near-Hamiltonian system^[1-3]. In the case that $\mu_4 = 0$, for system (1) $|_{-}$ Khibnik *et al.* give a conjecture that the double limit cycle bifurcation surface is the graph of a function $\mu_3 = \mu_3(\mu_1, \mu_2)$ ^[4]. Later, all global phase portraits for system (1) $|_{-}$ and a positive answer to this conjecture are given successfully^[5-7]. For system (1) $|_{+}$, Chen and Zhu provided the global bifurcation diagram and all global phase portraits^[8].

System (1) has an equivalent system

$$\begin{cases} \dot{x} = y - (b_1 x + b_2 x^2 + x^3), \\ \dot{y} = -(a_1 x + a_2 x^2 \mp x^3) \end{cases} \quad (2)$$

where $a_1, a_2, b_1, b_2 \in \mathbf{R}$. When $b_2 = a_2 = 0$, system (2) is Z_2 symmetric and all global phase portraits are given^[9,10]. When $b_1 = a_1 = 0$, dynamical behavior of (2) $|_{+}$ is analyzed^[11] and two conjectures are stated on bifurcation curves. All global phase portraits are obtained for (2) $|_{+}$ and these two conjectures already have positive answers^[12].

It is not hard to prove that system (2) $|_{-}$ with $b_1 = a_1 = 0$ has a saddle if there is a unique equilibrium, a cusp and a saddle if there are exactly two equilibria, neither limit cycles nor heteroclinic loops by the indices of these equilibria. Thus, we are interested in the case that $b_1 \neq 0, a_1 = 0$ for (2) $|_{-}$. In this paper we investigate the global dynamics of system (2) $|_{-}$ with $b_1 = 1, a_1 = 0, i. e.,$

$$\begin{cases} \dot{x} = y - (x + b x^2 + x^3), \\ \dot{y} = -(a x^2 - x^3) \end{cases} \quad (3)$$

where $a, b \in \mathbf{R}$. Since system (3) is invariant under

$$(x, y, a, b) \rightarrow (-x, -y, -a, -b),$$

we only need to consider (a, b) in $G := \{(a, b) \in \mathbf{R}^2 : a \geq 0\}$. After analyzing the equilibria, limit

cycles, homoclinic orbits and heteroclinic orbits as well as the connection among equilibria at finity and infinity, we give the global bifurcation diagram and global dynamics of system (3) for $(a, b) \in G$.

This paper is organized as follows. In Section 2, qualitative properties of all equilibria at finity and infinity are analyzed. In Section 3, we prove the nonexistence of limit cycles and the existence of heteroclinic orbits. Finally, in Section 4 we obtain the global bifurcation diagram and all global phase portraits of system (3) for $(a, b) \in G$.

2 Equilibria analysis

In this section, we find locations of equilibria and give their qualitative properties for system (3).

Lemma 2.1 When $a = 0$, system (3) has a unique equilibrium $O: (0, 0)$, which is a saddle. When $a > 0$, system (3) has exactly two equilibria $E: (a, a + b a^2 + a^3)$, $O: (0, 0)$ and E is a saddle, O is a saddle-node.

Proof Clearly, system (3) has a unique equilibrium at $O: (0, 0)$ when $a = 0$. The Jacobian matrix of the vector field at O is

$$J_0 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix},$$

which implies that O is degenerate. By transformation

$$x \rightarrow x + y, y \rightarrow y, t \rightarrow -t,$$

system (3) is equivalently transformed into

$$\begin{cases} \dot{x} = -(x + y)^3 := P(x, y), \\ \dot{y} = y + b(x + y)^2 + 2(x + y)^3 := Q(x, y) \end{cases} \quad (4)$$

The Implicit Function Theorem indicates that there exists a unique function

$$\varphi(x) = -b x^2 + o(x^2),$$

such that $Q(x, \varphi(x)) = 0$ and $\varphi(0) = 0$. Substituting $y = \varphi(x)$ into $P(x, y)$, we get

$$\dot{x} = -x^3 + o(x^3).$$

By Ref. [13, Theorem 7.1 in Chapter 2], the origin of system (4) is a saddle, so O is also a saddle of system (3) when $a = 0$.

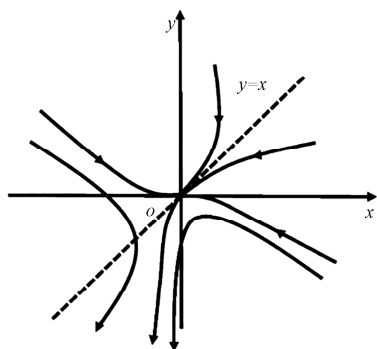


Fig. 1 Saddle-node O of system (3)

When $a > 0$, straight computation shows that system (3) has two equilibria $E: (a, a + b a^2 + a^3)$ and $O: (0, 0)$. Moreover, the determinant of the Jacobian matrix of the vector field at E is $-a^2$, so E is a saddle. The determinant and trace of the Jacobian matrix of the vector field at O are 0 and -1 , respectively. Furthermore, we use the transformation

$x \rightarrow x, y \rightarrow x + xy + (b - a)x^2, t \rightarrow -t$,
system (3) is equivalent to

$$\begin{cases} \dot{x} = -xy + a x^2 + x^3 := P(x, y), \\ \dot{y} = y + y^2 - (2 + 2ba - 2a^2)x^2 - \\ (3a - 2b)xy - x^2y - 2(b - a)x^3 := Q(x, y) \end{cases} \quad (5)$$

The Implicit Function Theorem indicates that there exists a unique function

$$\psi(x) = (2 + 2ba - 2a^2)x^2 + o(x^2)$$

such that $Q(x, \psi(x)) = 0$ and $\psi(0) = 0$. Substituting $y = \psi(x)$ into $P(x, y)$, we get

$$\dot{x} = a x^2 + o(x^2),$$

implying that the origin of system (5) is a saddle-node by Ref. [13, Theorem 7. 1 in Chapter 2]. Thus, O of system (3) is also a saddle-node as shown in Fig. 1.

In order to obtain the global structure of system (3), we discuss the qualitative properties of equilibria at infinity.

Lemma 2. 2 As shown in Fig. 2, system (3) has two equilibria at I_A^\pm infinity on the line $y = -x$ and two equilibria I_B^\pm at infinity on y -axis. Moreover, I_A^\pm are unstable nodes and I_B^\pm are degenerate stable nodes.

Proof By Poincaré transformation

$$x = 1/z, y = u/z,$$

system (3) is changed into

$$\begin{cases} \frac{du}{d\tau} = 1 + u - az + ubz + uz^2 - u^2z^2, \\ \frac{dz}{d\tau} = z + bz^2 + (1 - u)z^3 \end{cases} \quad (6)$$

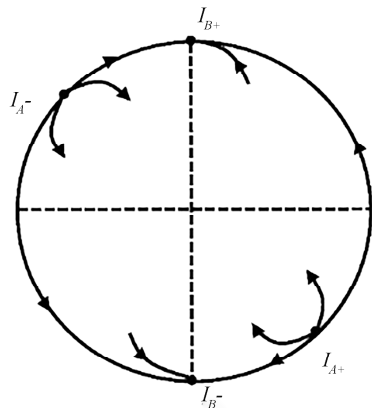


Fig. 2 Equilibria at infinity of system (3)

where $d\tau = dt/z^2$. Clearly, on the u -axis system (6) has a unique equilibrium $A(-1, 0)$, which is an unstable node. Thus system (3) has two unstable nodes I_A^\pm at infinity on the line $y = -x$.

By Poincaré transformation

$$x = v/z, y = 1/z,$$

system (3) is written as

$$\begin{cases} \frac{dv}{d\tau} = z^2 - v^3 - bv^2z + av^3z - vz^2 - v^4, \\ \frac{dz}{d\tau} = -v^3z + av^2z^2 \end{cases} \quad (7)$$

where $d\tau = dt/z^2$. In order to study equilibria I_B^\pm at infinity of system (3) on the y -axis, we only need to discuss the origin $B: (0, 0)$ of system (7). Since B of system (7) is degenerate, we use Briot-Bouquet transformation to analyze the number of orbits in exceptional directions^[14, 15]. By

$$v = r \cos \theta, z = r \sin \theta,$$

from (7) we get the following equation

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{H_1(\theta) + \tilde{H}(\theta, r)}{G_1(\theta) + \tilde{G}(\theta, r)} \quad (8)$$

where

$$G_1(\theta) = -\sin^3 \theta, H_1(\theta) = \cos \theta \sin^2 \theta$$

and $\tilde{H}(\theta, r), \tilde{G}(\theta, r) \rightarrow 0$ as $r \rightarrow 0$. Exceptional directions^[13] are determined by zeros of $G_1(\theta)$, i. e., 0 and π . Since $H_1(0) = H_1(\pi) = 0$, by Briot-Bouquet transformation

$$v \rightarrow v, z \rightarrow z_1 v,$$

system (7) is transformed into

$$\begin{cases} \frac{dv}{d\sigma} = -v^2 - v^3 + v z_1^2 - b v^2 z_1 + a v^3 z_1 - v^2 z_1^2, \\ \frac{dz_1}{d\sigma} = v z_1 - z_1^3 + b v z_1^2 + v z_1^3 \end{cases} \quad (9)$$

where $d\sigma = v d\tau$. System (9) still has one equilibrium $(0, 0)$ on the z_1 -axis, which is still degenerate. By transformation

$$v = r \cos \theta, z_1 = r \sin \theta,$$

from (9) we obtain equation (8), where

$$G_1(\theta) = -2 \cos^2 \theta \sin \theta$$

and

$$H_1(\theta) = \cos \theta (\sin^2 \theta - \cos^2 \theta).$$

It is not hard to check that $G_1(\theta) = 0$ has four real roots $0, \pi/2, \pi$ and $3\pi/2$, and

$$G'_1(0)H_1(0) = G'_1(\pi)H_1(\pi) = -2 < 0.$$

Thus, by Ref. [13, Theorem 3.7 in Chapter 2] system (9) has a unique orbit approaching $(0, 0)$ in direction $\theta = 0$ and a unique orbit leaving $(0, 0)$ in direction $\theta = \pi$ as $\sigma \rightarrow +\infty$.

Since $H_1(\pi/2) = H_1(3\pi/2) = 0$, by Briot-Bouquet transformation

$$v \rightarrow v_1 z_1, z_1 \rightarrow z_1,$$

system (9) is changed into

$$\begin{cases} \frac{dv_1}{ds} = 2 v_1 z_1 - 2 v_1^2 - 2b v_1^2 z_1 - v_1^3 z_1 + \\ a v_1^3 z_1^2 - 2 v_1^2 z_1^2, \\ \frac{dz_1}{ds} = v_1 z_1 - z_1^2 + b v_1 z_1^2 + v_1 z_1^3 \end{cases} \quad (10)$$

where $ds = z_1 d\sigma$. Equilibrium $(0, 0)$ of system (10) is still degenerate. By transformation

$$v_1 = r \cos \theta, z_1 = r \sin \theta,$$

from system (10) we obtain equation (8), where

$$G_1(\theta) = 3 \sin \theta \cos \theta (\cos \theta - \sin \theta),$$

$$H_1(\theta) = \cos \theta \sin^2 \theta - \sin^3 \theta + 2 \sin \theta \cos^2 \theta - 2 \cos^3 \theta.$$

We check that equation $G_1(\theta) = 0$ has exactly six real roots $0, \pi/4, \pi/2, \pi, 5\pi/4$ and $3\pi/2$. Moreover, we also have

$$G'_1(0)H_1(0) = G'_1(\pi)H_1(\pi) = -6 < 0$$

and

$$G'_1(\pi/2)H_1(\pi/2) = G'_1(3\pi/2)H_1(3\pi/2) = -3 < 0.$$

By Ref. [13, Theorem 3.7 in Chapter 2], system (10) has a unique orbit approaching $(0, 0)$ in direction $\theta = 0$, a unique orbit leaving $(0, 0)$ in direction $\theta = \pi$, a unique orbit approaching $(0, 0)$ in direction $\theta = \pi/2$ and a unique orbit leaving $(0, 0)$ in direction $\theta = 3\pi/2$ as $\tau \rightarrow +\infty$.

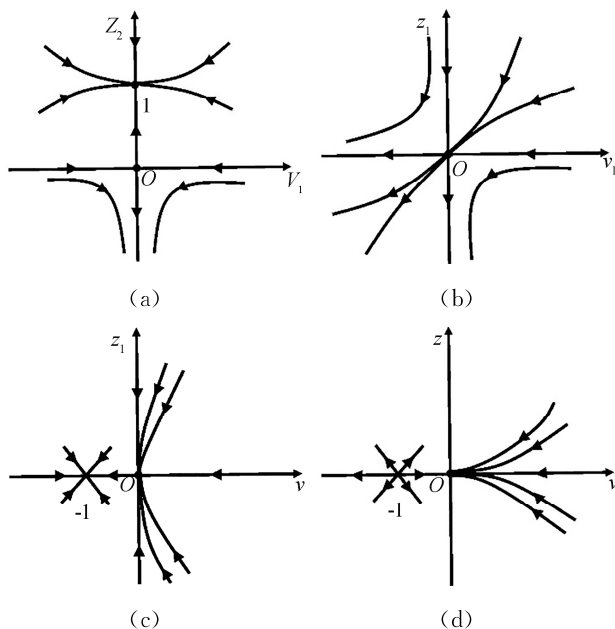


Fig. 3 Local phase portraits: (a) (v_1, z_2) plane; (b) (v_1, z_1) plane; (c) (v, z_1) plane; (d) (v, z) plane

Since $H_1(\pi/4) = H_1(5\pi/4) = 0$, by Briot-Bouquet transformation

$$v_1 \rightarrow v_1, z_1 \rightarrow z_2 v_1,$$

system (10) takes form

$$\begin{cases} \frac{dv_1}{ds_1} = -2 v_1 + 2 v_1 z_2 - 2b v_1^2 z_2 - \\ v_1^3 z_2 - 2 v_1^3 z_2^2 + a v_1^4 z_2^2, \\ \frac{dz_2}{ds_1} = 3 z_2 - 3 z_2^2 + 3b v_1 z_2^2 + \\ v_1^2 z_2^2 - a v_1^3 z_2^3 + 3 v_1^2 z_2^3 \end{cases} \quad (11)$$

which is further changed into

$$\begin{cases} \frac{dv_1}{ds_2} = -\frac{2}{3} v_1 z_2 + \frac{1}{3} v_1^2 [2b(z_2 + 1) + \\ v_1(z_2 + 1)] + \frac{2}{3} (z_2 + 1)^2 v_1^3 - \\ \frac{1}{3} a (z_2 + 1)^2 v_1^4, \\ \frac{dz_2}{ds_2} = z_2 + z_2^2 - \frac{1}{3} v_1 (3b + v_1) (z_2 + 1)^2 + \\ \frac{1}{3} a v_1^3 (z_2 + 1)^3 - v_1^2 (z_2 + 1)^3 \end{cases} \quad (12)$$

by

$$v_1 \rightarrow v_1, z_1 \rightarrow z_2 + 1, ds_2 = -3 ds_1 = -3 v_1 ds.$$

Similarly, there exists a function

$$\chi(v_1) = b v_1 + \left(\frac{4}{3} + b^2\right) v_1^2 + o(v_1^2)$$

such that

$$dz_2/ds_2 = 0, dv_1/ds_2 = v_1^3/9 + o(v_1^3)$$

along the curve $z_2 = \chi(v_1)$. By Ref. [13, Theorem 7.1 in Chapter 2], the origin of (12) is an unstable node. Thus we obtain the qualitative properties of $(0, 1)$ in (v_1, z_2) plane for system (11), as shown in Fig. 3a. Then, we obtain the qualitative properties of $(0, 0)$ in the (v_1, z_1) plane and in the (v, z_1) plane, respectively, as shown in Fig. 3b and Fig. 3c. Finally, we find that equilibrium B of system (7) is a degenerate stable node as shown in Fig. 3d. Then, system (3) has two degenerate stable nodes I_{B^\pm} at infinity on y -axis.

3 Limit cycles and heteroclinic orbits

In this section, for system (3) we study the existence of limit cycles, homoclinic orbits and the heteroclinic orbits.

Lemma 3.1 System (3) has neither limit cycles nor homoclinic orbits for all $(a, b) \in G$.

Proof Note that when $a = 0$ the unique equilibrium O is a saddle, which has index -1 . When $a > 0$, since O is saddle-node and E is a saddle, the index of O is 0 and the index of E is -1 by Ref. [16, Theorem 3.1 in Chapter 7]. Then, by Ref. [13, Chapter 3], neither limit cycles nor homoclinic loops surround O when $a = 0$, surround both O and E when $a > 0$.

Lemma 3.2 There exists a continuous function $\phi(a) > 0$ such that for fixed $a > 0$, system (3) has one heteroclinic orbit if and only if $b \geq \phi(a)$.

Proof By transformation

$$x \rightarrow x, y \rightarrow y + (x + bx^2 + x^3),$$

in sense of global equivalence system (3) is changed into

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(ax^2 - x^3) - (1 + 2bx + 3x^2)y \end{cases} \quad (13)$$

We check that system (13) has two equilibria at $O(0, 0)$ and $\hat{E}(a, 0)$, where $O(0, 0)$ is a saddle-

node and $\hat{E}(a, 0)$ is a saddle. For system (13), we have

$$\begin{vmatrix} y & -(ax^2 - x^3) - (1 + 2b'x + 3x^2)y \\ y & -(ax^2 - x^3) - (1 + 2bx + 3x^2)y \end{vmatrix} = 2(b' - b)xy^2 \leq 0$$

for all $x \in (0, a)$ when $b > b'$. Therefore, the field vector of system (13) is generalized rotated with respect to b in the strip $0 < x < a$. Thus, for fixed $a > 0$ in system (13), y_A decreases continuously and y_B increases continuously as b increases, where y_A, y_B are the ordinates of A, B , respectively. Here A, B are the intersection points of line $x = \delta$ and the stable manifold W_0^s of O , the unstable manifold W_0^u of \hat{E} respectively, where $\delta > 0$ is a sufficiently small constant, as shown in Fig. 4.

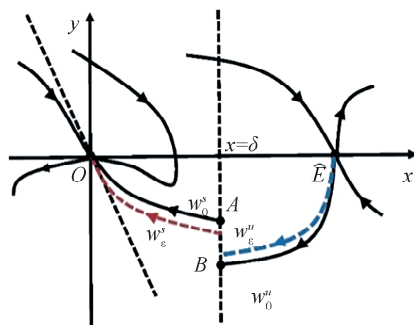


Fig. 4 Monotonic moving for b in system (13)

For system (13) with $a > 0$ and $0 < x < a$,

$$\begin{aligned} \frac{dy}{dx} \Big|_{y=-x} &= \frac{x^2(x-a)}{-x} - (1 + 2bx + 3x^2) = \\ &= -1 + x[-4x + a - 2b]. \end{aligned}$$

Clearly, $dy/dx < -1$ on $y = -x$ when $b > a/2$. So the ω -limit set of W_0^s is O and it approaches $(0, 0)$ in direction $\theta = 0$. Therefore,

$$y_A < y_B \text{ for } b > a/2 \quad (14)$$

When $b < 0$ and $1/b$ is sufficiently small, we have $dy/dx > -1$ on $y = -x$. In a small neighborhood of O , the horizontal isocline is of form

$$y = -ax^2 + (1 + 2ab)x^3 + o(x^3)$$

and there is an asymptote

$$x = \delta := (-b - \sqrt{b^2 - 3}),$$

as shown in Fig. 5. In interval $(0, a)$, δ is the unique zero of function

$$h(x) := 3x^2 + 2bx + 1.$$

Then, $dy/dx > 0$ for $x \in [\delta, a)$ below x -axis. Let

B, N be the intersection points of line $x=\delta$ and the unstable manifold W_0^u of \hat{E} , the line $y=-x$ respectively. Since $y_N \rightarrow 0$ and y_B decreases as $b \rightarrow -\infty$, where y_B, y_N are the ordinates of B, N respectively, there exists $b^* < 0$ such that $1/b^*$ is sufficiently small and $y_B < y_N$, as shown in Fig. 5a. Then $y_Q < y_P$, where y_Q, y_P are the ordinates of Q, P respectively. Here Q, P are the intersection points of $y=h(x)$ and the unstable manifold W_0^u of \hat{E} , the line $y=-x$ respectively. If the unstable manifold W_0^u of \hat{E} approaches $(0, 0)$ in direction $y=-x$, we get a contradiction in region OPQ since there is only one orbit tending to O along $y=-x$. Thus the ω -limit set of W_0^u is I_B^- , corresponding to local phase portrait as shown in Fig. 5b. So $y_A > y_B$ when $b=b^*$, where y_A is the ordinate of the intersection point A of line $x=\delta$ and the stable manifold W_0^s . Associated with system (14), there exists a unique value of b , denoted by $\phi(a)$, such that $y_A = y_B$. It means that when $b \geq \phi(a)$ there exists a heteroclinic orbit connecting O and \hat{E} in system (13). Thus for fixed $a > 0$ system (3) has one heteroclinic orbit if and only if $b \geq \phi(a)$.

4 Bifurcation diagram and global phase portraits

In this section, we give the bifurcation diagram and a complete classification of global phase portraits for system (3) with $(a, b) \in G$.

Theorem 4.1 The complete classification of global phase portraits are given in Fig. 6. The global bifurcation diagram of system (3) with $(a, b) \in G$ is given in Fig. 7, including the following bifurcation curves.

- (i) There exists a transcritical bifurcation curve $a=0$, and the degenerate saddle O breaks into one saddle-node and one saddle when a changes from 0 to a positive number;
- (ii) There exists a heteroclinic bifurcation curve $b=\phi(a)$, and the heteroclinic orbit connecting the saddle-node and the saddle breaks (resp. persists) when b decreases (resp. increases) from $\phi(a)$, where $\phi(a)$ is a non-monotonic continuous function defined for $a > 0$ and there exists a unique $a^* > 0$ such that $\phi(a^*)=0$ and $\phi(a) < 0$ (resp. > 0) when $0 < a < a^*$ (resp. $a > a^*$).

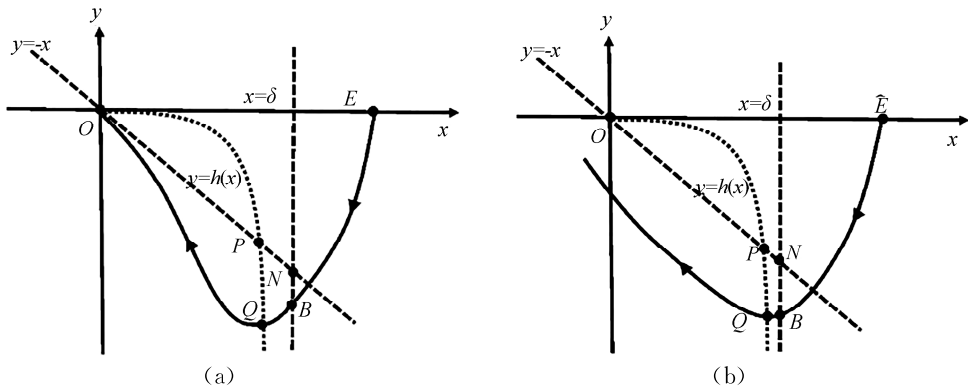


Fig. 5 Orbits of system (13)

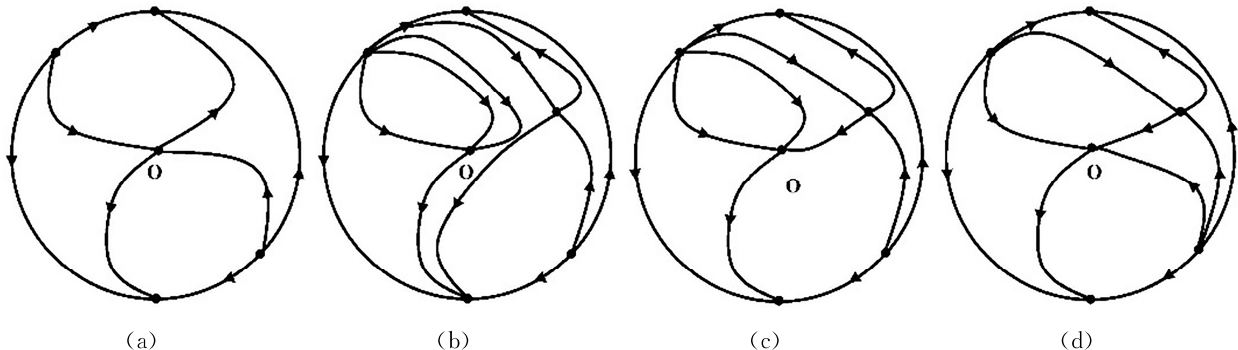


Fig. 6 Global phase portraits: (a) $a=0$; (b) $a>0; b<\phi(a)$; (c) $a>0, b=\phi(a)$; (d) $a>0, b>\phi(a)$

Proof For system (13), let γ_1 be the unstable manifold of O and γ_2 the stable boundary manifold of O in the second quadrant. By Lemmas 2.2, 3.1 in Ref. [13, Chapter 1] and $\dot{x} < 0$ below x -axis, the ω -limit set $\omega(\gamma_1)$ of γ_1 is exactly I_B^- . Similarly, the α -limit set $\alpha(\gamma_2)$ of γ_2 is I_A^- . Thus, equivalently for system (3), we get the global phase portraits when $a=0$, as shown in Fig. 6a. By Lemma 3.2, there exists a continuous function $\phi(a)$ such that when $b < \phi(a)$, the global phase diagram is shown in Fig. 6b. When $b = \phi(a)$, system (3) has one heteroclinic orbit connecting the saddle-node and the saddle, which is the stable boundary manifold of O as shown in Fig. 6c. When $b > \phi(a)$, the heteroclinic orbit persists between O and E , as shown in Fig. 6d. Thus the complete classification of all global phase portraits for system (3) in G are given in Fig. 6.

By Lemma 2.1, the degenerate saddle O breaks into one saddle-node and one saddle when a changes from 0 to a positive number, which means that curve $a=0$ is a transcritical bifurcation curve. By Lemma 3.2, system (3) has a heteroclinic orbit if and only if $a > 0, b \geq \phi(a)$ and the heteroclinic orbit connecting the saddle-node and the saddle breaks (resp. persists) when b decreases (resp. increases) from $\phi(a)$. Thus, the curve

$b = \phi(a)$ is a heteroclinic bifurcation curve. On the one hand, system (3) $|_{b=0}$ satisfies

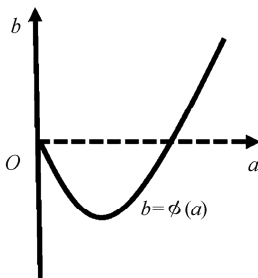


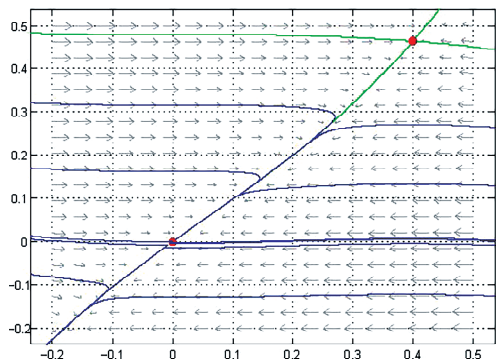
Fig. 7 Bifurcation diagram of system (3)

$$\begin{aligned} & \left| \frac{y - (x + x^3) - (a'x^2 - x^3)}{y - (x + x^3) - (ax^2 - x^3)} \right| = \\ & (a' - a)x^2[y - (x + x^3)] \geq 0 \end{aligned}$$

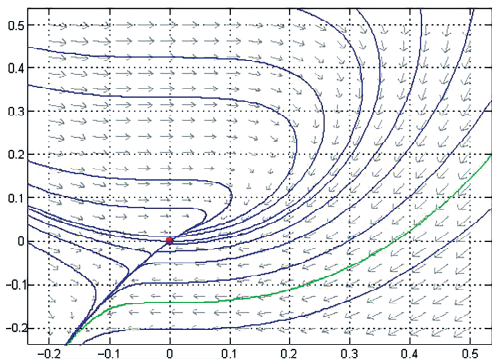
for all $x \in (0, a)$ and $y \leq (x + x^3)$ when $a > a'$. Thus the field vector of system (3) is generalized rotated with respect to a in the region

$$0 < x < a, \quad y \leq (x + x^3).$$

On the other hand, for $(a, b) = (0.4, 0) \in G$ there is a heteroclinic orbit connecting the saddle-node and the saddle as shown in Fig. 8a, which is consistent with Fig. 6d. For $(a, b) = (3.2, 0) \in G$, there is no heteroclinic orbit connecting the saddle-node and the saddle as shown in Fig. 8b, which is consistent with Fig. 6b. Thus the curve $b = \phi(a)$ has a unique intersection with the a -axis, as shown in Fig. 7.



(a)



(b)

Fig. 8 Phase portraits with $b=0$; (a) $a=0.4$; (b) $a=3.2$

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