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一类具曲率算子的非线性方程的波前解

鲁世平[1]，孔凡超[2]，李　洁[2]

(1. 南京信息工程大学数学理学院，南京 210044；2. 安徽师范大学数学计算机科学学院，芜湖 241003)

摘　要：运用单调动力系统定理，本文建立了如下一类具有曲率算子的非线性方程

$$\frac{\partial q(x,t)}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\frac{\partial q(x,t)}{\partial t}}{\sqrt{1+\left(\frac{\partial q(x,t)}{\partial t}\right)^2}}\right)-g(q(x,t))=0$$

波前解的存在性条件.

关键词：波前解；异宿轨道；平均曲率算子

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Traveling wavefronts for nonlinear equation with mean curvature-like operator

LU Shi-Ping[1]，*KONG Fan-Chao*[2]，*LI Jie*[2]

(1. College of Mathematics and Statistics, NUIST, Nanjing 210044, China;
(2. College of Mathematics and Computer Science, Anhui Normal University, Wuhu 241003, China)

Abstract: In this paper, we study the following nonlinear equation with mean curvature-like operator

$$\frac{\partial q(x,t)}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\frac{\partial q(x,t)}{\partial t}}{\sqrt{1+\left(\frac{\partial q(x,t)}{\partial t}\right)^2}}\right)-g(q(x,t))=0.$$

By using the theorem of the monotone dynamical system, the existence conditions of traveling wavefronts established.

Keywords: Traveling wavefront; Heteroclinic orbit; Mean curvature-like operator
(2010 MSC 34C37, 35C07)

1　Introduction

Mean curvature is an external bending measurement standard in differential geometry. It is a description of the surface embedded in the surrounding space (such as a two-dimensional surface embedded in three-dimensional Euclidean space), so it is widely used in the study of surface. In the past few decades, mean curvature equations and their modified forms which derived from differential geometry and physics have been paid more and more great attention (see Refs. [1-4] and the references therein).

In this paper, we consider traveling wavefronts for the following nonlinear equation with mean curvature-like operator:

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作者简介：鲁世平，安徽无为人，博士，硕士生导师，主要研究方向为常微分方程与动力系统. E-mail：lushiping88@sohu.com.

$$\frac{\partial q(x,t)}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\frac{\partial q(x,t)}{\partial t}}{\sqrt{1+(\frac{\partial q(x,t)}{\partial t})^2}}\right)-g(q(x,t))=0 \quad (1)$$

where $g\in C^1(\mathbf{R},\mathbf{R})$. Eq. (1) comes from gas dynamics and has been studied during the past thirty years, For example, combustible gas dynamics[5-9]. Suppose a compressible gas flows in a homogeneous isotropic rigid porous medium. Then the volumetric gas content θ, the velocity $\vec{v}$ and the density of the gas are governed by the continuity equation

$$\theta\frac{\partial\rho}{\partial t}+\nabla(\rho\vec{v})=0.$$

Because of the influence of many factors, such as the molecular and ion effects, instead, one has the following nonlinear relation:

$$\rho\vec{v}=-\lambda\frac{\nabla P}{\sqrt{1+|\nabla P|^2}},$$

where $\rho\vec{v}$ and $P=\rho\,\eta$ denote the momentum velocity and pressure respectively, $\lambda>0$ is physical constant. After changing variables and notations, Eq. (1) is derived.

In 2004, Li[10] discussed global existence and quenching phenomena for a parabolic equation of the mean curvature type with nonlinear convection term

$$u_t-\operatorname{div}\{\sigma(|\nabla u|^2)\nabla u\}+b(u)$$
$$\nabla u=0,x\in\Omega,t>0,$$
$$u(x,0)=u_0,x\in\Omega;u(x,t)=0,x\in\partial\Omega,t>0,$$

where $\sigma(|\nabla u|^2)=1/\sqrt{1+|\nabla u|^2}$. Such problems also have attracted the attention of Chen[11,12]. Recently, more and more authors paid attention to traveling wavefront[13-16]. However, to the best of our knowledge, the corresponding theory for traveling wavefronts of the nonlinear equation with mean curvature-like operator is not investigated till now. So, in this paper, we aim to study the existence of traveling wavefronts of the nonlinear equation with mean curvature-like operator.

This paper is organized as follows. In Section 2, we state some necessary definitions and lemmas. In Section 3, we prove the main results and we give an example of application in section 4.

2 Preliminaries

In this section, we provide some definitions and lemmas which will be used in this paper. Let $q(x,t)=u(\xi)=u(x-ct)$ with $c\in\mathbf{R}$, then Eq. (1) is transformed into the following form:

$$-cu'(\xi)+\left(\frac{u'(\xi)}{\sqrt{1+(u'(\xi))^2}}\right)'-g(u(\xi))=0 \quad (2)$$

Traveling wavefronts: A function $u(\xi)$ is called a traveling wave front with waves peed c, if there exist ξ_a,ξ_b with $-\infty\leqslant\xi_a<\xi_b<+\infty$, and a monotonic function $u(\xi)$ with $u\in C^1(\xi_a,\xi_b)$, and $\left(\frac{u'(\xi)}{\sqrt{1+(u'(\xi))^2}}\right)'\in C^1(\xi_a,\xi_b)$, such that (2) holds and

(a) If u is a monotonic increasing function, then $u(\xi_a)=0,u(\xi_b)=\frac{1}{2},u'(\xi_a)=u'(\xi_b)=0$;

(b) If u is a monotonic decreasing function, then $u(\xi_a)=\frac{1}{2},u(\xi_b)=0,u'(\xi_a)=u'(\xi_b)=0$.

Throughout this paper, for $y=(u,v)$ and $z=(u_1,v_1)$, we write $y\leqslant z$ if $u\leqslant u_1,v\leqslant v_1$, $y<z$ if $y\leqslant z$ but $y\neq z$, and $y\leqslant z$ if $y\leqslant z$ but $u\neq u_1$, $v\neq v_1$.

Let $\varphi_t(y)$ be the flow generated by the following autonomous system:

$$y'=f(y) \quad (3)$$

where $f=(f_1,f_2)\in C^1(\mathbf{R}^2,\mathbf{R}^2)$, $y=(u,v)\in\mathbf{R}^2$. We write $\gamma^+(y)=\{\varphi_t(y):t\geqslant 0\}$ for the positive orbit through the point y, and $\omega(y)=\bigcap_{t\geqslant 0}\bigcup_{s\geqslant t}\varphi_s(y)$ for the omega limit set of y. Let D be an open subset of $\mathbf{R}^2$.

Definition 2.1 A set $M\subseteq\mathbf{R}^2$ is said to be positively invariant, if $\varphi_t(M)\subseteq M$ for all $t\geqslant 0$.

Definition 2.2 A set $Q\subseteq\mathbf{R}^2$ is said to be p-convex, if for any $y,z\in Q$ satisfying $y\leqslant z$ and the segment joining them also belongs to Q.

Definition 2.3 System (3) is said to be cooperative on D, if D is p-convex and the following conditions hold:

$$\frac{\partial f_1(u,v)}{\partial v}\geqslant 0,\ \frac{\partial f_2(u,v)}{\partial u}\geqslant 0,(u,v)\in D.$$

***Remark* 1** According to Remark 1.4 in Ref.

[9], D can be a closed set if it satisfies the two conditions as following:

(i) D is the closure of an open set Q on which the system (3) is cooperative;

(ii) For all $x, y \in D$, satisfying $x < y$, there are two sequences $x_n, y_n \in Q$ such that

$x_n \to x, y_n \to y$ as $n \to +\infty$, $x_n < y_n$.

Lemma 2.4 [17] If System (3) is cooperative on D and $<_r$ stands for one of the relations $\leqslant$, $<$, $\preccurlyeq$, then $P_+ = \{y \in D: 0 <_r f(y)\}$ and $P_- = \{y \in D: f(y) <_r 0\}$ are positively invariant. If $y \in P_+$ (P_-), then $\varphi_t(y)$ is nondecreasing (nonincreasing) for $t \geqslant 0$. In addition, if $\gamma^+(y)$ has compact closure in D, then $\omega(y)$ is an equilibrium.

Remark 2 Suppose that all the conditions in Lemma 2.4 hold, let y be a point on the unstable manifold of a saddle, then $\varphi_t(y)$ is a heteroclinic orbit connecting the saddle and another equilibrium. If, in addition, $\varphi_t(y)$ is increasing for $t \leqslant 0$, then $\varphi_t(y)$ is increasing for $t \in \mathbf{R}$.

3 Main results

Let $v(\xi) = u(\xi) + \dfrac{u'(\xi)}{\sqrt{1+(u'(\xi))^2}}$. Then we can get from (2) that

$$\begin{cases} u'(\xi) = f_1(u(\xi), v(\xi)) = \dfrac{v(\xi)-u(\xi)}{\sqrt{1-(v(\xi)-u(\xi))^2}}, \\ v'(\xi) = f_2(u(\xi), v(\xi)) = \\ \quad (c+1) f_1(u(\xi), v(\xi)) + g(u(\xi)) \end{cases} \tag{4}$$

Throughout this paper, we assume that there exist $g_1 \in C^1(\mathbf{R}, \mathbf{R}^+)$, $\mathbf{R}^+ = (0, +\infty)$ and a given constant $0 < \delta < 1$ such that $g(u) = (\delta u - u^2) g_1(u)$. Obviously, system (4) has only two equilibria: $(0,0)$ and (δ, δ). If the system (4) admits an increasing heteroclinic orbit connecting the two equilibria, then Eq. (1) correspondingly admits an increasing traveling wavefront satisfying

$$\lim_{\xi \to -\infty} u(\xi) = 0, \quad \lim_{\xi \to +\infty} u(\xi) = \delta.$$

Theorem 3.1 Suppose that

$$c \leqslant -1 - \max_{u \in (0,\delta)} \left| \frac{\partial g(u)}{\partial u} \right|$$

and

$$g_1(0)\delta + \frac{c^2 + c\sqrt{c^2 + 4 g_1(0)\delta}}{2} > 0.$$

Then Eq. (1) admits an increasing traveling wavefront $q(x,t) = u(\xi)$ satisfying

$$\lim_{\xi \to -\infty} u(\xi) = 0, \quad \lim_{\xi \to +\infty} u(\xi) = \delta.$$

Proof Let $D = \{(u,v) \in \mathbf{R}^2 : 0 \leqslant u, v \leqslant \delta\}$. Obviously, D is p-convex. Since

$$\frac{f_1(u(\xi), v(\xi))}{\partial v} = \frac{1}{[1-(v(\xi)-u(\xi))^2]^{\frac{3}{2}}} > 0, \quad (u,v) \in D$$

and

$$\frac{\partial f_2(u(\xi), v(\xi))}{\partial u} = \frac{-c-1}{[1-(v(\xi)-u(\xi))^2]^{\frac{3}{2}}} + \frac{\partial g(u(\xi))}{\partial u(\xi)} \geqslant -c-1+\frac{\partial g(u(\xi))}{\partial u(\xi)} \geqslant \max_{u \in (0,\delta)} \left| \frac{\partial g(u)}{\partial u} \right| + \frac{\partial g(u(\xi))}{\partial u(\xi)} \geqslant 0,$$

by Definition 2.3, we know that the system (4) is cooperative on D. Next, we will show that D is positively invariant. Defined four segments constituting the boundary of D except two equilibria by D_1, D_2, D_3, D_4,

$$D_1 = \{(u,v) \in D, 0 < u \leqslant \delta, v = 0\},$$
$$D_2 = \{(u,v) \in D, u = 0, 0 < v \leqslant \delta\},$$
$$D_3 = \{(u,v) \in D, u = \delta, 0 \leqslant v < \delta\},$$
$$D_4 = \{(u,v) \in D, 0 \leqslant u < \delta, v = \delta\}.$$

For the points of D_1, we have

$$\begin{cases} u'(\xi) = f_1(u(\xi), 0) = \\ \quad \dfrac{-u(\xi)}{\sqrt{1-(-u(\xi))^2}} < 0, \\ v'(\xi) = f_2(u(\xi), 0) = \\ \quad (c+1) f_1(u(\xi), 0) + g(u(\xi)) > \\ \quad f_2(0,0) = 0 \end{cases} \tag{5}$$

For the points of D_2, we have

$$\begin{cases} u'(\xi) = f_1(0, v(\xi)) = \dfrac{v(\xi)}{\sqrt{1-(v(\xi))^2}} > 0, \\ v'(\xi) = f_2(0, v(\xi)) = \\ \quad (c+1) f_1(0, v(\xi)) < 0 \end{cases} \tag{6}$$

For the points of D_3, we have

$$\begin{cases} u'(\xi) = f_1(\delta, v(\xi)) = \dfrac{v(\xi)-\delta}{\sqrt{1-(v(\xi)-\delta)^2}} < 0, \\ v'(\xi) = f_2(\delta, v(\xi)) = (c+1) f_1(\delta, v(\xi)) > 0 \end{cases} \tag{7}$$

For the points of D_4, since

$$\frac{\partial f_2(u,\delta)}{\partial u} = \frac{-c-1}{[1-(\delta-u(\xi))^2]^{\frac{3}{2}}} + \frac{\partial g(u(\xi))}{\partial u(\xi)} \geqslant -c-1+\frac{\partial g(u(\xi))}{\partial u(\xi)} \geqslant 0,$$

we have

$$\begin{cases} u'(\xi) = f_1(u(\xi),\delta) = \\ \quad \dfrac{\delta - u(\xi)}{\sqrt{1-(\delta-u(\xi))^2}} > 0, \\ v'(\xi) = f_2(u(\xi),\delta) = \\ \quad (c+1)f_1(u(\xi),\delta) + g(u(\xi)) \leqslant \\ \quad f_2(\delta,\delta) = 0 \end{cases} \tag{8}$$

By (5～8), we see that the direction of the vector field (f_1,f_2) along the boundary of D except the two equilibria always point to the interior of D. The linearization system of (4) at (0,0) is

$$\begin{pmatrix} u'(\xi) \\ v'(\xi) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -c-1+g_1(0)\delta & c+1 \end{pmatrix} \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix}.$$

Calculate the eigenvalues of the linearization of the system (4) at (0,0), we obtain

$$\lambda_1 = \frac{c-\sqrt{c^2+4g_1(0)\delta}}{2},$$

$$\lambda_2 = \frac{c+\sqrt{c^2+4g_1(0)\delta}}{2}.$$

We can see that (0,0) is a saddle. Calculate the eigenvector of λ_2, i.e., a solution of

$$\begin{pmatrix} \lambda_2+1 & -1 \\ c+1-g_1(0)\delta & \lambda_2-c-1 \end{pmatrix} \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

it easy to see that

$$\begin{pmatrix} \lambda_2+1 & -1 \\ c+1-g_1(0)\delta & \lambda_2-c-1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{4} \\ \dfrac{1}{4}+\dfrac{\lambda_2}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{\lambda_2^2 - c\lambda_2 - g_1(0)\delta}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $l = (\frac{1}{4}, \frac{1}{4}+\frac{\lambda_2}{4})^T$ is a solution of Eq. (9). Applying Theorem 6.1 of Ref. [18], there is a one-dimensional unstable manifold tangent to l at (0, 0). Points on this unstable manifold are parametrically represented in a small neighborhood of (0, 0) by a function $\rho:\mathbf{R}\to\mathbf{R}^2$, where $\rho(m) = (0,0) + ml + o(|m|)$.

Now we show that, for sufficiently small $m > 0, \rho(m) \in P + \{(u,v), f_1(u,v) \geqslant 0, f_2(u,v) \geqslant 0\}$. Since

$$f_1(\frac{m}{4}, \frac{m}{4}+\frac{m}{4}\lambda_2) = \frac{\frac{m}{4}\lambda_2}{\sqrt{1-(\frac{m}{4}\lambda_2)^2}} > 0 \tag{10}$$

and

$$g_1(0)\delta + \frac{c^2+c\sqrt{c^2+4g_1(0)\delta}}{2} > 0,$$

for sufficiently small $m > 0$, we have

$$\frac{m}{4}[\frac{c^2+c\sqrt{c^2+4g_1(0)\delta}}{2} + (\delta-\frac{m}{4})g_1(\frac{m}{4})] > 0,$$

as $c \leqslant -1 - \max\limits_{u\in(0,\delta)} |\frac{\partial g(u)}{\partial u}|$. We can assert that $\frac{(c+1)\lambda_2}{\sqrt{1-(\frac{m}{4}\lambda_2)^2}} > c\lambda_2$. Then we have

$$f_2(\frac{m}{4}, \frac{m}{4}+\frac{m}{4}\lambda_2) = \frac{m}{4}[\frac{(c+1)\lambda_2}{\sqrt{1-(\frac{m}{4}\lambda_2)^2}}] + g(\frac{m}{4}) > c\lambda_2\frac{m}{4} + g(\frac{m}{4}) = \frac{m}{4}[\frac{c^2+c\sqrt{c^2+4g_1(0)\delta}}{2} + (\delta-\frac{m}{4})g(\frac{m}{4})] > 0 \tag{11}$$

Furthermore, since D is positively invariant, $\gamma^+(\rho(m))$ has compact closure in D. By Lemma 2.4, $\varphi_t(\rho(m))$ is increasing for $t \geqslant 0$ at $\leqslant 0$ and $\omega(\rho(m))$ is an equilibrium. Obviously, $\varphi_t(\rho(m))$ is also increasing for and $\omega(\rho(m)) = (\delta,\delta)$. By Remark 2, System (4) admits an increasing heteroclinic orbit connecting the two equilibria, which implies that Eq. (1) correspondingly admits an increasing traveling wavefront satisfying $\lim\limits_{\xi\to-\infty} u(\xi) = 0$ and $\lim\limits_{\xi\to+\infty} u(\xi) = \delta$. Hence Theorem 3.1 holds.

4 Example

Consider the following nonlinear equation with mean curvature-like operator

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial}{\partial x}(\frac{\frac{\partial q(x,t)}{\partial t}}{\sqrt{1+(\frac{\partial q(x,t)}{\partial t})^2}}) - (\frac{1}{3}q(x,t) - q^2(x,t))(1+q^2(x,t)) = 0 \tag{12}$$

Eq. (12) can be regarded as a equation of the form (1), where

$$g(q(x,t)) = (\frac{1}{3}q(x,t) - q^2(x,t))(1 + q^2(x,t)),$$

$$g_1(q(x,t)) = 1 + q^2(x,t), \delta = \frac{1}{3}.$$

Choosing $c = -\frac{37}{27}$, it is not difficult to see that

$$-1 - \max_{u\in(0,\frac{1}{2})} \left|\frac{\partial g(u)}{\partial u}\right| = -1 - \max_{u\in(0,\frac{1}{3})} \left|\frac{1}{3} - 4u^3 - \frac{5u}{3}\right| > -\frac{37}{27} = c$$

and

$$g_1(0)\delta + \frac{c^2 + c\sqrt{c^2 + 4g_1(0)\delta}}{2} = \frac{1}{3} + \frac{\frac{37^2}{27^2} - \frac{37}{27}\sqrt{\frac{37^2}{27^2} + \frac{4}{3}}}{2} \approx 0.04 > 0.$$

Therefore the conditions of Theorem 3.1 are satisfied. It follows that Eq. (12) has an increasing traveling wavefront $q(x,t) = u(\xi)$ satisfying

$$\lim_{\xi\to-\infty} u(\xi) = 0, \lim_{\xi\to+\infty} u(\xi) = \frac{1}{3}.$$

***Remark* 3** One can easily see that all the results in Refs. [1～19] and the references therein can not be applicable to Eq. (12) to obtain the result of existence of traveling wavefronts, Which implies that the results in our paper are essentially new.

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